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## On the geometric mean of the values of positive multiplicative arithmetical functions

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**Abstract.** In this paper we obtain asymptotic expansions for the geometric mean of the values of positive strongly multiplicative function f satisfying

$$f(p) = \alpha(d)p^d + O(p^{d-\delta})$$

for any prime p, with d real,  $\alpha(d)$  and  $\delta > 0$ .

## 1 Introduction

The arithmetic mean of the values of arithmetical functions is very well studied in the literature and some theories have been developed. For example, see [3], [6], [7], [8], [9], [14], [15], [20] and the references given there. In comparison, the geometric mean of the values of arithmetical functions has been studied only in some special cases. Let us denote by  $G_f(n)$  the geometric mean of the first n values of the positive arithmetic function f. In 2008 Deshouillers and Luca [4] studied the density modulo 1 of some sequences involving the values of the Euler function  $\varphi$ , including the sequence with general term  $G_{\varphi}(n)$ . Meanwhile, they proved that

$$G_{\varphi}(n) = \frac{1}{\mathrm{e}} \prod_{p} \left( 1 - \frac{1}{p} \right)^{\frac{1}{p}} n + O(\log n)$$

In 2013 the first author [12] studied uniform distribution modulo 1 of some sequences involving the values of the Euler function, where he improved the above error term up to  $O(\log \log n)$ . In 2012 Bosma and Kane [1], and later in 2018 Pomerance [17] considered a variant of the geometric mean of the first n values of the sum of divisors function to study

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the so-called "aliquot constant". In 2016 the first author [13] considered the geometric mean of the first n values of the function  $d(n) = \sum_{d|n} 1$ . He proved that given any positive integer r, there exist computable constants  $c_1, \ldots, c_r$  such that

$$G_{\rm d}(n) = 2^M \prod_{\substack{p^{\alpha} \\ \alpha \ge 2}} \log\left(1 + \frac{1}{\alpha}\right)^{\frac{1}{p^{\alpha}}} (\log n)^{\log 2} \left(1 + \sum_{j=1}^r \frac{c_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right)\right),$$

where M is the Meissel–Mertens constant. Recently, the second author [10] studied the geometric mean of the first n values of the Jordan totient function.

In this paper we are motivated by introducing a general theory for studying the geometric mean of the values of positive multiplicative functions f. Since  $\log f$  is additive, we get

$$n\log G_f(n) = \sum_{k \le n} \log f(k) = \sum_{\substack{p^{\alpha} \le n \\ \alpha \ge 1}} \left( \log f(p^{\alpha}) - \log f(p^{\alpha-1}) \right) \left\lfloor \frac{n}{p^{\alpha}} \right\rfloor$$

Consequently,

$$n\log G_f(n) = \sum_{p \leqslant n} \left[\frac{n}{p}\right] \log f(p) + \sum_{\substack{p^{\alpha} \leqslant n \\ \alpha \geqslant 2}} \left[\frac{n}{p^{\alpha}}\right] \log \frac{f(p^{\alpha})}{f(p^{\alpha-1})}.$$
(1)

In particular, if we further assume that f is strongly multiplicative, then

$$n\log G_f(n) = \sum_{p \leqslant n} \left[\frac{n}{p}\right] \log f(p).$$
(2)

With regards to this case, we prove the following general result.

**Theorem 1.1.** Let f be a positive strongly multiplicative function such that for any prime p it satisfies  $f(p) = \alpha(d) p^d + O(p^{d-\delta})$  with d real and  $\alpha(d), \delta > 0$ . Then for any positive integer r, there exist computable constants  $c_1, \ldots, c_r$  such that

$$G_f(n) = \alpha(d)^M e^{d(\gamma + E - 1)} \varrho_f n^d (\log n)^{\log \alpha(d)} \left( 1 + \sum_{j=1}^r \frac{c_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right) \right),$$

where

$$M = \lim_{x \to \infty} \sum_{p \leqslant x} \frac{1}{p} - \log \log x$$

is the Meissel-Mertens constant,  $\gamma$  is Euler's constant, E is the constant in Mertens' approximation defined by

$$E = \lim_{x \to \infty} \sum_{p \leqslant x} \frac{\log p}{p} - \log x,$$

and  $\rho_f$  is a constant depending on f, given by the following product running over primes

$$\varrho_f = \prod_p \left( \frac{f(p)}{\alpha(d) p^d} \right)^{\frac{1}{p}}.$$

As an example satisfying the conditions of Theorem 1.1, we consider the square-free kernel of n defined by  $\kappa(n) = \prod_{p|n} p$ .

**Corollary 1.2.** For any positive integer r, there exist computable constants  $c_1, \ldots, c_r$  such that

$$G_{\kappa}(n) = \mathrm{e}^{\gamma + E - 1}n + \sum_{j=1}^{r} c_j \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$

The proof of Theorem 1.1 depends on the approximation of the sum on the right hand side of (2). The following key result gives the required approximation, and implies Theorem 1.1 immediately.

**Theorem 1.3.** Let  $Q(x) = \alpha(d) x^d + \mathcal{E}(x)$  with d real and  $\alpha(d) > 0$ , and  $\mathcal{E}(x) = O(x^{d-\delta})$ for some fixed  $\delta > 0$ . Moreover, we assume that Q(n) > 0 for any positive integer n. Given any positive integer r, there exist computable constants  $\eta_0, \eta_1, \ldots, \eta_r$  such that

$$\sum_{p \leq n} \left[\frac{n}{p}\right] \log Q(p) = d n \log n + (\log \alpha(d)) n \log \log n$$

$$+\eta_0 n + \sum_{j=1}^r \eta_j \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$

More precisely

$$\eta_0 = M \log \alpha(d) + d(\gamma + E - 1) + C_Q,$$
(3)

where  $C_Q$  is an absolute constant in terms of Q defined by

$$C_Q = \sum_p \frac{1}{p} \log \frac{Q(p)}{\alpha(d) p^d}.$$
(4)

## 2 Proofs

We have divided the proof of Theorem 1.3 into a sequence of propositions. Let us break up the sum addressed in this theorem as follows

$$\sum_{p \leq n} \left[\frac{n}{p}\right] \log Q(p) = \left(\log \alpha(d)\right) S_1(n) + d S_2(n) + S_3(n),$$

where

$$S_1(n) = \sum_{p \leqslant n} \left[ \frac{n}{p} \right], \quad S_2(n) = \sum_{p \leqslant n} \left[ \frac{n}{p} \right] \log p, \quad S_3(n) = \sum_{p \leqslant n} \left[ \frac{n}{p} \right] \log \left( 1 + \frac{\mathcal{E}(p)}{\alpha(d)p^d} \right)$$

An approximation of  $S_1(n)$  is well-known due to the notion of the omega function

$$\omega(k) = \sum_{p|k} 1,\tag{5}$$

which counts the number of distinct prime divisors of the positive integer k. We observe that

$$\sum_{k \leqslant n} \omega(k) = \sum_{k \leqslant n} \sum_{p \mid k} 1 = \sum_{p \leqslant n} \sum_{\substack{k \leqslant n \\ p \mid k}} 1 = \sum_{p \leqslant n} \left\lfloor \frac{n}{p} \right\rfloor.$$

In 1970 Saffari [19] used Dirichlet's hyperbola method to prove

$$\frac{1}{n}\sum_{k\leqslant n}\omega(k) = \log\log n + M + \sum_{j=1}^{r}\frac{a_j}{\log^j n} + O\Big(\frac{1}{\log^{r+1} n}\Big),\tag{6}$$

for each integer  $r \ge 1$ , with

$$a_j = -\int_1^\infty \frac{\{t\}}{t^2} (\log t)^{j-1} \mathrm{d}t.$$
 (7)

More precisely, it is known [11] that  $a_1 = \gamma - 1$ . Hence, for any positive integer r we obtain

$$S_1(n) = n \log \log n + Mn + \sum_{j=1}^r a_j \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$
 (8)

We mention that later in 1976 Diaconis [5] reproved (6) by applying Perron's formula on the Dirichlet series  $\sum_{n=1}^{\infty} \omega(n) n^{-s}$  and using complex integration methods. Approximations of  $S_2(n)$  and  $S_3(n)$  are given in the following propositions.

**Proposition 2.1.** Given any positive integer r, there exist computable constants  $c_1, \ldots, c_r$  such that

$$S_2(n) = n \log n + (\gamma + E - 1) n + \sum_{j=1}^r c_j \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$
(9)

**Proposition 2.2.** For any fixed  $\delta > 0$ ,

$$S_3(n) = C_Q n + O\left(n^{1-\delta} + 1 + [\delta^{-1}] \log \log n\right).$$
(10)

Note that the big-O term in (10) is finally  $O(\frac{n}{\log^{r+1} n})$  for each r > 0. Thus, combining the asymptotic expansions (8), (9) and (10) completes the proof of Theorem 1.3.

Proof of Proposition 2.1. For each integer  $n \ge 2$ , let

$$\lambda(n) = \frac{\log n}{\log \kappa(n)}$$

be the index of composition of n. In 2005 De Koninck and Kátai [2, Theorem 3] proved that given any positive integer r, there exist computable constants  $d_1, \ldots, d_r$  such that

$$U(x) := \sum_{k \leq x} \frac{1}{\lambda(k)} = x + \sum_{j=1}^{r} d_j \frac{x}{\log^j x} + O\left(\frac{x}{\log^{r+1} x}\right).$$
(11)

The above asymptotic expansion is very useful to obtain an asymptotic expansion for  $S_2(n)$ . Indeed, we observe that

$$\sum_{k=1}^{n} \log \kappa(k) = \sum_{k=1}^{n} \log \prod_{p|k} p = \sum_{k=1}^{n} \sum_{p|k} \log p = \sum_{p \le n} \left[\frac{n}{p}\right] \log p = S_2(n).$$

Hence, by Abel summation we get

$$S_2(n) = \sum_{k=1}^n \log \kappa(k) = \sum_{k=2}^n \frac{1}{\lambda(k)} \log k = U(n) \log n - U(2^-) \log 2 - \int_2^n \frac{U(t)}{t} dt.$$

To deal with the last integral, we study the functions  $L_j(t)$  defined for each integer  $j \ge 1$ by the following anti-derivative

$$\mathcal{L}_j(t) := \int \frac{\mathrm{d}t}{\log^j t}.$$

Note that  $L_1(t)$  is the logarithmic integral function, which admits the following expansion

$$L_1(t) = li(t) = \sum_{i=1}^r (i-1)! \frac{t}{\log^i t} + O\left(\frac{t}{\log^{r+1} t}\right).$$
 (12)

Integrating by parts gives

$$\mathcal{L}_{j-1}(t) = \int \left(\frac{1}{\log^{j-1} t}\right) (dt) = \frac{t}{\log^{j-1} t} + (j-1) \int \frac{dt}{\log^j t}$$

Hence, for  $j \ge 2$  the functions  $L_j(t)$  satisfy the recurrence

$$\mathcal{L}_{j}(t) = \frac{1}{j-1}\mathcal{L}_{j-1}(t) - \frac{t}{(j-1)\log^{j-1}t}.$$

By repeatedly using this recurrence we deduce that

$$(j-1)! L_j(t) = li(t) - \sum_{i=1}^{j-1} (i-1)! \frac{t}{\log^i t}.$$

Hence, by using the expansion (12), for  $1 \leq j \leq r$  we obtain

$$\mathcal{L}_{j}(t) = \sum_{i=j}^{r} \frac{(i-1)!}{(j-1)!} \frac{t}{\log^{i} t} + O\left(\frac{t}{\log^{r+1} t}\right).$$
(13)

We deduce from the expansion (11) that

$$\int_{2}^{n} \frac{U(t)}{t} dt = \int_{2}^{n} \left( 1 + \sum_{j=1}^{r} d_{j} \frac{1}{\log^{j} t} + O\left(\frac{1}{\log^{r+1} t}\right) \right) dt$$
$$= n + \sum_{j=1}^{r} d_{j} L_{j}(n) - \left( 2 + \sum_{j=1}^{r} d_{j} L_{j}(2) \right) + O\left(\frac{n}{\log^{r+1} n}\right).$$

With r replaced by r + 1 in (11), we obtain

$$U(n)\log n = n\log n + d_1n + \sum_{j=1}^r d_{j+1}\frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$

Combining the above expansions yields that

$$S_2(n) = n \log n + (d_1 - 1)n + \sum_{j=1}^r \left( d_{j+1} \frac{n}{\log^j n} - d_j \mathcal{L}_j(n) \right) - C_r + O\left(\frac{n}{\log^{r+1} n}\right),$$

where  $C_r = 2 + U(2^-) \log 2 + \sum_{j=1}^r d_j L_j(2) = O_r(1)$ . Note that

$$\sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^{j} n} - d_{j} \mathcal{L}_{j}(n) \right)$$
$$= \sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^{j} n} - \sum_{i=j}^{r} d_{j} \frac{(i-1)!}{(j-1)!} \frac{n}{\log^{i} n} \right) + O\left(\frac{n}{\log^{r+1} n}\right).$$

Also, an easy computation shows that

$$\sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^{j} n} - \sum_{i=j}^{r} d_{j} \frac{(i-1)!}{(j-1)!} \frac{n}{\log^{i} n} \right) = \sum_{j=1}^{r} c_{j} \frac{n}{\log^{j} n} + O\left(\frac{n}{\log^{r+1} n}\right),$$

for some computable constants  $c_j$  in terms of the  $d_j$ s. Furthermore we let  $c_0 = d_1 - 1$ . Consequently,

$$S_2(n) = n \log n + c_0 n + \sum_{j=1}^r c_j \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right).$$

It remains to compute the value of  $c_0$ . To do this, we let

$$S_2(n) = n M(n) - R(n),$$
 (14)

where

$$M(x) := \sum_{p \leqslant x} \frac{\log p}{p}$$
, and  $R(n) := \sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} \log p.$ 

Theorem 6 of [18] asserts validity of the double sided inequality

$$\log x + E - \frac{1}{2\log x} < M(x) < \log x + E + \frac{1}{2\log x},$$
(15)

the left hand side for x > 1 and the right hand side for  $x \ge 319$ . We estimate R(n). Let

$$F_1(x) = \sum_{p \leqslant x} \left\{ \frac{x}{p} \right\}$$
 and  $F_2(x) = \sum_{p^{\alpha} \leqslant x} \left\{ \frac{x}{p^{\alpha}} \right\}.$ 

Lemma 3 of [16] asserts that

$$F_2(n) = (1 - \gamma)\frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right)$$

Note that

$$\mathcal{F}_{2}(n) - \mathcal{F}_{1}(n) = \sum_{\substack{p^{\alpha} \leqslant n \\ \alpha \geqslant 2}} \left\{ \frac{n}{p^{\alpha}} \right\} < \sum_{\substack{p^{\alpha} \leqslant n \\ \alpha \geqslant 2}} 1 = \sum_{\substack{p \leqslant n^{\frac{1}{\alpha}} \\ \alpha \geqslant 2}} 1 = \sum_{\substack{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}}} \pi(n^{\frac{1}{\alpha}})$$
$$\ll \sum_{\substack{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}}} \frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}} \leqslant \frac{n^{\frac{1}{2}}}{\log n} \sum_{\substack{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}}} \alpha \ll \sqrt{n} \log n.$$

Hence

$$\mathcal{F}_1(n) = (1 - \gamma) \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right).$$

Let  $\varpi(k)$  to be 1 when k is prime and 0 otherwise. Abel summation allows us to write

$$R(n) = \sum_{k=2}^{n} \left\{ \frac{n}{k} \right\} \varpi(k) \log k$$
  
=  $F_1(n) \log n - F_1(2^-) \log 2 - \int_2^n \left( \sum_{p \le t} \left\{ \frac{n}{p} \right\} \right) \frac{\mathrm{d}t}{t}$   
=  $(1 - \gamma)n + O\left(\frac{n}{\log n}\right) - \int_2^n O\left(\frac{t}{\log t}\right) \frac{\mathrm{d}t}{t} = (1 - \gamma)n + O\left(\frac{n}{\log n}\right).$ 

By using (15) and (14), we obtain

$$S_2(n) = n \log n + (\gamma + E - 1) n + O\left(\frac{n}{\log n}\right),$$

implying that  $c_0 = \gamma + E - 1$  and also  $d_1 = \gamma + E$ .

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Proof of Proposition 2.2. We have

$$S_3(n) = \sum_{p \le n} \left[ \frac{n}{p} \right] \log \frac{Q(p)}{\alpha(d) p^d}$$
$$= n \sum_p \frac{1}{p} \log \frac{Q(p)}{\alpha(d) p^d} - n \sum_{p > n} \frac{1}{p} \log \frac{Q(p)}{\alpha(d) p^d} - \sum_{p \le n} \left\{ \frac{n}{p} \right\} \log \frac{Q(p)}{\alpha(d) p^d}.$$

Note that

$$\log \frac{Q(p)}{\alpha(d) p^d} = \log \left( 1 + \frac{\mathcal{E}(p)}{\alpha(d) p^d} \right) = \log \left( 1 + O\left(\frac{1}{p^{\delta}}\right) \right) = O\left(\frac{1}{p^{\delta}}\right).$$

Hence

$$\sum_{p} \frac{1}{p} \log \frac{Q(p)}{\alpha(d) p^d} \ll \sum_{p} \frac{1}{p^{1+\delta}},$$

and this implies that in (4) the series defining  $C_Q$  is absolutely convergent. Moreover

$$n\sum_{p>n}\frac{1}{p}\log\frac{Q(p)}{\alpha(d)\,p^d} \ll n\sum_{p>n}\frac{1}{p^{1+\delta}} \ll n\int_n^\infty \frac{\mathrm{d}t}{t^{1+\delta}} \ll n^{1-\delta}.$$

Also, note that

$$\sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} \log \frac{Q(p)}{\alpha(d) p^d} \ll \sum_{p \leqslant n} \frac{1}{p^{\delta}}.$$

Hence, if  $\delta = 1$ , then

$$\sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} \log \frac{Q(p)}{\alpha(d) p^d} \ll \log \log n,$$

and if  $\delta \neq 1$ , then

$$\sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} \log \frac{Q(p)}{\alpha(d) p^d} \ll \sum_{p \leqslant n} \frac{1}{p^{\delta}} \ll \int_2^n \frac{\mathrm{d}t}{t^{\delta}} \ll n^{1-\delta}.$$

Combining the above approximations we get (10).

Proof of Theorem 1.1. We observe that for any positive integer r

$$\exp\left(\sum_{j=1}^{r} \frac{c_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right)\right) = 1 + \sum_{j=1}^{r} \frac{c_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right),\tag{16}$$

where the computable constants  $c_1, \ldots, c_r$  are not necessarily the same on both sides. The relation (2) and Theorem 1.3 give

$$\log G_f(n) = d \log n + (\log \alpha(d)) \log \log n + \eta_0 + \sum_{j=1}^r \frac{\eta_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right).$$

Taking exponents and using (16) completes the proof.

Proof of Corollary 1.2. By using (2) we observe that  $n \log G_{\kappa}(n) = S_2(n)$ . Thus

$$\log G_{\kappa}(n) = \frac{S_2(n)}{n} = \log n + (\gamma + E - 1) + \sum_{j=1}^{r} \frac{c_j}{\log^j n} + O\left(\frac{1}{\log^{r+1} n}\right).$$

Taking exponents and considering (16) completes the proof.

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