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Lie Symmetry Analysis of Seventh Order Caudrey-Dodd-Gibbon Equation

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Abstract. In the present paper, seventh order Caudrey-Dodd-Gibbon (CDG) equation is solved by Lie symmetry analysis. All the geometry vector fields of seventh order KdV equations are presented. Using Lie transformation seventh order CDG equation is reduced into ordinary differential equations. These ODEs are solved by power series method to obtain exact solution. The convergence of the power series is also discussed.

1 Introduction

Nonlinear PDEs with high non-linearity have a very deep impact not only in nonlinear sciences but also in applied mathematics as well as in theoretical physics. The solution of these equations helps us understand the complete physical phenomena involved therein. There are number of methods used to solve the nonlinear PDEs; commonly used methods are exp function method [1], G'/G method [6], tanh method [3], Lie symmetry method [4], [2] etc. It is well known that Lie group method is a powerful tool to construct the exact solution of nonlinear PDEs; depending on Lie group various kind of solution are obtained as traveling wave solutions, soliton solutions, fundamental solutions and so on. The general form of seventh order Korteweg-de Varies (KdV) equation is given by

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0, \qquad (1)$$

where a, b, c, d, e, f, g are constants.

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In the present paper, we deal with CDG equation which is a particular form of above equation with $a = 420, b = 0, c = 420, d = 210, e = 70, f = g = 28$ as follows

$$u_t + 420u^3u_x + 420uu_xu_{2x} + 210u^2u_{3x} + 70u_{2x}u_{3x} + 28u_xu_{4x} + 28uu_{5x} + u_{7x} = 0. \quad (2)$$

where u, x, t denotes the wavelength, space and time variable respectively u_{nx} denotes the n-th partial derivative with respect to x. This equation is formed in various areas of science and engineering e.g. fluid dynamics, plasma physics, laser optics, traffic flow and elastic media etc. The paper is organized as follows. Section 1 contains some background information related to Lie symmetry analysis. In section 2, with the help of Lie algebra the vector field of Eq. (2) is obtained. In section 3, symmetry reduction is done to obtain ODEs. Power series solution and convergence of the ODE are presented in section 4. Section 5, contains results and discussion. Finally, conclusion is drawn in section 6.

2 Group Analysis of CDG Equation

We consider one-parameter Lie group of infinitesimal transformations:

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + o(\epsilon^2), \\ x^* &= x + \epsilon\xi(x,t,u) + o(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + o(\epsilon^2), \end{aligned} \quad (3)$$

where $\epsilon \ll 1$ is a small parameter. The geometric vector field of a PDE is given by

$$V = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u. \quad (4)$$

If the vector field (4) generates a symmetry of the equation (2), then V must satisfy the Lie symmetry condition

$$Pr^7V(\Delta)|_{\Delta=0} = 0, \quad (5)$$

where

$$\Delta = u_t + 420u^3u_x + 420uu_xu_{2x} + 210u^2u_{3x} + 70u_{2x}u_{3x} + 28u_xu_{4x} + 28uu_{5x} + u_{7x}, \quad (6)$$

and

$$Pr^7V = \tau\partial_t + \xi\partial_x + \eta\partial_u + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} + \eta^{2x}\partial_{u_{2x}} + \eta^{3x}\partial_{u_{3x}} + \eta^{4x}\partial_{u_{4x}} + \eta^{5x}\partial_{u_{5x}} + \eta^{7x}\partial_{u_{7x}} \quad (7)$$

is the 7-th order prolongation of V, where, for each $k \in \{1,2,3,4,5,7\}$, the coefficient function η^{kx} is given by

$$\eta^{kx} = D_x^k(\eta - \tau u_t - \xi u_x) + \tau u_{kxt} + \xi u_{(k+1)x}. \quad (8)$$

Here, the symbol D_x stands for total differential operator and is given by

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots. \tag{9}$$

Using the Lie symmetry analysis method, we obtain

$$\tau = c_1 t + c_2, \quad \xi = \frac{1}{7}c_1 x + c_3, \quad \eta = -\frac{2}{7}c_1 u, \tag{10}$$

where c_1, c_2, c_3 are arbitrary constant. So we have the following geometric vector fields

$$V_1 = \frac{1}{7}x\partial_x + t\partial_t - \frac{2}{7}u\partial_u, \quad V_2 = \partial_t, \quad V_3 = \partial_x. \tag{11}$$

Further, it is necessary to show the vector fields of Eq. (2) are closed under the Lie bracket, we have

$$[V_i, V_i] = 0, \quad i = 1, 2, 3. \tag{12}$$

$$[V_1, V_2] = -[V_2, V_1] = V_2, \quad [V_1, V_3] = -[V_3, V_1] = \frac{1}{7}V_3, \quad [V_2, V_3] = -[V_3, V_2] = 0. \tag{13}$$

3 Similarity Reductions

In this section, we obtain the reduction equations with the help of similarity variables and find the exact solutions of these equations.
Case (i): For the generator V_1 we have

$$u = t^{-\frac{2}{7}}f(\zeta), \quad \text{where} \quad \zeta = xt^{-\frac{1}{7}}. \tag{14}$$

Substituting Eq. (14) into Eq. (2), we have the following ODE:

$$\begin{aligned}-(2f + \zeta f^{(1)}) + 420f^3f^{(1)} + 420ff^{(1)}f^{(2)} + 210f^2f^{(3)}\\ + 70f^{(2)}f^{(3)} + 28f^{(1)}f^{(4)} + 28ff^{(5)} + f^{(7)} = 0,\end{aligned} \tag{15}$$

where $f^{(n)} = \frac{d^nf}{d\zeta^n}$.
Case (ii): For the generator V_2, we have a trivial solution $u(x,t) = c$ where c is an arbitrary constant.
Case (iii): For the generator V_3, we have

$$u = f(\zeta), \quad \text{where} \quad \zeta = x. \tag{16}$$

Substituting Eq. (16) into Eq. (1), we have the following ODE:

$$420f^3f^{(1)} + 420ff^{(1)}f^{(2)} + 210f^2f^{(3)} + 70f^{(2)}f^{(3)} + 28f^{(1)}f^{(4)} + 28ff^{(5)} + f^{(7)} = 0. \tag{17}$$

4 The Exact Power Series Solutions

Now, our aim is to solve the ODEs (15) and (17) by using the power series method. Suppose that Eq.(15) has a solution of the form

$$f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n. \tag{18}$$

Putting Eq.(18) into Eq.(15), we obtain

$$\begin{aligned}
&-\frac{2}{7}c_0 - \frac{2}{7}\sum_{n=1}^{\infty} c_n\zeta^n - \frac{1}{7}\sum_{n=1}^{\infty} n c_n \zeta^n + 420 c_0^3 c_1 \\
&\quad + 420\sum_{n=1}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l}(l-j+1)c_j c_k c_{l-j+1} c_{n-l-k}\zeta^n + 840 c_0 c_1 c_2 \\
&\quad + 420\sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(j-k+2)(j-k+1)(k+1)c_{k+1}c_{n-j}c_{j-k+2}\zeta^n + 1260 c_0^2 c_3 \\
&\quad + 210\sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(n-j+3)(n-j+2)(n-j+1)c_k c_{j-k} c_{n-j+3}\zeta^n + 840 c_2 c_3 \\
&\quad + 70\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+3)(n-k+2)(n-k+1)(k+1)(k+2)c_{n-k+3}c_{k+2}\zeta^n \\
&\quad + 672 c_1 c_4 \\
&\quad + 28\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+4)(n-k+3)(n-k+2)(n-k+1)(k+1)c_{n-k+4}c_{k+1}\zeta^n \\
&\quad + 3360 c_0 c_5 \\
&\quad + 28\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+5)(n-k+4)(n-k+3)(n-k+2)(n-k+1)c_{n-k+5}c_k\zeta^n \\
&\quad + 5040 c_7 + (n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)c_{n+7}\zeta^n = 0.
\end{aligned} \tag{19}$$

Comparing the coefficients for $n = 0$ in Eq. (19), we have

$$c_7 = \frac{\frac{2}{7}c_0 - 420c_0^3 c_1 - 840 c_0 c_1 c_2 - 1260 c_0^2 c_3 - 840 c_2 c_3 - 672 c_1 c_4 - 3360 c_0 c_5}{5040}. \tag{20}$$

For $n \geq 1$, we have recursion formula

$$
\begin{aligned}
c_{n+7} = & -\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\
& \times \Bigg(420 \sum_{l=0}^{n} \sum_{k=0}^{n-l} \sum_{j=0}^{l} (l-j+1) c_j c_k c_{l-j+1} c_{n-l-k} \\
& + 420 \sum_{j=0}^{n} \sum_{k=0}^{j} (j-k+2)(j-k+1)(k+1) c_{k+1} c_{n-j} c_{j-k+2} \\
& + 210 \sum_{j=0}^{n} \sum_{k=0}^{j} (n-j+3)(n-j+2)(n-j+1) c_k c_{j-k} c_{n-j+3} \\
& + 70 \sum_{k=0}^{n} (n-k+3)(n-k+2)(n-k+1)(k+1)(k+2) c_{n-k+3} c_{k+2} \\
& + 28 \sum_{k=0}^{n} (n-k+4)(n-k+3)(n-k+2)(n-k+1)(k+1) c_{n-k+4} c_{k+1} \\
& + 28 \sum_{k=0}^{n} (n-k+5)(n-k+4)(n-k+3)(n-k+2)(n-k+1) c_{n-k+5} c_k - \frac{2}{7} c_n - \frac{1}{7} n c_n \Bigg).
\end{aligned} \tag{21}
$$

The power series solution is given by

$$
\begin{aligned}
f(\zeta) = & \, c_0 + c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + c_4 \zeta^4 + c_5 \zeta^5 + c_6 \zeta^6 \\
& + \frac{\frac{2}{7} c_0 - 420 c_0^3 c_1 - 840 c_0 c_1 c_2 - 1260 c_0^2 c_3 - 840 c_2 c_3 - 672 c_1 c_4 - 3360 c_0 c_5}{5040} \zeta^7 \\
& + \sum_{n=1}^{\infty} c_{n+7} \zeta^{n+7},
\end{aligned} \tag{22}
$$

where c_{n+7} is given by Eq. (21). Finally the solution $u(x,t)$ of Eq. (15) is given by

$$
\begin{aligned}
& u(x,t) \\
& = \Bigg[c_0 + c_1 (x t^{-\frac{1}{7}}) + c_2 (x t^{-\frac{1}{7}})^2 + c_3 (x t^{-\frac{1}{7}})^3 + c_4 (x t^{-\frac{1}{7}})^4 + c_5 (x t^{-\frac{1}{7}})^5 + c_6 (x t^{-\frac{1}{7}})^6 \\
& \quad + \frac{\frac{2}{7} c_0 - 420 c_0^3 c_1 - 840 c_0 c_1 c_2 - 1260 c_0^2 c_3 - 840 c_2 c_3 - 672 c_1 c_4 - 3360 c_0 c_5}{5040} (x t^{-\frac{1}{7}})^7 \\
& \quad + \sum_{n=1}^{\infty} c_{n+7} (x t^{-\frac{1}{7}})^{n+7} \Bigg] t^{-\frac{2}{7}}.
\end{aligned} \tag{23}
$$

Further, we have to solve Eq. (17) so, once again from Eq. (17) and Eq. (18)

$$
\begin{aligned}
&420c_0^3c_1+420\sum_{n=1}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l}(l-j+1)c_jc_kc_{l-j+1}c_{n-l-k}\zeta^n+840c_0c_1c_2\\
&+420\sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(j-k+2)(j-k+1)(k+1)c_{k+1}c_{n-j}c_{j-k+2}\zeta^n+1260c_0^2c_3\\
&+210\sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(n-j+3)(n-j+2)(n-j+1)c_kc_{j-k}c_{n-j+3}\zeta^n+840c_2c_3\\
&+70\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+3)(n-k+2)(n-k+1)(k+1)(k+2)c_{n-k+3}c_{k+2}\zeta^n+672c_1c_4\\
&+28\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+4)(n-k+3)(n-k+2)(n-k+1)(k+1)c_{n-k+4}c_{k+1}\zeta^n+3360c_0c_5\\
&+28\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+5)(n-k+4)(n-k+3)(n-k+2)(n-k+1)c_{n-k+5}c_k\zeta^n+5040c_7\\
&+(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)c_{n+7}\zeta^n=0. \qquad (24)
\end{aligned}
$$

Equating the coefficients of like power in Eq.(24) we have

$$
c_7=\frac{-420c_0^3c_1-840c_0c_1c_2-1260c_0^2c_3-840c_2c_3-672c_1c_4-3360c_0c_5}{5040}. \qquad (25)
$$

For $n\geq 1$, we have recursion formula

$$
\begin{aligned}
c_{n+7}=&-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \qquad (26)\\
&\times\Bigg(420\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l}(l-j+1)c_jc_kc_{l-j+1}c_{n-l-k}\\
&+420\sum_{j=0}^{n}\sum_{k=0}^{j}(j-k+2)(j-k+1)(k+1)c_{k+1}c_{n-j}c_{j-k+2}\\
&+210\sum_{j=0}^{n}\sum_{k=0}^{j}(n-j+3)(n-j+2)(n-j+1)c_kc_{j-k}c_{n-j+3}\\
&+70\sum_{k=0}^{n}(n-k+3)(n-k+2)(n-k+1)(k+1)(k+2)c_{n-k+3}c_{k+2}\\
&+28\sum_{k=0}^{n}(n-k+4)(n-k+3)(n-k+2)(n-k+1)(k+1)c_{n-k+4}c_{k+1}\\
&+28\sum_{k=0}^{n}(n-k+5)(n-k+4)(n-k+3)(n-k+2)(n-k+1)c_{n-k+5}c_k\Bigg).
\end{aligned}
$$

The power series solution is given by

$$f(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + c_5\zeta^5 + c_6\zeta^6 \\ + \frac{-420c_0^3c_1 - 840c_0c_1c_2 - 1260c_0^2c_3 - 840c_2c_3 - 672c_1c_4 - 3360c_0c_5}{5040}\zeta^7 \\ + \sum_{n=1}^{\infty} c_{n+7}\zeta^{n+7}, \tag{27}$$

where c_{n+7} is given by Eq. (26). Finally, the solution $u(x,t)$ of Eq. (17) is given by

$$u(x,t) = \Bigg[c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 \\ + \frac{-420c_0^3c_1 - 840c_0c_1c_2 - 1260c_0^2c_3 - 840c_2c_3 - 672c_1c_4 - 3360c_0c_5}{5040}x^7 \\ + \sum_{n=1}^{\infty} c_{n+7}x^{n+7}\Bigg]. \tag{28}$$

Now, we show the convergence of the power series solution (18) of Eq. (15) by using implicit function theorem. From Eq. (21) we have

$$c_{n+7} \leq M\Bigg(|c_n| + \sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l} |c_j||c_k||c_{l-j+1}||c_{n-l-k}| \\ + \sum_{j=0}^{n}\sum_{k=0}^{j} |c_{k+1}||c_{n-j}||c_{j-k+2}| + \sum_{j=0}^{n}\sum_{k=0}^{j} |c_k||c_{j-k}||c_{n-j+3}| \\ + \sum_{k=0}^{n} |c_{n-k+3}||c_{k+2}| + \sum_{k=0}^{n} |c_{n-k+4}||c_{k+1}| + \sum_{k=0}^{n} |c_{n-k+5}||c_k|\Bigg), \tag{29}$$

where $M = 420$. If we define a power series

$$\nu = P(\zeta) = \Sigma_{n=0}^{\infty} p_n\zeta^n, \tag{30}$$

with

$$p_0 = |c_0|, \quad p_1 = |c_1|, \quad p_2 = |c_2| \quad p_3 = |c_3| \\ p_4 = |c_4|, \quad p_5 = |c_5|, \quad p_6 = |c_6|, \quad p_7 = |c_7|. \tag{31}$$

Also, we have

$$p_{n+7} = 420\Bigg(p_n + \sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l} p_j p_k p_{l-j+1} p_{n-l-k} + \sum_{j=0}^{n}\sum_{k=0}^{j} p_{k+1} p_{n-j} p_{j-k+2} + \sum_{j=0}^{n}\sum_{k=0}^{j} p_k p_{j-k} p_{n-j+3} + \sum_{k=0}^{n} p_{n-k+3} p_{k+2} + \sum_{k=0}^{n} p_{n-k+4} p_{k+1} + \sum_{k=0}^{n} p_{n-k+5} p_k\Bigg). \tag{32}$$

It is obvious that

$$|c_n| \leq |p_n| \quad n = 0, 1, 2, \ldots \tag{33}$$

If we are able to prove that the series $\nu = P(\zeta) = \Sigma_{n=0}^{\infty} p_n \zeta^n$ is convergent then from Eq. (33) we conclude that series given by Eq. (18) is also convergent. In order to prove the convergence of the series $\nu = P(\zeta)$, first of all we show that the series has positive radius of convergence. We have

$$P(\zeta) = p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + p_7\zeta^7 + \sum_{n=1}^{\infty} p_{n+7}\zeta^{n+7}. \tag{34}$$

From Eqs. (32) and (34), we have

$$\begin{aligned} P(\zeta) &= p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + p_7\zeta^7 \qquad (35)\\ &\quad + 420\Bigg[\sum_{n=1}^{\infty} p_n\zeta^{n+7} + \sum_{n=1}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{l} p_j p_k p_{l-j+1} p_{n-l-k}\zeta^{n+7} \\ &\quad + \sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j} p_{k+1} p_{n-j} p_{j-k+2}\zeta^{n+7} + \sum_{n=1}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j} p_k p_{j-k} p_{n-j+3}\zeta^{n+7} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p_{n-k+3} p_{k+2}\zeta^{n+7} + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p_{n-k+4} p_{k+1}\zeta^{n+7} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p_{n-k+5} p_k\zeta^{n+7}\Bigg] \\ &= p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + p_7\zeta^7 \qquad (36)\\ &\quad + 420\Big[\zeta^6 P(\zeta)^4 + (2\zeta^4 - 2p_0\zeta^6)P(\zeta)^3 + \\ &\quad + (3\zeta^2 - 4p_0\zeta^4 - 2p_1\zeta^5 + (p_0^2 - p_2)\zeta^6 - p_3\zeta^7)P(\zeta)^2 \\ &\quad - (5p_0\zeta^2 + 4p_1\zeta^3 + (3p_2 - 3p_0^2)\zeta^4 + (3p_3 - 2p_0p_1)\zeta^5 + 2p_4\zeta^6 + p_5\zeta^7)P(\zeta) \\ &\quad + 2p_0\zeta^2 + 3p_0p_1\zeta^3 + (p_1^2 + 2p_0p_2 - p_0^3)\zeta^4 + (2p_0p_3 + p_1p_2 - p_0^2p_1)\zeta^5 \\ &\quad + (p_0p_4 + p_1p_3)\zeta^6 - p_0\zeta^7\Big]. \end{aligned}$$

Now, we consider the implicit functional equation

$$
\begin{aligned}
&T(\zeta,\nu)\\
&\quad=\nu-P\\
&\quad=\nu-p_0+p_1\zeta+p_2\zeta^2+p_3\zeta^3+p_4\zeta^4+p_5\zeta^5+p_6\zeta^6+p_7\zeta^7\\
&\qquad+420\Big[\zeta^6\nu^4+(2\zeta^4-2p_0\zeta^6)\nu^3\\
&\qquad\quad+(3\zeta^2-4p_0\zeta^4-2p_1\zeta^5+(p_0^2-p_2)\zeta^6-p_3\zeta^7)\nu^2\\
&\qquad\quad-(5p_0\zeta^2+4p_1\zeta^3+(3p_2-3p_0^2)\zeta^4+(3p_3-2p_0p_1)\zeta^5+2p_4\zeta^6p_5\zeta^7)\nu\\
&\qquad\quad+2p_0\zeta^2+3p_0p_1\zeta^3+(p_1^2+2p_0p_2-p_0^3)\zeta^4+(2p_0p_3+p_1p_2-p_0^2p_1)\zeta^5\\
&\qquad\quad+(p_0p_4+p_1p_3)\zeta^6-p_0\zeta^7\Big].
\end{aligned}
\tag{37}
$$

From the Eq. (37) it immediately follows that $T(\zeta,\nu)$ is analytic in (ζ,ν) - plane and also

$$T(0,p_0)=0 \qquad \left.\frac{\partial T}{\partial \nu}\right|_{(0,p_0)}=1\neq 0. \tag{38}$$

Hence, in the light of implicit function theorem, $\nu=P(\zeta)$ is analytic in a neighborhood of the point $(0,p_0)$ with the positive radius of convergence. Consequently, the power series solution of Eq. (15) converges in a neighborhood of the point $(0,p_0)$. Similarly, it can be shown that the power series solution of Eq. (17) given by Eq. (28) converges in a neighborhood of the point $(0,p_0)$. This completes the proof.

5 Results and Discussion

The soliton solutions of CDG equation were obtained by Wazwaz. In particular, one soliton solution of Eq.(2) is given by (Wazwaz [5])

$$u(x,t)=\frac{2e^{x-t}}{(1+e^{x-t})^2}. \tag{39}$$

We have obtained exact analytic solutions in the form of power series which are given by Eq. (23) and Eq. (28) respectively. The graphical representation of the solutions of Eq. (23) by taking particular values of constants $(c_1,c_2\ldots..c_6)$ with increasing values of t have been shown below. One can observe from these figures that u decreases rapidly when we increase t.

6 Conclusion

In the present paper, following the classical Lie symmetry method, the seventh order CDG equation is studied. In this method, we get infinitesimal generator of the equation.

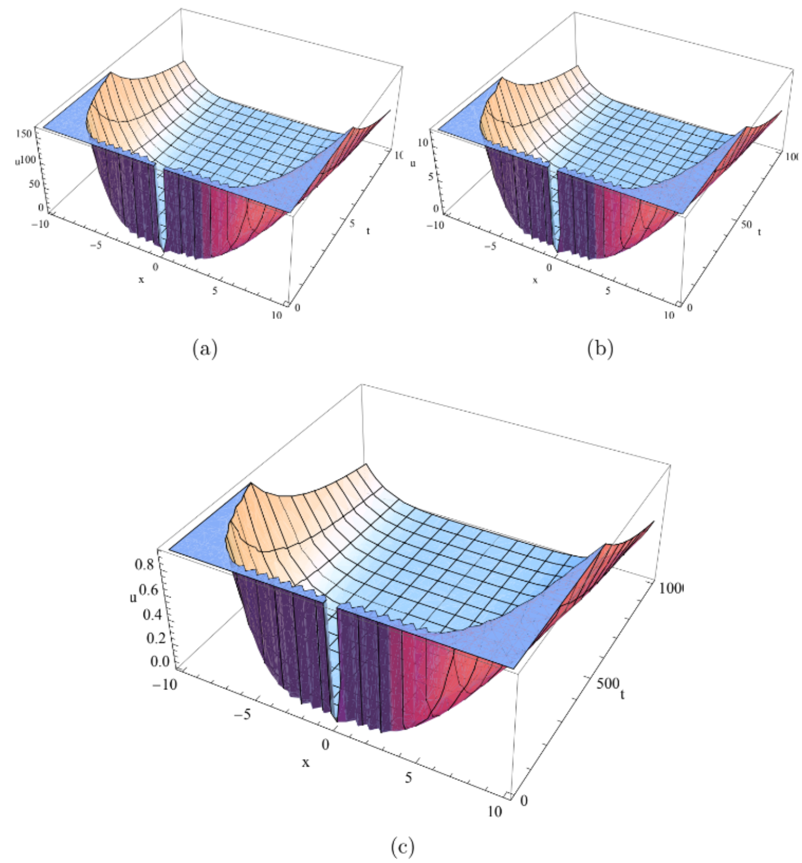

Figure 1: Profiles of $u(x,t)$ with various values of t.

Then, we discuss the Lie symmetry groups of the CDG equation and reduce the CDG equation into ODEs with the help of similarity variables. Finally, we solve these ODEs using power series method to obtain the exact solutions. In our best knowledge the solutions obtained in this paper are the new solution to CDG equation i.e. not obtained by anyone so far.

References

[1] Abdul Zahra, Khalid A. and Abdul Hussain, Mudhir A.: Exp-function method for solving nonlinear beam equation. *Int. Math. Forum* 6 (45-48) (2011) 2349–2359.

[2] Bluman, G. W. and Cole, J. D.: *Similarity methods for differential equations*. Applied Mathematical Sciences, Vol. 13. Springer-Verlag, New York-Heidelberg (1974).

[3] Malfliet, Willy and Hereman, Willy: The tanh method. I. Exact solutions of nonlinear evolution and wave equations. *Phys. Scripta* 54 (6) (1996) 563–568.

[4] Olver, Peter J.: *Applications of Lie groups to differential equations*. Graduate Texts in Mathematics. Springer-Verlag, New York (1993).

[5] Wazwaz, Abdul-Majid: One and two soliton solutions for the sinh-Gordon equation in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions. *Appl. Math. Lett.* 25 (12) (2012) 2354–2358.

[6] Wang, Mingliang and Zhang, Jinliang and Li, Xiangzheng: Application of the $\left(\frac{G'}{G}\right)$-expansion to travelling wave solutions of the Broer-Kaup and the approximate long water wave equations. *Appl. Math. Comput.* 206 (1) (2008) 321–326.

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