Communications in Mathematics **30** (2022), no. 2, 37–50
DOI: https://doi.org/10.46298/cm.10193

Computing subalgebras and $\mathbb{Z}_2$-gradings of simple Lie algebras over finite fields

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Abstract. This paper introduces two new algorithms for Lie algebras over finite fields and applies them to the investigate the known simple Lie algebras of dimension at most 20 over the field $\mathbb{F}_2$ with two elements. The first algorithm is a new approach towards the construction of $\mathbb{Z}_2$-gradings of a Lie algebra over a finite field of characteristic 2. Using this, we observe that each of the known simple Lie algebras of dimension at most 20 over $\mathbb{F}_2$ has a $\mathbb{Z}_2$-grading and we determine the associated simple Lie superalgebras. The second algorithm allows us to compute all subalgebras of a Lie algebra over a finite field. We apply this to compute the subalgebras, the maximal subalgebras and the simple subquotients of the known simple Lie algebras of dimension at most 16 over $\mathbb{F}_2$ (with the exception of the 15-dimensional Zassenhaus algebra).

1 Introduction

The classification of the finite-dimensional simple Lie algebras depends heavily on their underlying field. For algebraically closed fields of characteristic zero the classification has been achieved long ago and is folklore nowadays. For algebraically closed fields of characteristic $p \geq 5$ the classification has been completed more recently, see [16]. A classification over fields of characteristic $p \in \{2, 3\}$ has not been achieved so far. The available evidence suggests that the classification over fields of characteristic 2 will differ significantly from the other cases. This motivates the computational investigation of the known simple Lie algebras over the field with two elements with the aim to gain further insight into their structure.

In this paper we introduce two algorithms to investigate Lie algebras over finite fields. Our first method determines $\mathbb{Z}_2$-gradings. A well-known construction for $\mathbb{Z}_2$-gradings of

MSC 2020: 17B05, 17B50, 17B70, 17-08
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Lie algebras over fields $\mathbb{F}$ of characteristic different from 2 uses the Cartan decomposition of L: If $\theta\colon L \to L$ is an automorphism with $\theta^2 = 1$, then θ is diagonalizable with eigenvalues 1 and -1 and its eigenspaces form a $\mathbb{Z}_2$-grading. We introduce an alternative construction for fields of characteristic 2: If $l \in L$ with $(ad_L(l))^2 = ad_L(l)$, then $ad_L(l)$ is diagonalizable with eigenvalues 0 and 1 and its eigenspaces form a $\mathbb{Z}_2$-grading. We call l an *idempotent* and we introduce an algorithm to compute the idempotents in a Lie algebra L over a finite field $\mathbb{F}$ with $char(\mathbb{F}) = 2$. We use this algorithm to determine the $Aut(L)$-orbits of idempotents in the known simple Lie algebras of dimension at most 20 over the field $\mathbb{F}_2$ with two elements, see Section 4 for details. Based on this computation, we propose the following conjecture.

Conjecture 1.1. *Every finite-dimensional simple Lie algebra over a field of characteristic 2 has a non-central idempotent, and hence a non-degenerate $\mathbb{Z}_2$-grading.*

Lie algebras with $\mathbb{Z}_2$-grading are closely related to Lie superalgebras. The latter play a role in physics where they are used to describe the mathematics of supersymmetry, we refer to [11], [1] and [13] for details on Lie superalgebras. In [2] it has been shown that each $\mathbb{Z}_2$-grading of a simple Lie algebra over a field of characteristic 2 determines a simple Lie superalgebra. We recall this construction in Theorem 2.4 below for completeness, and use it to determine the simple Lie superalgebras arising from the known simple Lie algebras of dimension at most 20 over $\mathbb{F}_2$. See Section 4 for details.

Our second aim in this paper is to describe an algorithm that computes all subalgebras of a finite Lie algebra L up to the action of $Aut(L)$. We apply this algorithm to determine all subalgebras of the currently known simple Lie algebras of dimension at most 16 over the field $\mathbb{F}_2$ with two elements, except for the 15-dimensional Zassenhaus algebra, which we denote by $W(4)$. The Lie algebra $W(4)$ has more than two million orbits of subalgebras under the action of its automorphism group and it is the only case in our considered range that we could not complete. Our algorithm also allows to determine maximal subalgebras, and the simple subquotients of a Lie algebra over a finite field. We exhibit our computational results in Section 4 below.

Our algorithms are implemented in the computer algebra system GAP [19]. We use the FinLie package [6] for the computation of automorphism groups of Lie algebras over $\mathbb{F}_2$ and the list of known low-dimensional simple Lie algebras over the field $\mathbb{F}_2$, which is also a part of FinLie. This list contains the complete classification of Lie algebras of dimension at most 9 obtained by Vaughan-Lee [20], the list determined by Eick [5] using computational methods, three Lie algebras that have been computed recently using the methods of [5], and a Lie algebra described by Skryabin [17]. Generators for the three new Lie algebras can be found in Appendix 5 below.

2 Idempotents, gradings and Lie superalgebras

Let L be a Lie algebra and let B be an abelian additive group. The Lie algebra L is B-graded if L can be written as a direct sum of vector spaces

$$L = \bigoplus_{b \in B} L_b$$

so that $[L_a, L_b] \subseteq L_{a+b}$ holds for all $a, b \in B$. Such a grading is also called a *group grading.* We say that a group grading is *degenerate* if there exists $b \in B$ with $L_b = \{0\}$. Patera & Zassenhaus [15] initiated a first systematic study of arbitrary non-degenerate gradings. We also refer to [4] and [7] for further background on gradings.

Of particular interest are gradings with $B \cong \mathbb{Z}_2$ being cyclic of order 2. In the following section we introduce a construction for such a type of grading.

2.1 Idempotents and gradings

Let L be a Lie algebra over an arbitrary field $\mathbb{F}$. For $l \in L$ and $a \in \mathbb{F}$ we define the vector space $E_a(l) = \{h \in L \mid [l, h] = ah\}$ to be the eigenspace of $ad_L(l)$ to the eigenvalue a and

$$E(l) = \bigoplus_{a \in \mathbb{F}} E_a(l).$$

An element $l \in L$ is *diagonalizable* if $ad_L(l)$ is diagonalizable over $\mathbb{F}$. We denote by $V(l) = \langle a \in \mathbb{F} \mid E_a(l) \neq \{0\} \rangle$ the additive subgroup of $\mathbb{F}$ generated by the eigenvalues of $ad_L(l)$.

Lemma 2.1. *Let L be a Lie algebra over an arbitrary field and let $l \in L$.*

(a) *$E(l)$ is the direct sum of its non-zero subspaces $E_a(l)$ and $[E_a(l), E_b(l)] \subseteq E_{a+b}(l)$ holds for all $a, b \in \mathbb{F}$.*

(b) *Suppose that l is diagonalizable. Then $E(l)$ is a group grading of L for the group $V(l)$.*

Proof. (a) The set $E_a(l)$ is the eigenspace to the eigenvalue a of $ad_L(l)$. Hence $E_a(l)$ is a subspace and the eigenspaces form a direct sum. Now let $h \in E_a(l)$ and $k \in E_b(l)$. Then $[l, [h, k]] = [[l, h], k] + [h, [l, k]] = a[h, k] + b[h, k] = (a + b)[h, k]$. Hence $[h, k] \in E_{a+b}(l)$.

(b) Follows readily from (a). □

Lemma 2.1 leads to the following central observation.

Theorem 2.2. *Let L be a field of characteristic 2, and let l be a non-central idempotent of L. Then $L = L_0 \oplus L_1$ is a non-degenerate $\mathbb{Z}_2$-grading, where $L_0 = E_0(l)$ and $L_1 = E_1(l)$.*

Proof. Let $m = ad_L(l)$. As $m^2 = m$, it follows that $m(m-1) = 0$ and thus the minimal polynomial of m divides $x(x-1)$. This implies that m is diagonalizable with $L_0 \oplus L_1 = L$. Next $L_0 \neq \{0\}$, since $l \in L_0$ and $L_1 \neq \{0\}$, since $l \notin Z(L)$. In summary, we obtain a non-degenerate $\mathbb{Z}_2$-grading for L. □

Remark 2.3. Let $\alpha \in Aut(L)$ and let $l_1, l_2 \in L \setminus Z(L)$ be idempotents with $l_2 = \alpha(l_1)$. If $L = L_0 \oplus L_1$ is the grading associated to l_1 as in Theorem 2.2, then $\alpha(L_0) \oplus \alpha(L_1)$ is the grading associated to l_2, i.e., both gradings belong to the same $Aut(L)$-orbit.

The idempotents of a Lie algebra L over a field $\mathbb{F}$ with $char(\mathbb{F}) = 2$ can be computed easily. Let $b_1, \ldots, b_n$ be a basis of L and let $x_1, \ldots, x_n$ be indeterminates. If $x = x_1 b_1 + \ldots + x_n b_n$, then $ad_L(x) = x_1 ad_L(b_1) + \ldots + x_n ad_L(b_n)$. Hence

$$ad_L(x)^2 - ad_L(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j ad_L(b_i) ad_L(b_j) - \sum_{i=1}^{n} x_i ad_L(b_i).$$

The equation $ad_L(x)^2 - ad_L(x) = 0$ now translates to n^2 polynomial equations in the indeterminates $x_1, \ldots, x_n$ over the field $\mathbb{F}$. We obtain the idempotents of L by determining all $(x_1, \ldots, x_n) \in \mathbb{F}^n$ solving all of the n^2 equations.

2.2 Simple Lie superalgebras from simple Lie algebras

We first recall the definition of a Lie superalgebra over a field $\mathbb{F}$ of characteristic 2 from [13], see also [2] and [1]. Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic 2. Then L is a Lie superalgebra if there exists a $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$ and a map s (called *squaring*)

$$s\colon L_1 \to L_0,\ x \mapsto s(x),$$

such that $s(\alpha x) = \alpha^2 s(x)$ for all $\alpha \in \mathbb{F}, x \in L_1$, and $[x, y] = s(x + y) - s(x) - s(y)$ for all $x, y \in L_1$, and $[s(x), y] = [x, [x, y]]$ for all $x \in L_1, y \in L$. The later condition translates to $ad_L(s(x)) = (ad_L(x))^2$ for $x \in L_1$, and, if the center of L is trivial, then this allows to determine $s(x)$ for $x \in L_1$ if it exists. The following is in part also proved in [2]. We include a proof here for completeness.

Theorem 2.4. *Let L be a finite-dimensional $\mathbb{Z}_2$-graded Lie algebra with trivial center over a field $\mathbb{F}$ of characteristic 2. Then L embeds into a finite-dimensional Lie superalgebra S over $\mathbb{F}$ such that*

(a) *as a Lie algebra, L is an ideal of S and S/L is abelian of dimension at most* $\dim(L_1)$.
(b) *if L is a simple Lie algebra, then S is a simple Lie superalgebra.*

Proof. Let $\varphi\colon L \to End(L),\ l \mapsto ad_L(l)$. As the center of L is trivial, the homomorphism φ is injective and $L \cong \varphi(L)$. Therefore, in the following we identify L with $\varphi(L)$.

As a first step, we recall some details of the arithmetic of $End(L)$. The Lie bracket in $End(L)$ is given by $[x, y] = xy - yx = xy + yx$, since $char(\mathbb{F}) = 2$. For $x, y \in End(L)$ and $a, b \in \mathbb{F}$ we obtain that

$$\begin{aligned}(ax+by)^2 &= (ax)^2 + axby + byax + (by)^2 \\ &= a^2x^2 + b^2y^2 + ab[x,y],\end{aligned} \tag{1}$$

$$\begin{aligned}[x^2,y] &= x^2y + yx^2 \\ &= x^2y + xyx + xyx + yx^2 \\ &= [x,[x,y]], \text{ and} \\ \\ [x^2,y^2] &= [x,[x,y^2]] \\ &= [x,[y^2,x]] \\ &= [x,[y,[y,x]]].\end{aligned} \tag{2}$$

Next, we define $C = \langle x^2 \mid x \in L_1\rangle$ to be the vector subspace of $End(L)$ spanned by the squares of elements in L_1 and we set $S = L + C$. Then by construction S is a vector space. Equation (2) asserts that S is a Lie algebra and L is an ideal of S with S/L being abelian. Equation (1) yields that $\dim(S/L) \leq \dim(L_1)$.

We define $S_0 = L_0 + C$ and $S_1 = L_1$, and show that this yields a $\mathbb{Z}_2$-grading of S. Clearly, $[S_1,S_1] = [L_1,L_1] \subseteq L_0 \subseteq S_0$. Furthermore, $[S_0,S_0] \subseteq L_0$ by Equation (2) and similarly, $[S_0,S_1] \subseteq L_1 = S_1$. It remains to show that $S_0 \cap S_1 = \{0\}$. Let $x \in S_0 \cap S_1$. Then $x \in L_1$ and $x = u + c$ with $u \in L_0, c \in C$. If $y \in L_0$, then $[x,y] \in L_1$ and $[x,y] = [u+c,y] = [u,y] + [c,y] \in L_0$. If $y \in L_1$, then $[x,y] \in L_0$ and $[x,y] = [u+c,y] = [u,y] + [c,y] \in L_1$. Since in both cases $y \in L_0$ and $y \in L_1$, we obtain that $[x,y] = 0$, and therefore $x \in Z(L) = \{0\}$ follows. In summary, if $x \in S_0 \cap S_1$, then $x = 0$ and thus $S_0 \cap S_1 = \{0\}$.

As S has a squaring by construction, it follows that S is a Lie superalgebra. It remains to show that S is simple as Lie superalgebra. This is also shown in [2, Th. 3.3.1]. We recall a proof here for completeness. By the remarks at the end of [18, Chapter 2] a minimal 2-envelope G of L can be constructed as $G = L + \langle x^2 \mid x \in L\rangle$. We then have $L \subseteq S \subseteq G$. Let I be a superideal in S; that is, I is a Lie ideal in the Lie algebra S and $s(I \cap S_1) \subseteq I$. Let H be the ideal generated by I in G. Then $H \cap L$ is an ideal in L and because L is simple, we have $H \cap L = \{0\}$ or $H \cap L = L$. If $H \cap L = \{0\}$, then $[H,G] \subseteq H \cap [G,G] \subseteq H \cap L = \{0\}$. It follows that $H \subseteq Z(G) \subseteq L$, where the second inclusion follows from Theorem 5.8 (3) in [18, Chapter 2] and uses that G is a minimal 2-envelope of L. In conclusion $H = \{0\}$ and because $I \subseteq H$ also $I = \{0\}$. If $H \cap L = L$, then $L \subseteq I$ and thus also $C = \langle s(L_1)\rangle \subseteq \langle s(I \cap S_1)\rangle \subseteq I$ and we deduce $I = S$. $\square$

The proof of Theorem 2.4 readily translates into an algorithm for constructing a *superization* S based on a given $\mathbb{Z}_2$-grading of L; that is, an algorithm that yields an embedding of L into a simple Lie superalgebra S. We determine the superizations for gradings determined by idempotents for all known simple Lie algebras of dimension at most 20 over the field $\mathbb{F}_2$ in Section 4 below.

3 Computing all subalgebras

Let L be a finite-dimensional Lie algebra over a finite field $\mathbb{F}$. Our aim is to introduce a practical algorithm for computing all subalgebras of L. The automorphism group $Aut(L)$ acts on the set of all subalgebras of L. Given $A \leq Aut(L)$, our method constructs orbit representatives under the action of A. We remark that in our applications $A = Aut(L)$, but the algorithm works in the general setting. Section 4 exhibits the application of our method to the simple Lie algebras over $\mathbb{F}_2$ of dimension at most 16.

3.1 The subalgebra algorithm

The basic idea of our method is induction. In the initial step of the induction, we determine the A-orbits of 1-dimensional subspaces of L. If $U = \langle u \rangle$ is a 1-dimensional subspace of L, then $[u, u] = 0$ and thus U is a subalgebra of L. Hence all 1-dimensional subspaces are subalgebras. We denote the A-orbits of 1-dimensional subspaces with $O_1, \ldots, O_r$.

We now initialize a list $\mathcal{L}$ containing A-orbit representatives of all constructed subalgebras. We choose the orbit representatives in a canonical way by choosing the subalgebra with the lexicographically smallest upper triangular basis. With this convention, we obtain each A-orbit of subalgebras exactly once. We only store a representative of the orbit and we can easily check if two subalgebras belong to the same orbit. In the first step we add canonical A-orbit representatives for the 1-dimensional subspaces to $\mathcal{L}$.

In the induction step, we consider each subalgebra U in $\mathcal{L}$ in turn. If $\dim(U) = \dim(L)$ then $U = L$ and there is nothing to do. Thus assume that $\dim(U) < \dim(L)$. Then we determine $B = Stab_A(U)$ and loop over the orbits $O_1, \ldots, O_r$. Given O_i we determine the B-orbits in O_i. Then for each B-orbit representative W, say, we construct the subalgebra V generated by U and W. We add a canonical A-orbit representative of V to $\mathcal{L}$, if this is not already contained in $\mathcal{L}$.

Remark 3.1. We add a few remarks on efficiency of time and space of this method.

(a) Storing A-orbit representatives instead of all subalgebras reduces the space used to store the results.

(b) By using canonical representatives of A-orbits we can readily check if a new A-orbit representative is already existing in $\mathcal{L}$.

Let U be a subalgebra in $\mathcal{L}$. If there exists a subalgebra $V \subseteq L$ with $U \subseteq V$, then $[V, U] \subseteq V$ and hence the quotient space V/U is a submodule for the action of $ad(U)$ on L/U. The MeatAxe is able to determine all submodules of a given module over a finite field and this is a highly efficient algorithm, provided that there are only rather few submodules. If L/U as an $ad(U)$-module has only few submodules, then we use this alternative to construct these submodules. Once the submodules are available, we then check which of them are subalgebras.

3.2 Example: the 3-dimensional simple Lie algebra

Let L be the 3-dimensional simple Lie algebra over $\mathbb{F}_2$. This has a basis $\{b_1, b_2, b_3\}$ with $[b_1, b_2] = b_3, [b_1, b_3] = b_1$ and $[b_2, b_3] = b_1 + b_2$. Note that this is not the standard basis of the 3-dimensional Zassenhaus algebra, but the basis used in the FinLie package. Its automorphism group $A = Aut(L)$ is a non-abelian subgroup of order 6 in $GL(3, 2)$. It has 3 orbits of 1-dimensional subspaces of L:

(O_1) $\{\langle(1, 0, 1)\rangle\}$,
(O_2) $\{\langle(0, 0, 1)\rangle, \langle(1, 1, 0)\rangle, \langle(0, 1, 0)\rangle\}$,
(O_3) $\{\langle(0, 1, 1)\rangle, \langle(1, 1, 1)\rangle, \langle(1, 0, 0)\rangle\}$.

Hence there are 7 subalgebras of dimension 1 falling into 3 orbits under A. The list $\mathcal{L}$ is then initialized with the representatives $\langle(1, 0, 1)\rangle, \langle(0, 0, 1)\rangle, \langle(0, 1, 1)\rangle$.

In the next step of the algorithm, iterated extensions are determined. This starts with the subalgebra $U = \langle(1, 0, 1)\rangle$. Its stabilizer B is equal to A and hence we extend U twice: first with $W_1 = \langle(0, 0, 1)\rangle$ and second with $W_2 = \langle(0, 1, 1)\rangle$. This yields $V_1 = \langle U, W_1\rangle$ and $V_2 = \langle U, W_2\rangle$. Both are subalgebras and both have the same canonical representative $V = \langle(1, 0, 0), (0, 0, 1)\rangle$ under the action of A. Hence they both represent the same A-orbit of subalgebras of L and we add its canonical representative V to $\mathcal{L}$.

In the second iteration the subalgebra $\langle(0, 0, 1)\rangle$ is extended by subalgebras in O_2 and O_3 and in the third step the subalgebra $\langle(0, 1, 1)\rangle$ is extended by subalgebras in O_3. None of these extensions yields a new A-orbit of subalgebras, and hence we conclude that the list $\mathcal{L}$ is already complete.

We summarize the subalgebras of L in the graph exhibited in Figure 1. The vertices of the graph correspond to the subalgebras of L. The top vertex corresponds to L, the next layer contains the three subalgebras of dimension 2, the third layer contains the seven subalgebras of dimension 1, and at the bottom there is the trivial subalgebra. Two subalgebras U and V are joined by an edge if $U < V$ and there is no intermediate subalgebra between them. The circles around subalgebras of dimension 1 and 2 indicate $Aut(L)$-orbits.

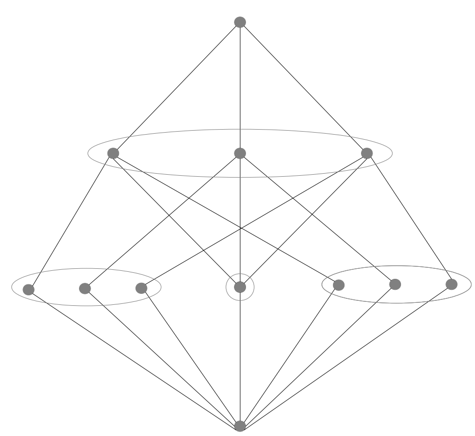

Figure 1: The Hasse diagram of the subalgebra lattice of $L_{3,1}$.

We note that the maximal nilpotent subalgebras of L are the subalgebras of dimension 1. Hence L is an example of a Lie algebra where the Cartan subalgebras are not all in one orbit under $Aut(L)$.

3.3 Alternative approaches

A first naive approach to compute all subalgebras is to determine all subspaces of L as an $\mathbb{F}$-vector space and then to select those subspaces that are closed under the Lie multiplication. This naive approach is practical if $|L| \leq 500$.

If L has a non-trivial ideal I, then this can be used for an inductive approach. First, construct all subalgebras of the quotient L/I. Second, consider each determined subalgebra U/I in turn and determine all proper supplements to I in U that are subalgebras. This idea is likely to more efficient than the method described above, but it does not apply to the case of simple Lie algebras and this is our desired application.

4 Results

Let $L_{d,i}$ denote the i-th simple Lie algebra over $\mathbb{F}_2$ of dimension d as contained in the FinLie package. When possible, we also exhibit name(s) for these Lie algebras as follows:

- A, B, C, D, E, F, G describe the simple constituents of the classical Lie algebras;
- W, S, H, K describe the simple constituents of the Lie algebras of Cartan type;
- P describes the Hamiltonian type Lie algebras, see [14];
- Q describes the Contact type Lie algebras, see [21];
- Kap_i $(1 \leq i \leq 4)$ describes four series of Lie algebras constructed by Kaplansky [12];
- Bro_i $(1 \leq i \leq 3)$ describes three series of Lie algebras constructed by Brown [3];
- V_7, V_8 and V_9 are simple Lie algebras determined by Vaughan-Lee [20].

4.1 Idempotents, gradings and superizations

The following tables contain for each known simple Lie algebra L over $\mathbb{F}_2$ up to dimension 20 its name(s) as far as available, the number of $Aut(L)$-orbits of idempotents of L, and the dimensions of the associated superizations. The orbits of idempotents are described by $m \times [d_0, d_1]$, which means that there are m orbits of idempotents such that for the associated $\mathbb{Z}_2$-grading $\dim(L_0) = d_0$ and $\dim(L_1) = d_1$ holds.

Lie alg	names	gradings	dim of superizations
$L_{3,1}$	$W(2)$	$1 \times [1,2]$	5
$L_{6,1}$	$W(2) \otimes \mathbb{F}_4$	$1 \times [2,4]$	10
$L_{7,1}$	$W(3)$	$2 \times [3,4]$	$9, 9$
$L_{7,2}$	$V_7, P(1,2)$	$1 \times [3,4]$	10
$L_{8,1}$	$A_2, W(1,1), Q(1,1,1)$	$1 \times [4,4]$	8
$L_{8,2}$	V_8	$1 \times [4,4]$	8
$L_{9,1}$	$W(2) \otimes \mathbb{F}_8, V_9$	$1 \times [3,6]$	15
$L_{10,1}$	$Kap_3(5)$	$1 \times [4,6]$, $1 \times [6,4]$	$12, 14$
$L_{12,1}$	$W(2) \otimes \mathbb{F}_{16}$	$1 \times [4,8]$	20
$L_{14,1}$	$W(3) \otimes \mathbb{F}_4$	$2 \times [6,8]$	$18, 18$
$L_{14,2}$	$V_7 \otimes \mathbb{F}_4$	$1 \times [6,8]$	20
$L_{14,3}$	$S(2,2)$	$3 \times [6,8]$	$16, 17, 17$
$L_{14,4}$	$P(1,1,1,1), Kap_1(4)$	$2 \times [6,8]$	$16, 17$
$L_{14,5}$	$A_3, B_3, C_3, G_2, S(1,1,1), H(1,1,1,1)$	$1 \times [6,8]$	14
$L_{14,6}$	$\mathrm{Bro}_2(1,1)$	$1 \times [6,8]$	16
$L_{15,1}$	$W(2) \otimes \mathbb{F}_{32}$	$1 \times [5,10]$	25
$L_{15,2}$	$W(4)$	$4 \times [7,8]$	$17, 17, 17, 17$
$L_{15,3}$	$Kap_3(6), Kap_2(4)$	$2 \times [7,8]$	$17, 19$
$L_{15,4}$	$P(2,1,1)$	$6 \times [7,8]$	$18, 19, 19, 19, 19$
$L_{15,5}$	$P(3,1)$	$3 \times [7,8]$	$18, 18, 18$
$L_{15,6}$	$P(2,2)$	$5 \times [7,8]$	$18, 19, 19, 19, 19$
$L_{15,7}$	from [5]	$6 \times [7,8]$	$18, 18, 19, 19, 19, 19$
$L_{15,8}$	from [5]	$6 \times [7,8]$	$17, 17, 18, 18, 19, 19$
$L_{15,9}$	new	$3 \times [7,8]$	$17, 17, 19$
$L_{15,10}$	new	$8 \times [7,8]$	$17, 17, 18, 18, 19, 19, 19, 19$
$L_{15,11}$	new	$3 \times [7,8]$	$17, 17, 19$
$L_{15,12}$	from [8],[17]	$4 \times [7,8]$	$18, 19, 19, 19$
$L_{16,1}$	$W(1,1) \otimes \mathbb{F}_4, A_2 \otimes \mathbb{F}_4, V_8 \otimes \mathbb{F}_4$	$1 \times [8,8]$	16
$L_{16,2}$	$W(2,1), Q(2,1,1)$	$4 \times [8,8]$	$16, 17, 17, 17$
$L_{16,3}$	from [5]	$4 \times [8,8]$	$16, 17, 17, 17$
$L_{16,4}$	from [5]	$10 \times [8,8]$	$16, 16, 17, 17, 17, 17, 17, 17, 17, 17$
$L_{16,5}$	from [5]	$6 \times [8,8]$	$17, 17, 17, 17, 17, 17$
$L_{16,6}$	from [5]	$4 \times [8,8]$	$16, 17, 17, 17$
$L_{18,1}$	$W(2) \otimes \mathbb{F}_{64}$	$1 \times [6,12]$	30
$L_{20,1}$	$Kap_3(5) \otimes \mathbb{F}_4$	$1 \times [8,12]$, $1 \times [12,8]$	$24, 28$

Table 1: Idempotents and superizations

Note that the FinLie package contains simple Lie algebras up to dimension 30 over $\mathbb{F}_2$. It is possible to compute idempotents and associated superizations for these algebras, but the automorphism group computation is not feasible in all cases.

Furthermore, we remark that in all cases except for $Kap_3(5)$ (and its tensor product with $\mathbb{F}_4$) the simple Lie algebra L uniquely determines the dimensions of L_0 and L_1 in the $\mathbb{Z}_2$-gradings associated to idempotents of L.

Remark 4.1. Let L be a simple Lie algebra over $\mathbb{F}_2$ and let $\mathbb{E}$ be a finite field extension of $\mathbb{F}_2$. If l is an idempotent of L, then $l \otimes 1$ is an idempotent of $L \otimes_{\mathbb{F}_2} \mathbb{E}$.

4.2 Subalgebras and maximal subalgebras

The following tables contain for each simple Lie algebra L over $\mathbb{F}_2$ up to dimension 16 (with the exception of $W(4) = L_{15,2}$) the number of $Aut(L)$-orbits of subalgebras of given dimension and the number of $Aut(L)$-orbits of maximal subalgebras of given dimension.

The Lie algebra $L_{15,2}$ is $W(4)$, and it is excluded in the list below. It has more than two million orbits of subalgebras and our algorithm did not succeed in listing all of them.

Lie alg	dim 1	dim 2	dim 3	dim 4	dim 5	dim 6	dim 7	dim 8	
$L_{3,1}$	3	1	0	0	0	0	0	0	all
	0	1	0	0	0	0	0	0	max
$L_{6,1}$	5	5	3	1	0	0	0	0	all
	0	0	1	1	0	0	0	0	max
$L_{7,1}$	39	85	79	48	9	1	0	0	all
	0	0	1	1	0	1	0	0	max
$L_{7,2}$	43	43	26	9	2	0	0	0	all
	0	0	4	5	2	0	0	0	max
$L_{8,1}$	6	10	10	7	4	1	0	0	all
	0	0	0	0	1	1	0	0	max
$L_{8,2}$	6	8	6	4	2	0	0	0	all
	0	0	0	1	2	0	0	0	max
$L_{9,1}$	7	9	6	3	3	1	0	0	all
	0	0	1	0	0	1	0	0	max
$L_{10,1}$	16	31	41	26	12	6	2	0	all
	0	0	0	0	0	2	2	0	max
$L_{12,1}$	11	28	19	7	5	12	5	1	all
	0	0	0	0	0	1	0	1	max

Table 2: Subalgebras of simple Lie algebras of dimension at most 12

Lie alg	dim 1	dim 2	dim 3	dim 4	dim 5	dim 6	dim 7	dim 8	dim 9	dim 10	dim 11	dim 12	
$L_{14,1}$	211	2712	8011	9548	9827	8345	6564	2778	316	28	5	1	all
	0	0	0	0	0	1	1	1	0	0	0	1	max
$L_{14,2}$	237	481	532	251	116	58	36	19	4	2	0	0	all
	0	0	0	0	0	7	1	6	0	2	0	0	max
$L_{14,3}$	135	790	1988	2545	2315	1489	822	298	56	11	2	1	all
	0	0	0	0	0	1	3	1	0	0	1	1	max
$L_{14,4}$	78	289	538	545	360	204	139	56	14	3	1	0	all
	0	0	0	0	0	1	3	2	0	3	1	0	max
$L_{14,5}$	6	14	25	32	28	19	13	10	4	1	1	0	all
	0	0	0	0	0	0	0	1	0	1	1	0	max
$L_{14,6}$	19	70	143	171	126	81	53	30	10	1	2	0	all
	0	0	0	0	0	1	1	0	0	1	2	0	max

Table 3: Subalgebras of simple Lie algebras of dimension 14

Lie alg	dim 1	dim 2	dim 3	dim 4	dim 5	dim 6	dim 7	dim 8	dim 9	dim 10	dim 11	dim 12	dim 13	
$L_{15,1}$	15	67	72	19	5	7	31	31	7	1	0	0	0	all
	0	0	1	0	0	0	0	0	0	1	0	0	0	max
$L_{15,3}$	37	156	318	447	366	257	159	94	30	10	3	1	0	all
	0	0	0	0	0	2	0	0	0	1	2	1	0	max
$L_{15,4}$	911	3177	5925	7027	5250	3260	1714	721	167	34	10	2	0	all
	0	0	14	0	0	1	15	13	2	0	4	2	0	max
$L_{15,5}$	455	4199	14178	23832	26523	22453	13857	5663	1555	316	53	8	2	all
	0	0	2	0	0	0	3	4	0	0	0	0	2	max
$L_{15,6}$	511	2975	7644	11384	9992	6933	4018	1948	625	143	24	5	1	all
	0	0	5	0	0	2	5	4	0	0	1	1	1	max
$L_{15,7}$	1663	4823	7807	8280	5229	2978	1547	826	162	21	4	1	0	all
	0	0	26	0	0	1	30	22	7	1	3	1	0	max
$L_{15,8}$	475	1885	3410	4010	2720	1631	861	472	97	18	5	1	0	all
	0	0	5	0	0	3	6	6	2	1	4	1	0	max
$L_{15,9}$	117	542	1146	1603	1245	807	464	273	81	17	4	2	0	all
	0	0	1	0	0	0	2	1	0	0	2	2	0	max
$L_{15,10}$	491	2307	4946	6504	5070	3214	1666	706	152	34	11	2	0	all
	0	0	6	0	0	1	9	6	1	1	3	2	0	max
$L_{15,11}$	77	304	608	781	575	385	223	125	41	10	2	1	0	all
	0	0	1	0	0	1	3	1	0	0	1	1	0	max
$L_{15,12}$	687	3449	8572	11602	8502	4533	2120	895	206	30	9	3	0	all
	0	0	6	0	0	0	10	8	1	0	2	3	0	max

Table 4: Subalgebras of simple Lie algebras of dimension 15 except $W(4) = L_{15,2}$

Lie alg	dim 1	dim 2	dim 3	dim 4	dim 5	dim 6	dim 7	dim 8	dim 9	dim 10	dim 11	dim 12	dim 13	dim 14	
$L_{16,1}$	13	51	56	70	49	52	41	42	16	6	2	1	0	0	all
	0	0	0	0	0	0	0	2	0	1	0	1	0	0	max
$L_{16,2}$	157	1445	5214	10302	11518	10008	6604	3878	1414	351	68	19	5	1	all
	0	0	0	0	0	0	0	2	4	1	0	1	0	1	max
$L_{16,3}$	168	1086	2658	4248	3838	2485	1420	879	344	74	16	5	2	0	all
	0	0	0	0	0	0	0	2	3	1	0	3	2	0	max
$L_{16,4}$	495	2988	6900	10355	8700	5457	2968	1752	651	109	18	6	1	0	all
	0	0	0	1	0	0	0	7	9	3	0	4	1	0	max
$L_{16,5}$	379	2267	5857	9320	8037	5220	2930	1832	696	137	23	8	2	0	all
	0	0	0	2	0	0	0	7	8	1	0	2	2	0	max
$L_{16,6}$	297	972	1311	1397	791	232	89	86	34	5	1	1	0	0	all
	0	0	0	3	0	0	0	9	6	4	1	1	0	0	max

Table 5: Subalgebras of simple Lie algebras of dimension 16

4.3 Subquotients

The following tables exhibit for each simple Lie algebra L over $\mathbb{F}_2$ up to dimension 16 (with the exception of $W(4) = L_{15,2}$) the simple Lie algebras which arise as proper subquotients. An entry "+" in the row of $L_{d,i}$ and column $L_{f,j}$ indicates that $L_{f,j}$ embeds as a proper subquotient into $L_{d,i}$.

	$L_{3,1}$	$L_{6,1}$	$L_{7,1}$	$L_{7,2}$	$L_{8,1}$	$L_{8,2}$	$L_{9,1}$	$L_{10,1}$
$L_{3,1}$								
$L_{6,1}$	+							
$L_{7,1}$	+							
$L_{7,2}$	+							
$L_{8,1}$	+							
$L_{8,2}$	+							
$L_{9,1}$	+							
$L_{10,1}$	+							
$L_{12,1}$	+	+						
$L_{14,1}$	+	+	+					
$L_{14,2}$	+	+		+				
$L_{14,3}$	+		+	+				
$L_{14,4}$	+		+	+	+			
$L_{14,5}$	+				+	+		
$L_{14,6}$	+		+		+			
$L_{15,1}$	+							
$L_{15,3}$	+	+			+			+
$L_{15,4}$	+		+	+				
$L_{15,5}$	+		+	+				
$L_{15,6}$	+	+	+	+				
$L_{15,7}$	+	+	+	+	+			
$L_{15,8}$	+	+	+	+	+			+
$L_{15,9}$	+		+		+			
$L_{15,10}$	+		+	+				+
$L_{15,11}$	+	+	+		+			
$L_{15,12}$	+		+	+				
$L_{16,1}$	+	+			+	+		
$L_{16,2}$	+		+	+	+			
$L_{16,3}$	+		+	+	+			
$L_{16,4}$	+		+	+	+	+		
$L_{16,5}$	+		+	+	+	+		
$L_{16,6}$	+		+	+	+	+		

Table 6: Simple subquotients of simple Lie algebras

5 Appendix

We list explicit generators for the three new simple Lie algebras that have been found using the random search methods described in [5]. We note that the Lie algebras described in [8] are isomorphic to the Lie algebra obtained in [17] over $\mathbb{F}_2$ and hence are already contained in our list of known simple Lie algebras.

Explicit generators for the Lie algebra number 9 of dimension 15:

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\
\cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot
\end{pmatrix}
$$

Explicit generators for the Lie algebra number 10 of dimension 15:

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 \\
1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & 1
\end{pmatrix}
$$

Explicit generators for the Lie algebra number 11 of dimension 15:

$$
\begin{pmatrix}
\cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1
\end{pmatrix},
\begin{pmatrix}
1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 \\
1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot
\end{pmatrix}
$$

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Received: August 30, 2021
Accepted for publication: January 6, 2022
Communicated by: Friedrich Wagemann