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Asymptotic formula for the multiplicative function $\frac{d(n)}{k^{\omega(n)}}$

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Abstract. For a fixed integer k, we define the multiplicative function

$$D_{k,\omega}(n) := \frac{d(n)}{k^{\omega(n)}},$$

where $d(n)$ is the divisor function and $\omega(n)$ is the number of distinct prime divisors of n. The main purpose of this paper is the study of the mean value of the function $D_{k,\omega}(n)$ by using elementary methods.

1 Introduction

Let $k \geq 2$ be a fixed integer. We recall that $d(n) := \sum_{d|n} 1$ is the number of divisors of n, and $\omega(n) := \sum_{p|n} 1$ is the number of distinct prime divisors of n. We define the function $D_{k,\omega}(n)$ by

$$D_{k,\omega}(n) := \frac{d(n)}{k^{\omega(n)}}. \tag{1}$$

Notice that for every fixed integer $k \geq 2$, the function $D_{k,\omega}(n)$ is multiplicative and for every prime number p and every integer m the relation

$$D_{k,\omega}(p^m) = \frac{m+1}{k}, \tag{2}$$

holds. By using (2), we get

$$D_{k,\omega}(n) = \prod_{p^m \| n} \frac{m+1}{k}$$

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where $p^m \| n$ means $p^m \mid n$ and $p^{m+1} \nmid n$. In the particular case $k = 2$, the function $D_{2,\omega}(n)$ is exactly $D(n) = \dfrac{d(n)}{d^*(n)}$, (see [2]). For $k \geq 3$, we can easily check that

$$\sum_{n \leq x} D_{k,\omega}(n) \ll_k x(\log x)^{2/k-1}. \tag{3}$$

Indeed, for any integer n, we have $D_{k,\omega}(n) \leq d(n) \ll_\varepsilon n^\varepsilon$. Furthermore, the hypotheses of Shiu's theorem are satisfied; see Theorem 1 in [7] and [6, p.1]. One gets

$$\sum_{n \leq x} D_{k,\omega}(n) \ll_k \frac{x}{\log x} \exp\Big(\sum_{p \leq x} \frac{2}{kp}\Big).$$

Now, by using Lemma 4.63 in [1], it follows that

$$\sum_{n \leq x} D_{k,\omega}(n) \ll_k \frac{x}{\log x} \exp\Big(\frac{2}{k} \log(2e^\gamma \log x)\Big) \ll_k x(\log x)^{2/k-1}.$$

2 Main result

In this section, we establish two results concerning the mean value of the function $D_{k,\omega}(n)$. We begin by giving a weaker result.

Theorem 2.1. *Let $k \geq 2$ be a fixed integer. For all $x \geq 1$ large enough, we have*

$$\sum_{n \leq x} D_{k,\omega}(n) = \frac{x(\log x)^{2/k-1}}{\Gamma(2/k)} \prod_p \Big(1 - \frac{1}{p}\Big)^{2/k} \Big(1 + \frac{2p-1}{kp(p-1)^2}\Big) + O(x(\log x)^{-1})(\log\log x)^{4/k}.$$

The proof of this result is based on Tulyaganov's theorem; this theorem is summarized as follows:

Theorem 2.2. *Let f be a complex valued multiplicative function. Suppose there exists $z \in \mathbb{C}$, independent of p, with $|z| \leq c_1$ and*

a)

$$\sum_{p \leq x} f(p) \log p = zx + O(xe^{-c_2\sqrt{\log x}})$$

b)

$$\sum_{p \leq x} |f(p)| \log p \ll x$$

c)

$$\sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{|f(p^\alpha)| \log p^\alpha}{p^\alpha} \ll (\log\log x)^2$$

d)

$$\sum_{p} \frac{|f(p)|^2 \log p}{p^2} < c_3$$

for some real numbers c_1, c_2 and c_3. Then, for all $x \geq 1$ sufficiently large, we have

$$\sum_{n \leq x} f(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} \prod_{p} \Big(1 - \frac{1}{p}\Big)^{z} \Big(1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^\alpha}\Big) \Big\{1 + O(\frac{(\log\log x)^2}{\log x})\Big\}$$
$$+ O(x(\log x)^{\max(0,\operatorname{Re} z-1)-1})(\log\log x)^{2(A-\max(0,\operatorname{Re} z-1))},$$

where $A > 0$ satisfies

$$\sum_{u<p\leq v} |f(p)| p^{-1} \leq A \log(\log v / \log u) + O(1).$$

Proof. This theorem is a consequence of Theorem 4 in [9], where we take $g = f$. □

To complete the demonstration of the main result we have the following lemmas.

Lemma 2.3. *For any fixed integer $k \geq 2$, we have the estimate*

$$\sum_{p \leq x} |D_{k,\omega}(p)| \log p \ll x.$$

Proof. By Chebyshev's estimates [3], we have

$$\sum_{p\leq x} |D_{k,\omega}(p)| \log p = \frac{2}{k} \sum_{p \leq x} \log p < \frac{2}{k}(1.000081x) \ll x.$$ □

Lemma 2.4. *For any fixed integer $k \geq 2$, there is a constant $c > 0$, such that*

$$\sum_{p \leq x} D_{k,\omega}(p) \log p = \frac{2}{k} x + O(xe^{-c\sqrt{\log x}}).$$

Proof. We have

$$\sum_{p\leq x} D_{k,\omega}(p) \log p = \frac{2}{k} \sum_{p\leq x} \log p = \frac{2}{k}\theta(x),$$

and by Theorem 6.9 in [5], there is a constant $c > 0$ such that

$$\theta(x) = x + O(xe^{-c\sqrt{\log x}}),$$

which implies the desired result. □

Lemma 2.5. *For any fixed integer $k \geq 2$, we have*

$$\sum_{p} \frac{|D_{k,\omega}(p)|^2}{p^2} \log p < \infty.$$

Proof. We first check the inequality $\sum_{m=2}^{\infty} \frac{\log m}{m(m-1)} \leq \log 4$, and using the following

$$\sum_p \frac{\log p}{p^2} < \sum_{m=2}^{\infty} \frac{\log m}{m^2} \leq \sum_{m=2}^{\infty} \frac{\log m}{m(m-1)},$$

then we have

$$\sum_p \frac{|D_{k,\omega}(p)|^2}{p^2} \log p = \frac{4}{k^2} \sum_p \frac{\log p}{p^2} < \frac{4 \log 4}{k^2}.$$

□

Lemma 2.6. *For any fixed integer $k \geq 2$, we have*

$$\sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{|D_{k,\omega}(p^\alpha)| \log(p^\alpha)}{p^\alpha} \leq \frac{28}{k}.$$

Proof. For every integer $k \geq 3$, we write

$$\begin{aligned}\sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{|D_{k,\omega}(p^\alpha)| \log(p^\alpha)}{p^\alpha} &= \frac{1}{k} \sum_{p \leq x} \log p \sum_{\alpha=2}^{\infty} \frac{\alpha(\alpha+1)}{p^\alpha} \\ &= \frac{1}{k} \sum_{p \leq x} \frac{\log p}{p} \sum_{\alpha=2}^{\infty} \frac{\alpha(\alpha+1)}{p^{\alpha-1}},\end{aligned}$$

and the infinite series $\sum_{\alpha=2}^{\infty} \frac{\alpha(\alpha+1)}{p^{\alpha-1}}$ converges to $\frac{2}{(1-1/p)^3} - 2$, since

$$\begin{aligned}\sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{|D_{k,\omega}(p^\alpha)| \log(p^\alpha)}{p^\alpha} &= \frac{2}{k} \sum_{p \leq x} \frac{3p^2 - 3p + 1}{p(p-1)^3} \log p \\ &\leq \frac{28}{k} \sum_{p \leq x} \frac{\log p}{p^2}.\end{aligned}$$

By Lemma 70.1 in [4], we have $\sum_p \frac{\log p}{p^\alpha} < \frac{1}{\alpha - 1}$ for all $\alpha > 1$, consequently

$$\sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{|D_{k,\omega}(p^\alpha)| \log(p^\alpha)}{p^\alpha} < \frac{28}{k}.$$

Finally, by Lemma 2.3, 2.4, 2.5 and 2.6 we have shown that the function $D_{k,\omega}(n)$ satisfies the conditions of Theorem 2.2. As we have

$$\sum_{u < p \leq v} \frac{|D_{k,\omega}(p)|}{p} = \frac{2}{k} \sum_{u < p \leq v} \frac{1}{p} \leq \frac{2}{k} \log \frac{\log v}{\log u} + O(1),$$

then the constant A in Theorem 2 is $\frac{2}{k}$.

□

The next result is improved over the previous one.

Theorem 2.7. *Let $k \geq 2$ be a fixed integer. For all $x \geq 1$ large enough, we have*

$$\sum_{n \leq x} D_{k,\omega}(n) = \frac{x(\log x)^{2/k-1}}{\Gamma(2/k)} \prod_p \Big(1 - \frac{1}{p}\Big)^{2/k} \Big(1 + \frac{2p-1}{k(p-1)^2}\Big) + O_k(x(\log x)^{2/k-2}).$$

The demonstration is based on the following lemmas:

Lemma 2.8. *Let $k \geq 2$ be a fixed integer. For every $s := \sigma + it \in \mathbb{C}$ such that $\sigma > 1$ and $L(s, D_{k,\omega}(n)) := \sum_{n=1}^{\infty} \dfrac{D_{k,\omega}(n)}{n^s}$, we have*

$$L(s, D_{k,\omega}(n)) = \zeta(s)^{2/k} L(s, g_k),$$

or $L(s, g_k)$ is a series of Dirichlet absolutely convergent in the half-plane $\sigma > \frac{1}{2}$.

Proof. If $\sigma > 1$, then

$$\begin{aligned} L(s, D_{k,\omega}(n)) &= \prod_p \Big(1 + \sum_{\alpha=1}^{\infty} \frac{D_{k,\omega}(p^\alpha)}{p^{\alpha s}}\Big) \\ &= \prod_p \Big(1 + \sum_{\alpha=1}^{\infty} \frac{\alpha+1}{k p^{\alpha s}}\Big) \\ &= \prod_p \Big(1 + \frac{2p^s - 1}{k(p^s-1)^2}\Big), \end{aligned}$$

on the other hand we have

$$\Big(1 + \frac{2p^s - 1}{k(p^s-1)^2}\Big) = ((1 - p^{-s})^{-2/k}) \Big(1 + \frac{h(s)}{k(p^s-1)^2}\Big),$$

such that

$$h(s) = (1 - p^{-s})^{2/k}(kp^{2s} - 2(k-1)p^s + k - 1) - k(p^s-1)^2.$$

Since

$$(1 - p^{-s})^{2/k} = 1 - \frac{2}{kp^s} - \frac{k-2}{k^2 p^{2s}} - O\Big(\frac{k}{p^{3\sigma}}\Big),$$

he comes

$$\begin{aligned} h(s) &= \Big(1 - \frac{2}{kp^s} - \frac{k-2}{k^2 p^{2s}} - O\Big(\frac{k}{p^{3\sigma}}\Big)\Big)(kp^{2s} - 2(k-1)p^s + k - 1) - k(p^s-1)^2 \\ &= 2\Big(1 - \frac{1}{k}\Big) + O(p^{-\sigma}), \end{aligned}$$

which implies the announced result. ∎

Lemma 2.9 ([8]). *Let $A > 0$. Uniformly for $x \geq 2$ and $z \in \mathbb{C}$ such that $|z| \leq A$, we have*

$$\sum_{n \leq x} \tau_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O_A(x(\log x)^{\operatorname{Re} z-2}).$$

$\tau_z(n)$ is the multiplicative function defined by $\tau_z(p^\alpha) = \binom{z+\alpha-1}{\alpha}$.

Proof of Theorem 3. According to the Lemma 2.8, we have $D_{k,\omega} = \tau_{2/k} * g_k$. Then, by Lemma 2.9

$$\begin{aligned}
\sum_{n \leq x} D_{k,\omega}(n) &= \sum_{d \leq x} g_k(d) \sum_{m \leq \frac{x}{d}} \tau_{2/k}(m) \\
&= \sum_{d \leq x} g_k(d) \Big(\frac{x(\log \frac{x}{d})^{2/k-1}}{d\Gamma(2/k)} + O_k\big(\frac{x}{d}(\log \frac{x}{d})^{2/k-2}\big)\Big) \\
&= \sum_{d \leq x} g_k(d) \Big(\frac{x(\log x)^{2/k-1}}{d\Gamma(2/k)} + O_k((\log x)^{2/k-2} \log d) \\
&\qquad + O_k\big(\frac{x}{d}(\log \frac{x}{d})^{2/k-2}\big)\Big) \\
&= \frac{x(\log x)^{2/k-1}}{\Gamma(2/k)} \sum_{d \leq x} \frac{g_k(d)}{d} + O_k\Big(x(\log x)^{2/k-2} \sum_{d \leq x} \frac{|g_k(d)|(1+\log d)}{d}\Big).
\end{aligned}$$

The series $L(s, g_k)$ is absolutely convergent on the half-plane $\sigma > \frac{1}{2}$, then for all $\varepsilon > 0$

$$\sum_{d \leq x} |g_k(d)| \ll_{k,\varepsilon} x^{1/2+\varepsilon},$$

hence by partial summation

$$\sum_{d \leq x} \frac{|g_k(d)|(1+\log d)}{d} \ll_{k,\varepsilon} x^{-1/2+\varepsilon}$$

and therefore

$$\sum_{n \leq x} D_{k,\omega}(n) = L(1, g_k) \frac{x(\log x)^{2/k-1}}{\Gamma(2/k)} + O_k(x(\log x)^{2/k-2}) + O_{k,\omega}(x^{1/2+\varepsilon}).$$

Which completes the demonstration. □

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