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Polynomial complex Ginzburg-Landau equations in almost periodic spaces

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Abstract. We consider complex Ginzburg-Landau equations with a polynomial nonlinearity in the real line. We use splitting-methods to prove well-posedness for a subset of almost periodic spaces. Specifically, we prove that if the initial condition has multiples of an irrational phase, then the solution of the equation maintains those same phases.

1 Introduction

We consider the 1-dimensional autonomous system

$$\begin{cases} \partial_t u = (\alpha + i\beta)\partial_{xx} u + \gamma u + (a+ib)B(u), \\ u(0) = u_0, \end{cases} \tag{1}$$

where $u(x,t)$ is a complex valued function with $x \in \mathbb{R}$, $t > 0$, $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $a > 0$, $b > 0$ and B a continuous map. The linear term represented by $(\alpha + i\beta)\partial_{xx}$ characterizes the Complex Ginzburg-Landau equations. For $\beta = 0$ (1) reduces to a nonlinear heat equation and for $\alpha = 0$ to a nonlinear Schrödinger equation. A large amount of work has been done to prove well-posedness of (1) with different nonlinearities (see, for instance, [1], [16], [17]). In our case, we study well-posedness of problem (1) with polynomial nonlinearities. These nonlinearities are considered in Fisher-Kolmogorov equations and Fitzhugh-Nagumo equations [2], [14], [19]. Almost periodic spaces were introduced by Bohr [9] and further developed by Stepanoff [20], Weyl [21] and Besicovitch [4], [3]. These spaces are well-studied for the Complex Ginzburg-Landau equations with different nonlinearities [18], [15]. We consider u_0 with specific irrational phases, and we prove that the

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time evolution by (1) maintains the same phases. We use splitting-methods for evolution equations developed for numerical purposes [12], [10]. These methods were used to prove well-posedness of Complex Ginzburg-Landau equations and Reaction-diffusion equations in other spaces [7], [8], [6], [5].

The paper is organized as follows: In Section 2 we set notations and preliminary results. In Section 3 we analyze the nonlinear problem. Finally, in Section 4 using splitting methods, we combine results from Sections 2 and 3 to obtain that the solution of (1) is in a subset of an almost periodic space.

2 Notations and Preliminaries.

We introduce some definitions and preliminary results.

Definition 2.1. We define $C_{\mathrm{u}}(\mathbb{R})$ as the set of uniformly continuous and bounded functions on $\mathbb{R}$ equipped with the norm,

$$\|u\|_{\infty} = \sup_{x\in\mathbb{R}} |u(x)|.$$

Definition 2.2. We define the set of almost periodic functions [9] as,

$$P(\mathbb{R}) = \{u \in C_u(\mathbb{R}) : \ u = \sum_{j=1}^{\infty} a_j e^{ix\lambda_j} : \lambda_j \in \mathbb{R}\}$$

equipped with the uniform continuous norm.

Definition 2.3. We define the following subset of almost periodic functions,

$$A_\lambda(\mathbb{R}) = \{u \in C_u(\mathbb{R}) : \ u = \sum_{j=1}^{\infty} a_j e^{ixj\lambda}; \sum_{j=1}^{\infty} |a_j| < \infty \ : \lambda \in \mathbb{R}\} \tag{2}$$

equipped with the norm:

$$\|u\|_{A_\lambda} = \sum_{j=1}^{\infty} |a_j|. \tag{3}$$

Theorem 2.4. *$A_\lambda(\mathbb{R})$ is a Banach space.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ a Cauchy sequence such that $u_n \in A_\lambda(\mathbb{R}) \subset P(\mathbb{R})$. Then $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence, that is, $\|u_n\|_{A_\lambda} = \sum_{j=1}^{\infty} |a_j^n| \leq K$ for all $n \in \mathbb{N}$. Additionally, $\{u_n\}_{n\in\mathbb{N}}$ is a convergent sequence in $P(\mathbb{R})$, i.e. $u_n \to u$ with $u \in P(\mathbb{R})$ and $\|u\|_{A_\lambda} = \sum_{j=1}^{\infty} |a_j|$. Therefore, we have that $\sum_{j=1}^{N} |a_j| = \lim_{n\to\infty} \sum_{j=1}^{N} |a_j^n| \leq K$, which implies that

$$\lim_{N\to\infty} \sum_{j=1}^{N} |a_j| = \sum_{j=1}^{\infty} |a_j| \leq K.$$

On the other hand, we can define an inner product for $u, v \in P(\mathbb{R})$:

$$\langle u(x), v(x)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x)v(x)dx.$$

In particular, we have a normalized orthogonal system in the following sense (see [9]):

$$\langle e^{i\lambda_1 x}, e^{-i\lambda_2 x}\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i\lambda_1 x}e^{-i\lambda_2 x} = \begin{cases} 0 & , \text{for } \lambda_1 \neq \lambda_2 \\ 1 & , \text{for } \lambda_1 = \lambda_2. \end{cases}$$

For $u \in P(\mathbb{R}),\ u = \sum_{j=1}^{\infty} a_j e^{i\lambda_j x}$, we have,

$$\langle u(x), e^{-i\lambda_j x}\rangle = a_j.$$

If $u_n \in A_\lambda(\mathbb{R})$ is a Cauchy sequence, then

$$a_j^n = \langle u_n(x), e^{-ixj\lambda}\rangle \xrightarrow{n\to\infty} \langle u(x), e^{-ixj\lambda}\rangle = a_j.$$

As (2) and (3) are met then $u \in A_\lambda(\mathbb{R})$. □

Remark 2.5. As $\ell^1(\mathbb{N})$ is a Banach Algebra then, if $u \in A_\lambda(\mathbb{R})$ and $v \in A_\lambda(\mathbb{R})$ we have that,

$$\|uv\|_{A_\lambda} \leq \|u\|_{A_\lambda}\|v\|_{A_\lambda}.$$

The following definitions and proofs can be extended to $x \in \mathbb{R}^d$ (See [13]).

Definition 2.6. We denote $U(t)$ as the one parameter semigroup that solves the underlying linear equation

$$\partial_t u = (\alpha + i\beta)\partial_{xx}u + \gamma u. \tag{4}$$

The operator can be represented by the convolution in x

$$U(t) = (4\pi t(\alpha + i\beta))^{-1/2} e^{(-x^2/[4t(\alpha+i\beta)])+\gamma t} * u_0 = G_t(x) * u_0$$

and the kernel G_t satisfies:

$$|G_t(x)| = \left(4\pi t(\alpha^2+\beta^2)(4\pi t(\alpha^2+\beta^2)^{1/2}\right)^{-1/2} e^{(-x^2/[4t(\alpha^2+\beta^2)])+\gamma t}.$$

Clearly, $G_t(x) \in L^1(\mathbb{R})$.

Proposition 2.7. *For each $t \geq 0$, define $U(t)u_0 = G_t * u_0$. The one-parameter family of operators $\{U(t)\}_{t\geq 0}$ ia a strongly continuous semigroup in $C_u(\mathbb{R})$.*

Proof. The semigroup property, $U(t)U(t')u = U(t+t')u$ is proven similarly to the heat kernel. We show that, $U(t)u$ converges to u for all $u \in C_{\mathrm{u}}(\mathbb{R})$ when $t \to 0$. Indeed, we have,

$$\begin{aligned}|(U(t)u)(x) - u(x)| &\leq \int_{\mathbb{R}} G_t(y)|u(x-y) - u(x)|dy \\ &= \int_{|y|<\delta} G_t(y)|u(x-y) - u(x)|dy + \int_{|y|\geq\delta} G_t(y)|u(x-y) - u(x)|dy.\end{aligned}$$

The first integral of the right side of the equality can be estimated as follows:

$$\begin{aligned}\int_{|y|<\delta} G_t(y)|u(x-y) - u(x)|dy &\leq \int_{\mathbb{R}} G_t(y) \max_{|y|<\delta} |u(x-y) - u(x)|dy \\ &= \max_{|y|<\delta} |u(x-y) - u(x)|.\end{aligned}$$

This can be small enough because, $|y| < \delta$ and u is uniformly continuous. For the second term we proceed in the following way,

$$\int_{|y|\geq\delta} G_t(y)|u(x-y) - u(x)|dy \leq 2\|u\|_\infty \int_{|y|\geq\delta} G_t(y)dy.$$

Since $(-x^2/[4t(\alpha^2+\beta^2)]) \to \infty$ when $t \to 0^+$ and $G_t \in L^1(\mathbb{R})$, the right side of the previous equality tends to 0. The next property proves that U is well defined, that is $U(t)u \in C_{\mathrm{u}}(\mathbb{R})$.

$$\begin{aligned}|(U(t)u)(x_1) - (U(t)u)(x_2)| &\leq \int_{\mathbb{R}} G_t(y)|u(x_1-y) - u(x_2-y)|dy \\ &\leq \varepsilon \int_{\mathbb{R}} G_t(y)dy = \varepsilon,\end{aligned}$$

In the last inequality we used that u is uniformly continuous. □

Lemma 2.8. *If $u_0 \in A_\lambda(\mathbb{R})$ then $U(t)u_0 \in A_\lambda(\mathbb{R})$ for $t > 0$*

Proof. As $u_0 \in A(\mathbb{R})$ then we have

$$U(t)u_0 = G_t * u_0 = \int_{\mathbb{R}} (4\pi t(\alpha + i\beta))^{-1/2} e^{(-y^2(\alpha - i\beta)/[4t(\alpha^2+\beta^2)])+\gamma t} u_0(x-y)dy$$

where

$$u_0(x-y) = \sum_{j=1}^{\infty} a_j e^{ij\lambda(x-y)} \tag{5}$$

then we have,

$$\begin{aligned}U(t)u_0 &= \int_{\mathbb{R}} (4\pi t(\alpha+i\beta))^{-1/2} e^{(-y^2(\alpha-i\beta)/[4t(\alpha^2+\beta^2)])+\gamma t} \sum_{j=1}^{\infty} a_j e^{ij\lambda(x-y)} dy \\ &= \int_{\mathbb{R}} (4\pi t(\alpha+i\beta))^{-1/2} \sum_{j=1}^{\infty} a_j e^{(-y^2(\alpha-i\beta)/[4t(\alpha^2+\beta^2)])+\gamma t+ij\lambda(x-y)} dy \\ &= (4\pi t(\alpha+i\beta))^{-1/2} \int_{\mathbb{R}} \sum_{j=1}^{\infty} a_j B(y) e^{ij\lambda(x-y)} dy,\end{aligned}$$

with

$$B(y) = e^{(-y^2(\alpha-i\beta)/[4t(\alpha^2+\beta^2)])+\gamma t}.$$

Using dominated convergence theorem and that

$$\left|\sum_{j=1}^{n} a_j B(y) e^{ij\lambda(x-y)}\right| \leq \sum_{j=1}^{n} |a_j B(y)| = |B(y)| \sum_{j=1}^{n} |a_j| \leq K|B(y)|$$

we have,

$$U(t)u_0 = (4\pi t(\alpha+i\beta))^{-1/2} \sum_{j=1}^{\infty} \int_{\mathbb{R}} a_j B(y) e^{ij\lambda(x-y)} dy.$$

Finally, we know that

$$\int_{\mathbb{R}} e^{-ax^2-bx+c} dx = \frac{e^{b^2/(4a)+c}\sqrt{\pi}}{\sqrt{a}}$$

with $Re(a) > 0$. In our case we have

$$a = (\alpha-i\beta)/[4t(\alpha^2+\beta^2)]) = 1/(4t(\alpha+i\beta)), \quad b = ij\lambda \quad \text{and} \quad c = ixj\lambda+\gamma t.$$

Then we have,

$$\begin{aligned}U(t)u_0 =& (4\pi t(\alpha+i\beta))^{-1/2} \sum_{j=1}^{\infty} a_j \frac{e^{-t(j\lambda)^2(\alpha+i\beta)+ixj\lambda+\gamma t}\sqrt{\pi}}{\sqrt{(1/(4t(\alpha+i\beta))}} \\ =& \sum_{j=1}^{\infty} a_j e^{-t(j\lambda)^2(\alpha+i\beta)+ixj\lambda+\gamma t} \\ =& \sum_{j=1}^{\infty} A_j e^{ixj\lambda},\end{aligned}$$

where $A_j = a_j e^{-t(j\lambda)^2(\alpha+i\beta)+\gamma t}$, so we have (2). On the other hand we have,

$$\sum_{j=1}^{\infty} |A_j| = \sum_{j=1}^{\infty} |a_j e^{-t(j\lambda)^2(\alpha+i\beta)+\gamma t}| = \sum_{j=1}^{\infty} |a_j e^{-t(j\lambda)^2\alpha+\gamma t}| < \infty$$

Then we have (3) and $U(t)u_0 \in A_\lambda(\mathbb{R})$ □

Next, we consider integral solutions of the problem (1).

We say that $u \in C([0,T], C_{\mathrm{u}}(\mathbb{R}))$ is a mild solution of (1) if and only if u verifies

$$u(t) = U(t)u_0 + \int_0^t U(t-t')B(u(t'))dt'. \tag{6}$$

If F is a locally Lipschitz map, for any $z_0 \in C_{\mathrm{u}}(\mathbb{R})$ there exists a unique solution of the equation

$$\begin{cases} \partial_t z = B(z), \\ z(0) = z_0, \end{cases} \tag{7}$$

defined in the interval $[0, T^*(z_0))$. Moreover, there exists a function $\bar{T} : [0,\infty) \to [0,\infty)$, which is non-increasing and such that $T^*(z_0) \geq \bar{T}(|z_0|)$. The solution of (7) is solution of the integral equation

$$z(t) = z_0 + \int_0^t B(z(t'))dt'. \tag{8}$$

Also, one of the following alternatives holds:

- $T^*(z_0) = \infty$;
- $T^*(z_0) < \infty$ and $|z(t)| \to \infty$ when $t \uparrow T^*(z_0)$.

We will denote by $\mathsf{N}(t,.) : C_{\mathrm{u}}(\mathbb{R}) \to C_{\mathrm{u}}(\mathbb{R})$ the flow generated by the ordinary equation, i.e.: for any $x \in \mathbb{R}$, $\mathsf{N}(t, u_0)(x)$ is the solution of the problem (7) with initial data $z_0 = u_0(x)$. Therefore, if $u(t) = \mathsf{N}(t, u_0)$

$$u(x,t) = u_0(x) + \int_0^t B(u(x,t'))dt'.$$

We recall well-known local existence results for evolution equations.

Theorem 2.9. *There exists a function $T^* : C_u(\mathbb{R}) \to \mathbb{R}_+$ such that for $u_0 \in C_u(\mathbb{R})$, exists a unique $u \in C([0, T^*(u_0)), C_u(\mathbb{R}))$ mild solution of (1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$;

- *$T^*(u_0) < \infty$ and $\lim_{t\uparrow T^*(u_0)} |u(t)| = \infty$.*

Proof. See Theorem 4.3.4 in [11]. □

Proposition 2.10. *Under conditions of the theorem above, we have the following statements:*

1. *$T^* : C_u(\mathbb{R}) \to \mathbb{R}_+$ is lower semi-continuous;*
2. *If $u_{0,n} \to u_0$ in $C_u(\mathbb{R})$ and $0 < T < T^*(u_0)$, then $u_n \to u$ in the Banach space $C([0,T], C_u(\mathbb{R}))$.*

Proof. See Proposition 4.3.7 in [11]. □

3 Nonlinear equation

In order to apply the Lie-Trotter method, we prove that if the initial state $u_0 \in A_\lambda(\mathbb{R})$ the solution of the nonlinear ordinary equation $z(t) \in A_\lambda(\mathbb{R})$. We consider the equation with a cubic nonlinearity and then we extend the result to a nth-degree nonlinearity.

We study first, the solution for the nonlinear equation (7) with a cubic nonlinearity, that is

$$\begin{cases} \partial_t z = -(a+ib)z^3, \\ z(0) = z_0. \end{cases} \tag{9}$$

Lemma 3.1. *If $u_0(x) = z_0 \in A_\lambda(\mathbb{R})$ then the solution of the equation (9), $z(t) \in A_\lambda(\mathbb{R})$ for $t \in (0, T^*(z_0))$.*

Proof. We prove that $F : A_\lambda(\mathbb{R}) \to A_\lambda(\mathbb{R})$ and that F is a locally Lipschitz map in $A_\lambda(\mathbb{R})$. Let $u \in A_\lambda(\mathbb{R})$ and $k = -(a+ib)$ then we have,

$$\begin{aligned} F(u) = ku^3 = k \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_j a_k a_l e^{ix(j+k+l)\lambda} \\ = k \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} a_j a_{m-j-l} a_l e^{ixm\lambda} = \sum_{m=1}^{\infty} A_m e^{ixm\lambda} \end{aligned}$$

with $m = j+k+l$ and $A_m = (a_m)^3$. Also we have that $\sum_{m=1}^{\infty} |A_m| = \sum_{m=1}^{\infty} |a_j a_k a_l| < \infty$. On the other hand, if $u, v \in A_\lambda(\mathbb{R})$ we can see that,

$$\begin{aligned} \|F(u) - F(v)\|_{A_\lambda} &= \left\|u^3 - v^3\right\|_{A_\lambda} \\ &\leq \frac{1}{2} \left\|(u^2 - v^2)\right\|_{A_\lambda} \|(u+v)\|_{A_\lambda} + \left\|(u^2 + v^2)\right\|_{A_\lambda} \|(u-v)\|_{A_\lambda}. \end{aligned}$$

We use that, $\|u+v\|_{A_\lambda(\mathbb{R})} \leq \|u\|_{A_\lambda(\mathbb{R})} + \|v\|_{A_\lambda(\mathbb{R})} = \sum_{j=1}^{\infty} |a_j| + \sum_{k=1}^{\infty} |a_k| < \infty$, a similar procedure proves that $\|u^2 + v^2\|_{A_\lambda} < \infty$ and $\|u^2 - v^2\|_{A_\lambda} < \infty$. Then,

$$\|F(u) - F(v)\|_{A_\lambda} \leq \frac{1}{2} C \|u - v\|_{A_\lambda}.$$

□

We generalize the ODE with an n-th degree nonlinearity for $n > 2$.

$$\begin{cases} \partial_t z = -(a+ib)z^n, \\ z(0) = z_0. \end{cases} \tag{10}$$

Lemma 3.2. *If $u_0(x) = z_0 \in A_\lambda(\mathbb{R})$ then the solution of the equation (10), $z(t) \in A_\lambda(\mathbb{R})$ for $t \in (0, T^*(z_0))$.*

Proof. The proof is similar to the previous proof, using that,

$$u^n = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} a_{j_1} a_{j_2} \cdots a_{j_n} e^{ix\lambda \sum_{i=1}^{\infty} j_i}$$

and

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2 b^{n-3} + ab^{n-2} + b^{n-1}\right). \qquad \square$$

4 Splitting method

This section is based on the splitting method developed in [12]. We apply the Lie-Trotter method to the linear and nonlinear problem. The temporal variable must be broken down into regular intervals and the evolution of the linear and nonlinear problems are considered alternately. This is described by two sequences $\{V_{h,k}\}$ for the linear equation and $\{W_{h,k}\}$ for the nonlinear equation. Using Theorem 3.9 from [12], this approximate solution converges to the solution of problem (1), when the time intervals $h = t/n \to 0$.

Let X be a Banach space and we define $\alpha : \mathbb{R} \to \mathbb{R}$ a periodic function of period 1 as:

$$\alpha(t) = \begin{cases} 2 & \text{, if } k \le t < k + 1/2, \\ 0 & \text{, if } k - 1/2 \le t < k, \end{cases}$$

for $k \in \mathbb{Z}$.

Given $h > 0$, we define the function $\alpha_h : \mathbb{R} \to \mathbb{R}$ as $\alpha_h(t) = \alpha(t/h)$. Clearly $0 \le \alpha_h \le 2$, α_h is h-periodic and its mean value is 1.

We consider $\tau_h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\tau_h(t, t') = \int_{t'}^{t} \alpha_h(t'') dt''.$$

We define $\Omega = \{(t, t') \in \mathbb{R}^2 : 0 \le t' \le t\}$ and $U_h : \Omega \to \mathcal{B}(X)$ given by $U_h(t, t') = U(\tau_h(t, t'))$.

We consider the system,

$$\begin{cases} \partial_t u_h + \alpha_h(t) \partial_{xx} u_h(x,t) = (2 - \alpha_h(t)) F(u_h(x,t)), \\ u_h(x,0) = u_{h0}(x), \end{cases}$$

where $u(x,t) \in X,\ t > 0$ and $F : X \to X$ is a continuous function.

Similarly, we define the integral equation:

$$u_h(t) = U_h(t,0)u_{h0} + \int_0^t (2 - \alpha_h(t'))U_h(t,t')F(u_h(t'))dt'. \tag{11}$$

The following two theorems are a consequence of sections 2 and 3 of [12].

Theorem 4.1. *Let u_h the solution of (11), if $W_{h,k} = u_h(kh)$ y $V_{h,k} = u_h(kh - h/2)$, then*

$$V_{h,k+1} = U(h)U_{h,k}, \tag{12a}$$

$$W_{h,k+1} = N(kh + h, kh + h/2, V_{h,k+1}), \tag{12b}$$

where N is the flux associated to $2F$, that is:

$$\begin{cases} \dot{w} = 2F(w(t)), \\ w(0) = w_0. \end{cases}$$

Proof. For $t_1 \in (0,t)$ it verifies

$$u_h(t) = U_h(t,t_1)u_{h0}(t_1) + \int_{t_1}^t (2 - \alpha_h(t'))U_h(t,t')F(u_h(t'))dt'$$

using that $t_1 = kh$ y $t = kh + h/2$, we have

$$V_{h,k+1} = U_h(kh + h/2, kh)W_{h,k} + \int_{kh}^{kh+h/2} (2 - \alpha_h(t'))U_h(kh + h/2, t')F(u_h(t'))dt',$$

given that $\alpha_h(t) = 2$ for $t \in [kh, kh+h/2)$, we have $\tau_h(kh+h/2, kh) = h$ and therefore (12a). Similarly, $\alpha_h(t) = 0$ for $t \in [kh + h/2, kh + h)$, then $\tau_h(t, kh + h/2) = 0$ and therefore

$$u_h(t) = V_{h,k+1} + 2\int_{kh+h/2}^t F(u_h(t'))dt',$$

evaluating in $t = kh + h$, we obtain (12b). □

Theorem 4.2. *Let $u \in C([0,T^*), X)$ the solution of the integral problem (6)*

$$u(t) = U(t)u_0 + \int_0^t U(t - t')F(u(t'))dt',$$

$T \in (0,T^*)$ and $\varepsilon > 0$. There exists $h^* > 0$ such that if $0 < h < h^*$, then u_h the solution of (11) with $u_h(x,0) = u_0(x)$, is defined in the interval $[0,T]$ and verifies

$$\|u(t) - u_h(t)\|_X \leq \varepsilon \quad \text{for} \quad t \in [0,T].$$

Proof. See Theorem 3.9 from [12]. □

We now apply Lemma 2.8 from Section 2 related to linear equation and Lemmas 3.1 and 3.2 from Section 3 related to the nonlinear equations. In order to obtain well-posedness results for the solution $u(t)$ of equation (1) in $A_\lambda(\mathbb{R})$, we use Theorem 4.2 to join the linear and nonlinear results. The following theorem is proved for the cubic case but the other cases are similar.

Theorem 4.3. *Let $u_0 \in A_\lambda(\mathbb{R})$, then the solution of* (1) $u(t) \in A_\lambda(\mathbb{R})$ *for* $t \in (0, T^*(u_0))$.

Proof. For $t \in [0, T^*(u_0))$, let $n \in \mathbb{N}$, $h = t/n$ and $\{W_{h,k}\}_{0\le k\le n}, \{V_{h,k}\}_{1\le k\le n}$ be the sequences given by $W_{h,0} = u_0$,

$$V_{h,k+1} = U(h)W_{h,k}, \tag{13a}$$

$$W_{h,k+1} = \mathsf{N}(h, V_{h,k+1}), \quad k = 0, \ldots, n-1. \tag{13b}$$

We claim that $W_{h,k+1} \in A_\lambda(\mathbb{R})$ for $k = 0, \ldots, n$. Clearly, the assertion is true for $k = 0$. If $W_{h,k} \in A_\lambda(\mathbb{R})$, from Lemma 2.8, we have $U(h)W_{h,k} \in A_\lambda(\mathbb{R})$. Using Lemma 3.1, we can see that

$$W_{h,k+1} = \mathsf{N}(h, V_{h,k}) \in A_\lambda(\mathbb{R}).$$

By theorem 4.2 we have that $W_{h,n} \to u(t)$ when $n \to \infty$.

As $A_\lambda(\mathbb{R})$ is a Banach space, we obtain the result. □

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