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On the p-biharmonic submanifolds and stress p-bienergy tensors

Khadidja Mouffoki and Ahmed Mohammed Cherif

Abstract. In this paper, we consider p-biharmonic submanifolds of a space form. We give the necessary and sufficient conditions for a submanifold to be p-biharmonic in a space form. We present some new properties for the stress p-bienergy tensor.

1 Introduction

Consider a smooth map $\varphi:(M,g)\longrightarrow (N,h)$ between Riemannian manifolds, and let $p\geq 2$, for any compact domain D of M the p-energy functional of φ is defined by

$$E_p(\varphi; D) = \frac{1}{p} \int_D |d\varphi|^p v_g, \tag{1}$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$, and v^g is the volume element on (M,g). A map is called *p*-harmonic if it is a critical point of the *p*-energy functional over any compact subset D of M. Let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then

$$\frac{d}{dt}E_p(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_p(\varphi), v)v_g, \tag{2}$$

where $v = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ ,

$$\tau_p(\varphi) = \operatorname{div}^M(|d\varphi|^{p-2}d\varphi). \tag{3}$$

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Let $\tau(\varphi)$ be the tension field of φ defined by

$$\tau(\varphi) = \operatorname{trace}_{g} \nabla d\varphi = \sum_{i=1}^{m} \left\{ \nabla_{e_{i}}^{\varphi} d\varphi(e_{i}) - d\varphi(\nabla_{e_{i}}^{M} e_{i}) \right\}. \tag{4}$$

(see [2]), where $\{e_1, \ldots, e_m\}$ is an orthonormal frame on (M, g), $m = \dim M$, ∇^M is the Levi-Civita connection of (M, g), and ∇^{φ} denotes the pull-back connection on $\varphi^{-1}TN$. If $|d\varphi|_x \neq 0$ for all $x \in M$, the map φ is p-harmonic if and only if (see [1], [3], [7])

$$|d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\operatorname{grad}^{M}|d\varphi|) = 0.$$
(5)

Example 1.1. Let $n \geq 2$. The inversion map

$$\varphi: \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\},$$

$$x \longmapsto \frac{x}{|x|^l}$$

is p-harmonic if and only if $l = \frac{n+p-2}{p-1}$.

A natural generalization of p-harmonic maps is given by integrating the square of the norm of $\tau_p(\varphi)$. More precisely, the p-bienergy functional of φ is defined by

$$E_{2,p}(\varphi;D) = \frac{1}{2} \int_{D} |\tau_p(\varphi)|^2 v_g. \tag{6}$$

We say that φ is a *p*-biharmonic map if it is a critical point of the *p*-bienergy functional, that is to say, if it satisfies the Euler-Lagrange equation of the functional (6), that is (see [11])

$$\tau_{2,p}(\varphi) = -|d\varphi|^{p-2}\operatorname{trace}_{g} R^{N}(\tau_{p}(\varphi), d\varphi)d\varphi - \operatorname{trace}_{g} \nabla^{\varphi}|d\varphi|^{p-2}\nabla^{\varphi}\tau_{p}(\varphi) - (p-2)\operatorname{trace}_{g} \nabla\langle\nabla^{\varphi}\tau_{p}(\varphi), d\varphi\rangle|d\varphi|^{p-4}d\varphi = 0.$$
 (7)

Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame on (M, g), we have

$$\operatorname{trace}_{g} R^{N}(\tau_{p}(\varphi), d\varphi) d\varphi = \sum_{i=1}^{m} R^{N}(\tau_{p}(\varphi), d\varphi(e_{i})) d\varphi(e_{i}),$$

$$\operatorname{trace}_{g} \nabla^{\varphi} |d\varphi|^{p-2} \nabla^{\varphi} \tau_{p}(\varphi) = \sum_{i=1}^{m} \left(\nabla^{\varphi}_{e_{i}} |d\varphi|^{p-2} \nabla^{\varphi}_{e_{i}} \tau_{p}(\varphi) - |d\varphi|^{p-2} \nabla^{\varphi}_{\nabla^{M}_{e_{i}} e_{i}} \tau_{p}(\varphi) \right),$$

$$\langle \nabla^{\varphi} \tau_{p}(\varphi), d\varphi \rangle = \sum_{i=1}^{m} h \left(\nabla^{\varphi}_{e_{i}} \tau_{p}(\varphi), d\varphi(e_{i}) \right),$$

$$\operatorname{trace}_{g} \nabla \langle \nabla^{\varphi} \tau_{p}(\varphi), d\varphi \rangle |d\varphi|^{p-4} d\varphi = \sum_{i=1}^{m} \left(\nabla^{\varphi}_{e_{i}} \langle \nabla^{\varphi} \tau_{p}(\varphi), d\varphi \rangle |d\varphi|^{p-4} d\varphi(e_{i}) - \langle \nabla^{\varphi} \tau_{p}(\varphi), d\varphi \rangle |d\varphi|^{p-4} d\varphi(\nabla^{M}_{e_{i}} e_{i}) \right).$$

The p-energy functional (resp. p-bienergy functional) includes as a special case (p = 2) the energy functional (resp. bi-energy functional), whose critical points are the usual harmonic maps (resp. bi-harmonic maps), for more details on the concept of harmonic and bi-harmonic maps see [6], [9].

p-harmonic maps are always p-biharmonic maps by definition. In particular, if (M,g) is a compact orientable Riemannian manifold without boundary, and (N,h) is a Riemannian manifold with non-positive sectional curvature. Then, every p-biharmonic map from (M,g) to (N,h) is p-harmonic.

Example 1.2 ([11]). Let M the manifold $\mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}$ equipped with the Riemannian metric $g = (x_1^2 + x_2^2)^{-\frac{1}{p}} (dx_1^2 + dx_2^2 + dx_3^2)$, and let N the manifold \mathbb{R}^2 equipped with the Riemannian metric $h = dy_1^2 + dy_2^2$. The map

$$\varphi:(M,g)\to(N,h)$$
 defined by $\varphi(x_1,x_2,x_3)=\left(\sqrt{x_1^2+x_2^2},\ x_3\right)$

is proper *p*-biharmonic.

A submanifold in a Riemannian manifold is called a p-biharmonic submanifold if the isometric immersion defining the submanifold is a p-biharmonic map. We will call proper p-biharmonic submanifolds a p-biharmonic submanifols which is non p-harmonic.

In this paper, we will focus our attention on p-biharmonic submanifolds of space form, we give the necessary and sufficient conditions for submanifolds to be p-biharmonic. Then, we apply this general result to many particular cases. We also consider the stress p-bienergy tensor associated to the p-bienergy functional, and we give the relation between the divergence of the stress p-bienergy tensor and the p-bitension field (7). Finally, we classify maps between Riemannian manifolds with vanishing stress p-bienergy tensor.

2 Main Results

Let M be a submanifold of space form N(c) of dimension m, $\mathbf{i}: M \hookrightarrow N(c)$ be the canonical inclusion, and let $\{e_1, \ldots, e_m\}$ be an orthonormal frame with respect to induced Riemannian metric g on M by the inner product \langle,\rangle on N(c). We denote by ∇^N (resp. ∇^M) the Levi-Civita connection of $N^n(c)$ (resp. of (M,g)), by grad the gradient operator in (M,g), by B the second fundamental form of the submanifold (M,g), by A the shape operator, by B the mean curvature vector field of (M,g), and by ∇^{\perp} the normal connection of (M,g) (see for example [2]). Under the notation above we have the following results.

Theorem 2.1. The canonical inclusion i is p-biharmonic if and only if

$$\begin{cases}
-\Delta^{\perp} H + \operatorname{trace}_{g} B(\cdot, A_{H}(\cdot)) - m \left(c - (p-2)|H|^{2}\right) H = 0; \\
2\operatorname{trace}_{g} A_{\nabla^{\perp} H}(\cdot) + \left(p - 2 + \frac{m}{2}\right) \operatorname{grad}^{M} |H|^{2} = 0,
\end{cases} (8)$$

where Δ^{\perp} is the Laplacian in the normal bundle of (M,g).

Proof. First, the p-tension field of \mathbf{i} is given by

$$\tau_p(\mathbf{i}) = |d\mathbf{i}|^{p-2}\tau(\mathbf{i}) + (p-2)|d\mathbf{i}|^{p-3}d\mathbf{i}(\operatorname{grad}^M|d\mathbf{i}|),$$

since $\tau(\mathbf{i}) = mH$ (see [1], [2]), and $|d\mathbf{i}|^2 = m$, we get $\tau_p(\mathbf{i}) = m^{\frac{p}{2}}H$. Let $\{e_i, \dots, e_m\}$ be an orthonormal frame such that $\nabla_{e_i}^M e_j = 0$ at $x \in M$ for all $i, j = 1, \dots, m$, then calculating at x

$$\operatorname{trace}_{g} R^{N}(\tau_{p}(\mathbf{i}), d\mathbf{i}) d\mathbf{i} = \sum_{i=1}^{m} R^{N}(\tau_{p}(\mathbf{i}), d\mathbf{i}(e_{i})) d\mathbf{i}(e_{i})$$
$$= m^{\frac{p}{2}} \sum_{i=1}^{m} R^{N}(H, e_{i}) e_{i}.$$

By the following equation $R^N(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$, with $\langle H,e_i\rangle = 0$, for all $X,Y,Z\in\Gamma(TN(c))$ and $i=1,\ldots,m$, the last equation becomes

$$\operatorname{trace}_{g} R^{N}(\tau_{p}(\mathbf{i}), d\mathbf{i}) d\mathbf{i} = m^{\frac{p+2}{2}} cH.$$
(9)

We compute the term $\operatorname{trace}_q(\nabla^{\mathbf{i}})^2 \tau_p(\mathbf{i})$ at x

$$\sum_{i=1}^{m} \nabla_{e_{i}}^{\mathbf{i}} \nabla_{e_{i}}^{\mathbf{i}} H = \sum_{i=1}^{m} \nabla_{e_{i}}^{\mathbf{i}} \left(-A_{H}(e_{i}) + (\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp} \right)
= -\sum_{i=1}^{m} \nabla_{e_{i}}^{M} A_{H}(e_{i}) - \sum_{i=1}^{m} B(e_{i}, A_{H}(e_{i}))
- \sum_{i=1}^{m} A_{(\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp}}(e_{i}) + \sum_{i=1}^{m} \left(\nabla_{e_{i}}^{\mathbf{i}} (\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp} \right)^{\perp},$$
(10)

since $\langle A_H(X), Y \rangle = \langle B(X,Y), H \rangle$ for all $X, Y \in \Gamma(TM)$, we get

$$\sum_{i=1}^{m} \nabla_{e_i}^{M} A_H(e_i) = \sum_{i,j=1}^{m} \left\langle \nabla_{e_i}^{M} A_H(e_i), e_j \right\rangle e_j$$

$$= \sum_{i,j=1}^{m} e_i (\left\langle A_H(e_i), e_j \right\rangle) e_j$$

$$= \sum_{i,j=1}^{m} e_i (\left\langle B(e_i, e_j), H \right\rangle) e_j$$

$$= \sum_{i,j=1}^{m} e_i (\left\langle \nabla_{e_j}^{N} e_i, H \right\rangle) e_j,$$

since $\nabla_X^N \nabla_Y^N Z = R^N(X,Y)Z + \nabla_Y^N \nabla_X^N Z + \nabla_{[X,Y]}^N Z$, for all $X,Y,Z \in \Gamma(TN(c))$, we conclude

$$\sum_{i=1}^{m} \nabla_{e_{i}}^{M} A_{H}(e_{i}) = \sum_{i,j=1}^{m} \left\langle \nabla_{e_{i}}^{N} \nabla_{e_{j}}^{N} e_{i}, H \right\rangle e_{j} + \sum_{i,j=1}^{m} \left\langle \nabla_{e_{j}}^{N} e_{i}, \nabla_{e_{i}}^{\mathbf{i}} H \right\rangle e_{j}$$

$$= \sum_{i,j=1}^{m} \left\langle R^{N}(e_{i}, e_{j}) e_{i}, H \right\rangle e_{j} + \sum_{i,j=1}^{m} \left\langle \nabla_{e_{j}}^{N} \nabla_{e_{i}}^{N} e_{i}, H \right\rangle e_{j}$$

$$+ \sum_{i,j=1}^{m} \left\langle B(e_{i}, e_{j}), (\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp} \right\rangle e_{j},$$

since $R^N(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$, for all $X,Y,Z\in\Gamma(TN(c))$, we have

$$\sum_{i=1}^{m} \nabla_{e_{i}}^{M} A_{H}(e_{i}) = \sum_{i,j=1}^{m} e_{j} \left(\left\langle \nabla_{e_{i}}^{N} e_{i}, H \right\rangle \right) e_{j} - \sum_{i,j=1}^{m} \left\langle \nabla_{e_{i}}^{N} e_{i}, \nabla_{e_{j}}^{\mathbf{i}} H \right\rangle e_{j}
+ \sum_{i,j=1}^{m} \left\langle A_{(\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp}}(e_{i}), e_{j} \right\rangle e_{j}
= m \sum_{j=1}^{m} e_{j} \left(\left\langle H, H \right\rangle \right) e_{j} - m \sum_{j=1}^{m} \left\langle H, \nabla_{e_{j}}^{\mathbf{i}} H \right\rangle e_{j}
+ \sum_{i=1}^{m} A_{(\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp}}(e_{i})
= \frac{m}{2} \sum_{j=1}^{m} e_{j} \left(\left\langle H, H \right\rangle \right) e_{j} + \sum_{i=1}^{m} A_{(\nabla_{e_{i}}^{\mathbf{i}} H)^{\perp}}(e_{i}). \tag{11}$$

From equations (10) and (11), we obtain

$$\operatorname{trace}_{g}(\nabla^{\mathbf{i}})^{2} \tau_{p}(\mathbf{i}) = -\frac{m^{\frac{p+2}{2}}}{2} \operatorname{grad}^{M} |H|^{2} - 2m^{\frac{p}{2}} \operatorname{trace}_{g} A_{(\nabla^{\perp} H)}(\cdot) - m^{\frac{p}{2}} \operatorname{trace}_{g} B(\cdot, A_{H}(\cdot)) + m^{\frac{p}{2}} \Delta^{\perp} H.$$
(12)

Now, we compute the term ${\rm trace}_g \, \nabla \langle \nabla^{\bf i} \tau_p({\bf i}), d{\bf i} \rangle d{\bf i}$ at x

$$\sum_{i,j=1}^{m} \nabla_{e_i}^{\mathbf{i}} \langle \nabla_{e_j}^{\mathbf{i}} \tau_p(\mathbf{i}), d\mathbf{i}(e_j) \rangle d\mathbf{i}(e_i) = m^{\frac{p}{2}} \sum_{i,j=1}^{m} \nabla_{e_i}^{\mathbf{i}} \langle \nabla_{e_j}^{\mathbf{i}} H, e_j \rangle e_i,$$

by the compatibility of pull-back connection $\nabla^{\mathbf{i}}$ with the Riemannian metric of N(c), and

the definition of the mean curvature vector field H of (M, g), we have

$$\sum_{j=1}^{m} \langle \nabla_{e_j}^{\mathbf{i}} H, e_j \rangle = \sum_{j=1}^{m} \left\{ e_j \langle H, e_j \rangle - \langle H, \nabla_{e_j}^{\mathbf{i}} e_j \rangle \right\}$$

$$= -\sum_{j=1}^{m} \langle H, B(e_j, e_j) \rangle$$

$$= -m|H|^2,$$

by the last two equations, we have the following

$$\operatorname{trace}_{q} \nabla \langle \nabla^{\mathbf{i}} \tau_{p}(\mathbf{i}), d\mathbf{i} \rangle d\mathbf{i} = -m^{\frac{p+2}{2}} \operatorname{grad}^{M} |H|^{2} - m^{\frac{p+4}{2}} |H|^{2} H. \tag{13}$$

The Theorem 2.1 followed by (7), (9), (12), and (13).

If p=2 and $N=\mathbb{S}^n$, we arrive at the following Corollary.

Corollary 2.2. Let M be a submanifold of sphere \mathbb{S}^n of dimension m, then the canonical inclusion $\mathbf{i}: M \hookrightarrow \mathbb{S}^n$ is biharmonic if and only if

$$\begin{cases} \frac{m}{2} \operatorname{grad}^{M} |H|^{2} + 2 \operatorname{trace}_{g} A_{(\nabla^{\perp} H)}(\cdot) = 0, \\ -m H + \operatorname{trace}_{g} B(\cdot, A_{H}(\cdot)) - \Delta^{\perp} H = 0. \end{cases}$$

This result was deduced by B-Y. Chen and C. Oniciuc [4], [12].

Theorem 2.3. If M is a hypersurface with nowhere zero mean curvature of $N^{m+1}(c)$, then M is p-biharmonic if only if

$$\begin{cases}
-\Delta^{\perp} H + (|A|^2 + m(p-2)|H|^2 - mc)H = 0; \\
2A(\operatorname{grad}^M |H|) + (2(p-2) + m)|H|\operatorname{grad}^M |H| = 0.
\end{cases} (14)$$

Proof. Consider $\{e_1, \ldots, e_m\}$ to be a local orthonormal frame field on (M, g), and let η the unit normal vector field at (M, g) in $N^{m+1}(c)$. We have

$$H = \langle H, \eta \rangle \eta$$

$$= \frac{1}{m} \sum_{i=1}^{m} \langle B(e_i, e_i), \eta \rangle \eta$$

$$= \frac{1}{m} \sum_{i=1}^{m} g(A(e_i), e_i) \eta$$

$$= \frac{1}{m} (\operatorname{trace}_g A) \eta.$$

Let $i = 1, \ldots, m$, we compute

$$A_{H}(e_{i}) = \sum_{j=1}^{m} g(A_{H}(e_{i}), e_{j})e_{j}$$

$$= -\sum_{j=1}^{m} \langle \nabla_{e_{i}}^{N} H, e_{j} \rangle e_{j}$$

$$= -\sum_{j=1}^{m} e_{i} \langle H, e_{j} \rangle e_{j} + \sum_{j=1}^{m} \langle H, B(e_{i}, e_{j}) \rangle e_{j}$$

$$= \langle H, \eta \rangle \sum_{j=1}^{m} \langle \eta, B(e_{i}, e_{j}) \rangle e_{j},$$

by the last equation and the formula $\langle \eta, B(e_i, e_j) \rangle = g(Ae_i, e_j)$, we obtain the following equation $A_H(e_i) = \langle H, \eta \rangle A(e_i)$. So that

$$\sum_{i=1}^{m} B(e_i, A_H(e_i)) = \sum_{i=1}^{m} B(e_i, \langle H, \eta \rangle A(e_i))$$

$$= \langle H, \eta \rangle \sum_{i=1}^{m} B(e_i, A(e_i))$$

$$= \langle H, \eta \rangle \sum_{i=1}^{m} g(A(e_i), A(e_i)) \eta$$

$$= |A|^2 H. \tag{15}$$

In the same way, with $\eta = H/|H|$, we find that

$$\sum_{i=1}^{m} A_{\nabla_{e_i}^{\perp} H}(e_i) = \sum_{i,j=1}^{m} \langle A_{\nabla_{e_i}^{\perp} H}(e_i), e_j \rangle e_j$$

$$= -\sum_{i,j=1}^{m} \langle \nabla_{e_i}^{N} \nabla_{e_i}^{\perp} H, e_j \rangle e_j$$

$$= -\sum_{i,j=1}^{m} \langle e_i \langle H, \eta \rangle \nabla_{e_i}^{N} \eta, e_j \rangle e_j$$

$$= A(\operatorname{grad}^{M} |H|). \tag{16}$$

The Theorem 2.3 followed by equations (15), (16), and Theorem 2.1. \Box

Corollary 2.4. (i) A submanifold M with parallel mean curvature vector field in $N^n(c)$ is p-biharmonic if and only if

$$\operatorname{trace}_{g} B(\cdot, A_{H}(\cdot)) = m(c - (p - 2)|H|^{2})H, \tag{17}$$

(ii) A hypersurface M of constant non-zero mean curvature in $N^{m+1}(c)$ is proper p-biharmonic if and only if

$$|A|^2 = mc - m(p-2)|H|^2. (18)$$

Example 2.5. We consider the hypersurface

$$\mathbb{S}^{m}(a) = \{(x^{1}, \dots, x^{m}, x^{m+1}, b) \in \mathbb{R}^{m+2} : \sum_{i=1}^{m+1} (x^{i})^{2} = a^{2}\} \subset \mathbb{S}^{m+1},$$

where $a^2 + b^2 = 1$. We have

$$\eta = \frac{1}{r}(x^1, \cdots, x^{m+1}, -\frac{a^2}{b}),$$

with $r^2 = \frac{a^2}{b^2}$ (r > 0), is a unit section in the normal bundle of $\mathbb{S}^m(a)$ in \mathbb{S}^{m+1} . Let $X \in \Gamma(T\mathbb{S}^m(a))$, we compute

$$\nabla_X^{\mathbb{S}^{m+1}} \eta = \frac{1}{r} \nabla_X^{\mathbb{R}^{m+2}} (x^1, \cdots, x^{m+1}, -\frac{a^2}{b}) = \frac{1}{r} X.$$

Thus, $\nabla^{\perp}\eta=0$ and $A=-\frac{1}{r}Id$. This implies that $H=-\frac{1}{r}\eta$, and so $\mathbb{S}^m(a)$ has constant mean curvature $|H|=\frac{1}{r}$ in \mathbb{S}^{m+1} . Since $|A|^2=\frac{m}{r^2}$, according to Corollary 2.4. we conclude that $\mathbb{S}^m(a)$ is proper p-biharmonic in \mathbb{S}^{m+1} if and only if $p=1/b^2$.

3 Stress *p*-bienergy tensors

Let $\varphi:(M,g)\to (N,h)$ be a smooth map between two Riemannian manifolds and $p\geq 2$. Consider a smooth one-parameter variation of the metric g, i.e. a smooth family of metrics (g_t) $(-\epsilon < t < \epsilon)$ such that $g_0=g$, write $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, then $\delta g \in \Gamma(\odot^2 T^*M)$ is a symmetric 2-covariant tensor field on M (see [2]). Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t=g_{ij}(t,x)\,dx^i\,dx^j$, we now compute

$$\frac{d}{dt}E_{2,p}(\varphi;D)\Big|_{t=0} = \frac{1}{2} \int_{D} \delta(|\tau_p(\varphi)|^2) v_g + \frac{1}{2} \int_{D} |\tau_p(\varphi)|^2 \delta(v_{g_t}). \tag{19}$$

The calculation of the first term breaks down in three lemmas.

Lemma 3.1. The vector field $\xi = (\operatorname{div}^M \delta g)^{\sharp} - \frac{1}{2}\operatorname{grad}^M(\operatorname{trace} \delta g)$ satisfies

$$\delta(|\tau_{p}(\varphi)|^{2}) = -(p-2)|d\varphi|^{p-4}\langle \varphi^{*}h, \delta g \rangle h(\tau(\varphi), \tau_{p}(\varphi))$$

$$-2|d\varphi|^{p-2}\langle h(\nabla d\varphi, \tau_{p}(\varphi)), \delta g \rangle - 2|d\varphi|^{p-2}h(d\varphi(\xi), \tau_{p}(\varphi))$$

$$-(p-2)(p-4)|d\varphi|^{p-5}\langle \varphi^{*}h, \delta g \rangle h(d\varphi(\operatorname{grad}^{M}|d\varphi|), \tau_{p}(\varphi))$$

$$-2(p-2)|d\varphi|^{p-3}\langle d|d\varphi| \odot h(d\varphi, \tau_{p}(\varphi)), \delta g \rangle$$

$$-(p-2)|d\varphi|^{p-4}h(d\varphi(\operatorname{grad}^{M}\langle \varphi^{*}h, \delta g \rangle), \tau_{p}(\varphi)).$$

where φ^*h is the pull-back of the metric h, and \langle , \rangle is the induced Riemannian metric on $\otimes^2 T^*M$.

Proof. In local coordinates (x^i) on M and (y^{α}) on N, we have

$$\delta(|\tau_p(\varphi)|^2) = \delta(\tau_p(\varphi)^\alpha \tau_p(\varphi)^\beta h_{\alpha\beta}) = 2\delta(\tau_p(\varphi)^\alpha) \tau_p(\varphi)^\beta h_{\alpha\beta}. \tag{20}$$

By the definition of $\tau_p(\varphi)$ we get

$$\delta(\tau_p(\varphi)^{\alpha}) = \delta(|d\varphi|^{p-2}\tau(\varphi)^{\alpha} + \theta^{\alpha})
= \delta(|d\varphi|^{p-2})\tau(\varphi)^{\alpha} + |d\varphi|^{p-2}\delta(\tau(\varphi)^{\alpha}) + \delta(\theta^{\alpha}).$$
(21)

where $\tau(\varphi)^{\alpha} = g^{ij} \left(\varphi_{i,j}^{\alpha} + {}^{N} \Gamma_{\mu\sigma}^{\alpha} \varphi_{i}^{\mu} \varphi_{j}^{\sigma} - {}^{M} \Gamma_{ij}^{k} \varphi_{k}^{\alpha} \right)$ is the component of the tension field $\tau(\varphi)$, and $\theta^{\alpha} = (p-2) |d\varphi|^{p-3} g^{ij} |d\varphi|_{i} \varphi_{j}^{\alpha}$.

The first term in the right-hand side of (21) is given by

$$\delta(|d\varphi|^{p-2}) \tau(\varphi)^{\alpha} = (p-2)|d\varphi|^{p-4} \delta(\frac{|d\varphi|^2}{2}) \tau(\varphi)^{\alpha}$$
$$= -\frac{p-2}{2} |d\varphi|^{p-4} \langle \varphi^* h, \delta g \rangle \tau(\varphi)^{\alpha}. \tag{22}$$

The second term on the right-hand side of (21) is (see [10])

$$|d\varphi|^{p-2}\delta(\tau(\varphi)^{\alpha}) = -|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^{\alpha}_{ij} - |d\varphi|^{p-2}\xi^{k}\varphi^{\alpha}_{k}, \tag{23}$$

Now, we compute the third term on the right-hand side of (21)

$$\delta(\theta^{\alpha}) = (p-2)(p-3)|d\varphi|^{p-5}\delta(\frac{|d\varphi|^2}{2})g^{ij}|d\varphi|_i\varphi_j^{\alpha} + (p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|_i\varphi_j^{\alpha} + (p-2)|d\varphi|^{p-3}g^{ij}\delta(|d\varphi|_i)\varphi_j^{\alpha}.$$
(24)

By using $\delta(\frac{|d\varphi|^2}{2}) = -\frac{1}{2}\langle \varphi^* h, \delta g \rangle$ with $\delta(|d\varphi|_i) = (\delta(|d\varphi|))_i$, the equation (24) becomes

$$\delta(\theta^{\alpha}) = -\frac{(p-2)(p-3)}{2} |d\varphi|^{p-5} \langle \varphi^* h, \delta g \rangle g^{ij} |d\varphi|_i \varphi_j^{\alpha}$$

$$+ (p-2) |d\varphi|^{p-3} \delta(g^{ij}) |d\varphi|_i \varphi_j^{\alpha}$$

$$- \frac{p-2}{2} |d\varphi|^{p-4} g^{ij} \langle \varphi^* h, \delta g \rangle_i \varphi_j^{\alpha}$$

$$+ \frac{p-2}{2} |d\varphi|^{p-5} g^{ij} |d\varphi|_i \langle \varphi^* h, \delta g \rangle \varphi_j^{\alpha}.$$

$$(25)$$

Note that

$$2\delta(|d\varphi|^{p-2})\tau(\varphi)^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta} = -(p-2)|d\varphi|^{p-4}\langle\varphi^{*}h,\delta g\rangle\tau(\varphi)^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$
$$= -(p-2)|d\varphi|^{p-4}\langle\varphi^{*}h,\delta g\rangle h(\tau(\varphi),\tau_{p}(\varphi)),$$
(26)

$$2|d\varphi|^{p-2}\delta(\tau(\varphi)^{\alpha})\tau_{p}(\varphi)^{\beta}h_{\alpha\beta} = -2|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^{\alpha}_{ij}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$
$$-2|d\varphi|^{p-2}\xi^{k}\varphi_{k}^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$
$$= -2|d\varphi|^{p-2}\langle h(\nabla d\varphi,\tau_{p}(\varphi)),\delta g\rangle$$
$$-2|d\varphi|^{p-2}h(d\varphi(\xi),\tau_{p}(\varphi)), \tag{27}$$

and the following

$$2\delta(\theta^{\alpha})\tau_{p}(\varphi)^{\beta}h_{\alpha\beta} = -(p-2)(p-3)|d\varphi|^{p-5}\langle\varphi^{*}h,\delta g\rangle g^{ij}|d\varphi|_{i}\varphi_{j}^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$

$$+2(p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|_{i}\varphi_{j}^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$

$$-(p-2)|d\varphi|^{p-4}g^{ij}\langle\varphi^{*}h,\delta g\rangle_{i}\varphi_{j}^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$

$$+(p-2)|d\varphi|^{p-5}g^{ij}|d\varphi|_{i}\langle\varphi^{*}h,\delta g\rangle\varphi_{j}^{\alpha}\tau_{p}(\varphi)^{\beta}h_{\alpha\beta}$$

$$= -(p-2)(p-3)|d\varphi|^{p-5}\langle\varphi^{*}h,\delta g\rangle h(d\varphi(\operatorname{grad}^{M}|d\varphi|),\tau_{p}(\varphi))$$

$$-2(p-2)|d\varphi|^{p-3}\langle d|d\varphi| \odot h(d\varphi,\tau_{p}(\varphi)),\delta g\rangle$$

$$-(p-2)|d\varphi|^{p-4}h(d\varphi(\operatorname{grad}^{M}\langle\varphi^{*}h,\delta g\rangle),\tau_{p}(\varphi))$$

$$+(p-2)|d\varphi|^{p-5}\langle\varphi^{*}h,\delta g\rangle h(d\varphi(\operatorname{grad}^{M}|d\varphi|),\tau_{p}(\varphi)). \tag{28}$$

Substituting (21), (26), (27) and (28) in (20), the Lemma 3.1 follows.

Lemma 3.2 ([5]). Let D be a compact domain of M. Then

$$\int_{D} |d\varphi|^{p-2} h(d\varphi(\xi), \tau_{p}(\varphi)) v_{g} = \int_{D} \left\langle -\operatorname{sym}\left(\nabla |d\varphi|^{p-2} h(d\varphi, \tau_{p}(\varphi))\right) + \frac{1}{2} \operatorname{div}^{M}\left(|d\varphi|^{p-2} h(d\varphi, \tau_{p}(\varphi))^{\sharp}\right) g, \delta g \right\rangle v_{g}.$$

Lemma 3.3. We set $\omega = |d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi))$. Then

$$-\int_{D} |d\varphi|^{p-4} h(d\varphi(\operatorname{grad}^{M}\langle \varphi^{*}h, \delta g \rangle), \tau_{p}(\varphi)) v_{g} = \int_{D} \langle \varphi^{*}h, \delta g \rangle \operatorname{div} \omega v_{g}.$$

Proof. Note that

$$\operatorname{div}(\langle \varphi^* h, \delta g \rangle \omega) = \langle \varphi^* h, \delta g \rangle \operatorname{div} \omega + \omega(\operatorname{grad}^M \langle \varphi^* h, \delta g \rangle),$$

and consider the divergence Theorem, Lemma 3.3 follows.

Theorem 3.4. Let $\varphi:(M,g)\to (N,h)$ be a smooth map such that $|d\varphi|_x\neq 0$ for all $x\in M$, and let $\{g_t\}$ a one parameter variation of g. Then

$$\frac{d}{dt}E_{2,p}(\varphi;D)\Big|_{t=0} = \frac{1}{2} \int_{D} \langle S_{2,p}(\varphi), \delta g \rangle v_g,$$

where $S_{2,p}(\varphi) \in \Gamma(\odot^2 T^*M)$ is given by

$$S_{2,p}(\varphi)(X,Y) = -\frac{1}{2} |\tau_p(\varphi)|^2 g(X,Y) - |d\varphi|^{p-2} \langle d\varphi, \nabla^{\varphi} \tau_p(\varphi) \rangle g(X,Y)$$

$$+ |d\varphi|^{p-2} h(d\varphi(X), \nabla_Y^{\varphi} \tau_p(\varphi)) + |d\varphi|^{p-2} h(d\varphi(Y), \nabla_X^{\varphi} \tau_p(\varphi))$$

$$+ (p-2) |d\varphi|^{p-4} \langle d\varphi, \nabla^{\varphi} \tau_p(\varphi) \rangle h(d\varphi(X), d\varphi(Y)).$$

 $S_{2,p}(\varphi)$ is called the stress p-bienergy tensor of φ .

Proof. By using $\delta(v_{g_t}) = \frac{1}{2} \langle g, \delta g \rangle v_g$ (see [2]), Lemmas 3.1, 3.2, and 3.3, the equation (19) becomes

$$S_{2,f}(\varphi) = -(p-2)|d\varphi|^{p-4}h(\tau(\varphi), \tau_p(\varphi))\varphi^*h$$

$$-2|d\varphi|^{p-2}h(\nabla d\varphi, \tau_p(\varphi)) + 2\operatorname{sym}\left(\nabla|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))\right)$$

$$-\operatorname{div}^M\left(|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))^{\sharp}\right)g$$

$$-(p-2)(p-4)|d\varphi|^{p-5}h(d\varphi(\operatorname{grad}^M|d\varphi|), \tau_p(\varphi))\varphi^*h$$

$$-2(p-2)|d\varphi|^{p-3}d|d\varphi| \odot h(d\varphi, \tau_p(\varphi))$$

$$+(p-2)\operatorname{div}^M\left[|d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi))\right]\varphi^*h + \frac{1}{2}|\tau_p(\varphi)|^2g. \tag{29}$$

Note that, for all $X, Y \in \Gamma(TM)$, we have

$$2\operatorname{sym}\left(\nabla|d\varphi|^{p-2}h(d\varphi,\tau_{p}(\varphi))\right)(X,Y) = 2|d\varphi|^{p-2}h(\nabla d\varphi(X,Y),\tau_{p}(\varphi)) + |d\varphi|^{p-2}h(d\varphi(X),\nabla_{Y}^{\varphi}\tau_{p}(\varphi)) + |d\varphi|^{p-2}h(d\varphi(Y),\nabla_{X}^{\varphi}\tau_{p}(\varphi)) + X(|d\varphi|^{p-2})h(d\varphi(Y),\tau_{p}(\varphi)) + Y(|d\varphi|^{p-2})h(d\varphi(X),\tau_{p}(\varphi)),$$

$$(30)$$

and the following formula

$$-2d|d\varphi| \odot h(d\varphi, \tau_p(\varphi))(X, Y) = -X(|d\varphi|)h(d\varphi(Y), \tau_p(\varphi)) -Y(|d\varphi|)h(d\varphi(X), \tau_p(\varphi)).$$
 (31)

Calculating in a normal frame at x, we have

$$\operatorname{div}^{M}\left(|d\varphi|^{p-2}h(d\varphi,\tau_{p}(\varphi))^{\sharp}\right) = \sum_{i=1}^{m} e_{i}(g(|d\varphi|^{p-2}h(d\varphi,\tau_{p}(\varphi))^{\sharp},e_{i}))$$

$$= \sum_{i=1}^{m} e_{i}(|d\varphi|^{p-2}h(d\varphi(e_{i}),\tau_{p}(\varphi)))$$

$$= \sum_{i=1}^{m} e_{i}(|d\varphi|^{p-2})h(d\varphi(e_{i}),\tau_{p}(\varphi))$$

$$+ \sum_{i=1}^{m} |d\varphi|^{p-2}h(\nabla_{e_{i}}^{\varphi}d\varphi(e_{i}),\tau_{p}(\varphi))$$

$$+ \sum_{i=1}^{m} |d\varphi|^{p-2}h(d\varphi(e_{i}),\nabla_{e_{i}}^{\varphi}\tau_{p}(\varphi))$$

$$= (p-2)|d\varphi|^{p-3}h(d\varphi(\operatorname{grad}^{M}|d\varphi|),\tau_{p}(\varphi))$$

$$+|d\varphi|^{p-2}h(\tau(\varphi),\tau_{p}(\varphi))$$

$$+|d\varphi|^{p-2}\langle d\varphi,\nabla^{\varphi}\tau_{p}(\varphi)\rangle. \tag{32}$$

From the definition of $\tau_p(\varphi)$, and equation (32), we get

$$\operatorname{div}^{M}\left(|d\varphi|^{p-2}h(d\varphi,\tau_{p}(\varphi))^{\sharp}\right) = |\tau_{p}(\varphi)|^{2} + |d\varphi|^{p-2}\langle d\varphi,\nabla^{\varphi}\tau_{p}(\varphi)\rangle. \tag{33}$$

With the same method of (32), we find that

$$\operatorname{div}^{M}\left(|d\varphi|^{p-4}h(d\varphi,\tau_{p}(\varphi))\right) = (p-4)|d\varphi|^{p-5}h(d\varphi(\operatorname{grad}^{M}|d\varphi|),\tau_{p}(\varphi)) + |d\varphi|^{p-4}h(\tau(\varphi),\tau_{p}(\varphi)) + |d\varphi|^{p-4}\langle d\varphi,\nabla^{\varphi}\tau_{p}(\varphi)\rangle.$$
(34)

Substituting (30), (31), (33) and (34) in (29), the Theorem 3.4 follows. \Box

By using the definition of divergence for symmetric (0, 2)-tensors (see [2], [5]) we have the following result.

Theorem 3.5. Let $\varphi:(M,g)\to (N,h)$ be a smooth map such that $|d\varphi|_x\neq 0$ for all $x\in M$. Then

$$\operatorname{div}^{M} S_{2,p}(\varphi)(X) = -h(\tau_{2,p}(\varphi), d\varphi(X)), \quad \forall X \in \Gamma(TM).$$

Remark 3.6. When p=2, we have $S_{2,p}(\varphi)=S_2(\varphi)$, where $S_2(\varphi)$ is stress bienergy tensor in [10].

Corollary 3.7. Let $\varphi:(M,g)\to (N,h)$ be a smooth map. (1) Then $S_{2,m}(\varphi)=0$ implies that φ is m-harmonic, where $m=\dim M$. (2) If M is compact without boundary, and $p\neq \frac{m}{2}$. Then $S_{2,p}(\varphi)=0$ implies φ is p-harmonic.

Proof. Let $\{e_i\}$ be an orthonormal frame on (M,g). (1) We have

$$0 = \sum_{i=1}^{m} S_{2,p}(\varphi)(e_i, e_i) = -\frac{m}{2} |\tau_p(\varphi)|^2 + (p-m)|d\varphi|^{p-2} \langle d\varphi, \nabla^{\varphi} \tau_p(\varphi) \rangle.$$

For p=m, the last equation becomes $-\frac{m}{2}|\tau_m(\varphi)|^2=0$. So φ is m-harmonic map. (2) We set $\theta(X)=h(|d\varphi|^{p-2}d\varphi(X),\tau_p(\varphi))$, for all $X\in\Gamma(TM)$. The trace of $S_{2,p}(\varphi)$ gives the equality

$$0 = \sum_{i=1}^{m} S_{2,p}(\varphi)(e_i, e_i) = (\frac{m}{2} - p)|\tau_p(\varphi)|^2 + (p - m)\operatorname{div}^M \theta.$$

By using the Green Theorem, we get

$$\left(\frac{m}{2} - p\right) \int_{M} |\tau_{p}(\varphi)|^{2} v^{g} = 0.$$

Since $p \neq \frac{m}{2}$, we obtain $|\tau_p(\varphi)|^2 = 0$, that is φ is p-harmonic map.

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