

A regularity criterion in multiplier spaces to Navier-Stokes equations via the gradient of one velocity component

Ahmad M. Alghamdi, Sadek Gala and Maria Alessandra Ragusa

Abstract. In this paper, we study regularity of weak solutions to the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$. The main goal is to establish the regularity criterion via the gradient of one velocity component in multiplier spaces.

1 Introduction

In this paper we consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1)$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the unknown velocity vector and $\pi = \pi(x, t)$ denotes the hydrostatic pressure respectively. While u_0 is the prescribed initial data for the velocity with properties $\nabla \cdot u_0 = 0$.

The global existence of smooth solutions for the 3D incompressible Navier-Stokes equations is one of the most outstanding open problems in fluid mechanics. Different criteria for regularity of the weak solutions have been proposed and many interesting results were

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established (see, for example, [3], [8], [9], [17], [16], [14], [19], [24], [27], [32], [33], [34], [35], [37] and references therein).

Recently, many authors became interested in the regularity criteria involving only one velocity component, or its gradient, even though most of which are not scaling invariant (see, for example, [4], [5], [7], [10], [11], [15], [18], [22], [29] and the references cited therein). In particular, Zhou [30] showed that the solution is regular if

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{1}{2}, \quad 6 < q \leq \infty. \quad (2)$$

Later, Cao and Titi [5] obtained the regularity criterion

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{2}{3} + \frac{2}{3q}, \quad q > \frac{7}{2}. \quad (3)$$

Motivated by the above work, Zhou and Pokorný [37] showed the following regularity condition

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{4} + \frac{1}{2q}, \quad q > \frac{10}{3}, \quad (4)$$

while the limiting case $u_3 \in L^\infty(0, T; L^{\frac{10}{3}}(\mathbb{R}^3))$ was covered in [18]. Inspired by the work [3], we are interested in criteria involving the gradient of one velocity component ∇u_3 . In fact, He [15] first verified the following regularity result

$$\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty. \quad (5)$$

The above result was significantly improved by Pokorný [23] and Zhou [31] independently (see also [36]). More precisely, it reads as follows

$$\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq q < \infty. \quad (6)$$

Very recently, Ye [26] improves the previous work of Zhou and Pokorný [36] by using of a new anisotropic Sobolev inequality, and proved the following regularity criterion

$$\nabla u_3 \in L^{\frac{16q}{15q-23}}(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad q \in [2, 3]. \quad (7)$$

Note that

$$\frac{2}{\frac{16q}{15q-23}} + \frac{3}{q} = \frac{15q+1}{8q} > \frac{23}{12}, \quad \text{for any } 2 \leq q < 3.$$

Consequently, (7) can be regarded as a further improvement of [36]. Moreover, the endpoint case $q = 3$ recovers the result of [28]. For some other interesting regularity criteria, we refer the readers to [25], [26] and references therein.

The purpose of this work is to extend the regularity criterion of weak solutions in terms of one gradient of velocity component to the multiplier space which is larger than the Lebesgue space. The method is based on the following interpolation inequality

$$\|\varphi\|_{L^\gamma} \leq C \|\partial_3 \varphi\|_{L^\mu}^{\frac{1}{3}} \|\nabla_h \varphi\|_{L^\lambda}^{\frac{2}{3}},$$

where μ, λ and γ satisfy

$$1 \leq \mu, \quad \lambda < +\infty, \quad \frac{1}{\mu} + \frac{2}{\lambda} > 1 \quad \text{and} \quad 1 + \frac{3}{\gamma} = \frac{1}{\mu} + \frac{2}{\lambda}.$$

The detailed proof of this inequality can be found in the appendix of Cao and Wu [6].

In order to prove our theorem, let us recall the definition of weak solutions.

Definition 1.1. Let $T > 0, u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. A measurable function $u(x, t)$ is called a weak solution to the Navier-Stokes equations (1) on $[0, T]$ if the following conditions hold:

1. $u(x, t) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;
2. system (1) is satisfied in the sense of distributions;
3. the energy inequality, that is,

$$\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2.$$

By a strong solution, we mean that a weak solution u of the Navier-Stokes equations (1) satisfies

$$(u(x, t), \theta(x, t)) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)).$$

It is well known that the strong solution is regular and unique.

For $\alpha \in \mathbb{R}$, the Homogeneous Sobolev Space $\dot{H}^\alpha(\mathbb{R}^3)$ is the space of tempered distributions f for which

$$\|f\|_{\dot{H}^\alpha} = \sqrt{\int_{\mathbb{R}^3} |\xi|^{2\alpha} |\widehat{f}(\xi)|^2 d\xi} < +\infty.$$

For Homogeneous Sobolev Spaces, we refer to the book [2]. For instance, the following basic interpolation inequality holds:

Lemma 1.2. For $0 < \alpha \leq \beta$, the space $L^2 \cap \dot{H}^\beta$ is a subset of \dot{H}^α , and we have

$$\|f\|_{\dot{H}^\alpha} \leq \|f\|_{L^2}^{1-\frac{\alpha}{\beta}} \|f\|_{\dot{H}^\beta}^{\frac{\alpha}{\beta}}. \quad (8)$$

Proof. This is a particular case of [2], Proposition 1.32. □

We say that a function belongs to the multiplier spaces $\dot{X}_{1+\alpha} := M(\dot{H}^\alpha(\mathbb{R}^3) \rightarrow \dot{H}^{-1}(\mathbb{R}^3))$ if it maps, by pointwise multiplication, \dot{H}^α in \dot{H}^{-1} :

$$\dot{X}_{1+\alpha} = \{f \in \mathcal{S}'(\mathbb{R}^3) : \|fg\|_{\dot{H}^{-1}} \leq \|g\|_{\dot{H}^\alpha}\}.$$

$\dot{H}^\alpha(\mathbb{R}^3)$ denotes the homogeneous Sobolev space. The space $\dot{X}_{1+\alpha}$ has been characterized in [20], [21] (see also [13]). Now our regularity criterion for system (1) reads

Theorem 1.3. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that u is a weak solution to system (1). If ∇u_3 satisfies the following condition*

$$\nabla u_3 \in L^{\frac{8}{3-4\alpha}}(0, T; \dot{X}_{1+\alpha}(\mathbb{R}^3)), \text{ for some } 0 \leq \alpha < \frac{3}{4}, \quad (9)$$

then the solution u is regular on $(0; T]$.

Remark 1.4. Since $L^{\frac{3}{1+\alpha}}(\mathbb{R}^3) \subset \dot{X}_{1+\alpha}(\mathbb{R}^3)$ (see e.g. [35] for details), it is clear that our result improves that in [26] and extend the regularity criterion (4) from Lebesgue space L^α to multiplier space $\dot{X}_{1+\alpha}$.

Thanks to

$$\|f\|_{BMO} \leq C \|\nabla f\|_{\dot{X}_1}$$

(see e.g. [12, Proposition 2]), where BMO denotes the homogeneous space of bounded mean oscillations, it is easy to deduce the following regularity criterion.

Corollary 1.5. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that u is a weak solution to system (1). If u_3 satisfies the following condition*

$$u_3 \in L^{\frac{8}{3}}(0, T; BMO(\mathbb{R}^3)), \quad (10)$$

then the solution u is regular on $(0; T]$.

Remark 1.6. Since $L^\infty(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3)$, our result recovers the limiting case $q = \infty$ in (4), that is,

$$u_3 \in L^{\frac{8}{3}}(0, T; L^\infty(\mathbb{R}^3)).$$

Consequently, (10) can be regarded as a further improvement of the previous work [37].

2 Proof of main result.

In this section, under the assumptions of the Theorem 1.3, we prove our main result. Before proving our result, we recall the following multiplicative Sobolev imbedding inequality in the whole space \mathbb{R}^3 (see, for example [5]):

$$\|f\|_{L^6} \leq C \|\nabla_h f\|_{L^2}^{\frac{2}{3}} \|\partial_3 f\|_{L^2}^{\frac{1}{3}}, \quad (11)$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient operator. We are now give the proof of our main theorem.

Proof. To prove our result, it suffices to show that for any fixed $T > T^*$, there holds

$$\sup_{0 \leq t \leq T^*} \|\nabla u(t)\|_{L^2}^2 \leq C_T,$$

where T^* , which denotes the maximal existence time of a strong solution and C_T is an absolute constant which only depends on T and u_0 .

The method of our proof is based on two major parts. The first one establishes the bounds of $\|\nabla_h u\|_{L^2}^2$, while the second gives the bounds of the H^1 -norm of velocity u in terms of the results of part one.

Step I. Taking the inner product of (1)₁ with $-\Delta_h u$, we obtain after integrating by parts that

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \|\nabla \nabla_h u\|_{L^2}^2 = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta_h u dx = I. \quad (12)$$

where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ is the horizontal Laplacian. For the notational simplicity, we set

$$\begin{aligned} \mathcal{L}^2(t) &= \sup_{\tau \in [\Gamma, t]} \|\nabla_h u(\tau)\|_{L^2}^2 + \int_{\Gamma}^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 d\tau, \\ \mathcal{J}^2(t) &= \sup_{\tau \in [\Gamma, t]} \|\nabla u(\tau)\|_{L^2}^2 + \int_{\Gamma}^t \|\Delta u(\tau)\|_{L^2}^2 d\tau, \end{aligned}$$

for $t \in [\Gamma, T^*)$. In view of (9), we choose $\epsilon > 0$ to be precisely determined subsequently and then select $\Gamma < T^*$ sufficiently close to T^* such that for all $\Gamma \leq t < T^*$,

$$\int_{\Gamma}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \epsilon \ll 1. \quad (13)$$

Integrating by parts and using the divergence-free condition, it follows that

$$\begin{aligned} I &\leq \int_{\mathbb{R}^3} |\nabla u_3| |\nabla u| |\nabla_h u| dx \\ &\leq \| |\nabla u_3| |\nabla u| \|_{\dot{H}^{-1}} \|\nabla_h u\|_{\dot{H}^1} \\ &\leq C \|\nabla u_3\|_{\dot{X}_{1+\alpha}} \|\nabla u\|_{\dot{H}^\alpha} \|\nabla \nabla_h u\|_{L^2} \\ &\leq C \|\nabla u_3\|_{\dot{X}_{1+\alpha}} \|\nabla u\|_{L^2}^{1-\alpha} \|\nabla^2 u\|_{L^2}^\alpha \|\nabla \nabla_h u\|_{L^2} \\ &\leq C \|\nabla u_3\|_{\dot{X}_{1+\alpha}}^2 \|\nabla u\|_{L^2}^{2(1-\alpha)} \|\Delta u\|_{L^2}^{2\alpha} + \frac{1}{2} \|\nabla \nabla_h u\|_{L^2}^2, \end{aligned}$$

by Young's inequality and (8). Inserting the above estimate into (12) and integrating with respect to time, we deduce for every $\tau \in [\Gamma, t]$:

$$\begin{aligned} &\sup_{\tau \in [\Gamma, t]} \|\nabla_h u(\tau)\|_{L^2}^2 + \int_{\Gamma}^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 d\tau \\ &\leq \|\nabla_h u(\Gamma)\|_{L^2}^2 + C \int_{\Gamma}^t \|\nabla u_3(\tau)\|_{\dot{X}_{1+\alpha}}^2 \|\nabla u(\tau)\|_{L^2}^{2(1-\alpha)} \|\Delta u(\tau)\|_{L^2}^{2\alpha} d\tau \\ &\leq \|\nabla_h u(\Gamma)\|_{L^2}^2 + C \left(\sup_{\tau \in [\Gamma, t]} \|\nabla u(\tau)\|_{L^2}^{\frac{3}{2}-2\alpha} \right) \int_{\Gamma}^t \|\nabla u_3(\tau)\|_{\dot{X}_{1+\alpha}}^2 \|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta u(\tau)\|_{L^2}^{2\alpha} d\tau \\ &\leq C + C \left(\sup_{\tau \in [\Gamma, t]} \|\nabla u(\tau)\|_{L^2}^{\frac{3}{2}-2\alpha} \right) \left(\int_{\Gamma}^t \|\nabla u_3(\tau)\|_{\dot{X}_{1+\alpha}}^{\frac{8}{3-4\alpha}} d\tau \right)^{\frac{3}{4}-\alpha} \left(\int_{\Gamma}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_{\Gamma}^t \|\Delta u(\tau)\|_{L^2}^2 d\tau \right)^\alpha \\ &\leq C + C \mathcal{J}^{\frac{3}{2}-2\alpha}(t) \left(\int_{\Gamma}^t \|\nabla u_3(\tau)\|_{\dot{X}_{1+\alpha}}^{\frac{8}{3-4\alpha}} d\tau \right)^{\frac{3}{4}-\alpha} \epsilon^{\frac{1}{4}} \mathcal{J}^{2\alpha}(t) \\ &\leq C + C \epsilon^{\frac{1}{4}} \mathcal{J}^{\frac{3}{2}}(t), \end{aligned}$$

which leads to

$$\mathcal{L}^2(t) \leq C + C\epsilon^{\frac{1}{4}} \mathcal{J}^{\frac{3}{2}}(t). \quad (14)$$

Step II. Now, we will establish the bounds of H^1 -norm of the velocity field. In order to do it, taking the inner product of $(1)_1$ with $-\Delta u$ in $L^2(\mathbb{R}^3)$. Then, integration by parts gives the following identity:

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx$$

Integrating by parts and using the divergence-free condition, one can easily deduce that (see e.g. [37])

$$\begin{aligned} \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx &\leq C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2 \\ &\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^6}^{\frac{3}{2}} \\ &\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

by Hölder's inequality, Nirenberg-Gagliardo's interpolation inequality and (11). Integrating this last inequality in time, we deduce that for all $\tau \in [\Gamma, t]$

$$\begin{aligned} \mathcal{J}^2(t) &\leq \|\nabla u(\Gamma)\|_{L^2}^2 \\ &\quad + C \sup_{\tau \in [\Gamma, t]} \|\nabla_h u(\tau)\|_{L^2} \left(\int_{\Gamma}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_{\Gamma}^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_{\Gamma}^t \|\Delta u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\ &\leq \|\nabla u(\Gamma)\|_{L^2}^2 + 2C\mathcal{L}(t)\epsilon^{\frac{1}{4}} \mathcal{L}(t) \mathcal{J}^{\frac{1}{2}}(t) \\ &= \|\nabla u(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}} \mathcal{L}^2(t) \mathcal{J}^{\frac{1}{2}}(t). \end{aligned} \quad (15)$$

Inserting (14) into (15) and taking ϵ small enough, then it is easy to see that for all $\Gamma \leq t < T^*$, there holds

$$\mathcal{J}^2(t) \leq \|\nabla u(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}}(t) + C\epsilon^{\frac{1}{2}} \mathcal{J}^2(t) < \infty,$$

which proves

$$\sup_{\Gamma \leq t < T^*} \|\nabla u(t)\|_{L^2}^2 < +\infty.$$

This completes the proof of Theorem 1.3. \square

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