Communications in Mathematics **32** (2024), no. 1, 63–71
DOI: https://doi.org/10.46298/cm.9722

Structure of finite groups with restrictions on the set of conjugacy classes sizes

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Abstract. Let $N(G)$ be the set of conjugacy classes sizes of G. We prove that if $N(G) = \Omega \times \{1, n\}$ for specific set Ω of integers, then $G \simeq A \times B$ where $N(A) = \Omega$, $N(B) = \{1, n\}$, and n is a power of prime.

Introduction

Let A, B be finite groups and $G := A \times B$. It is easy to check that $N(G) = N(A) \times N(B)$. We are interested in the converse of this assertion.

Question 0.1. *Let G be a group such that $N(G) = \Omega \times \Delta$. Which Δ and Ω guarantee that $G \simeq A \times B$, where A and B are subgroups such that $N(A) = \Omega$ and $N(B) = \Delta$?*

A. Camina proved in [4] that, if $N(G) = \{1, p^m\} \times \{1, q^n\}$, where p and q are distinct primes, then G is nilpotent. In particular, $G = P \times Q$ for a Sylow p-subgroup P and a Sylow q-subgroup Q. Later A. Beltran and M. J. Felipe (see [1] and [2]) proved a more general result asserting that, if $N(G) = \{1, m\} \times \{1, n\}$, where m and n are positive coprime integers, then G is nilpotent, $n = p^a$ and $m = q^b$ for some distinct primes p and q.

In [13], C. Shao and Q. Jiang showed that if $N(G) = \{1, m_1, m_2\} \times \{1, m_3\}$, where m_1, m_2, m_3 are positive integers such that m_1 and m_2 do not divide each other and m_1m_2 is coprime to m_3, then $G \simeq A \times B$, where A and B are such that $N(A) = \{1, m_1, m_2\}$ and $N(B) = \{1, m_3\}$. In all these cases, the sets of prime divisors of the orders of A and B do not intersect. It was proved in [11] that if $N(G) = N(\mathrm{Alt}_5) \times N(\mathrm{Alt}_5)$ and $Z(G) = 1$ then $G \simeq \mathrm{Alt}_5 \times \mathrm{Alt}_5$.

In [7] a directed graph was introduced on the set $N(G) \setminus \{1\}$. Given $\Theta \subseteq \mathbb{N}$, with $|\Theta| < \infty$, define the directed graph $\Gamma(\Theta)$, with the vertex set Θ and edges $\overrightarrow{ab}$ whenever a divides b. Set $\Gamma(G) = \Gamma(N(G) \setminus \{1\})$.

MSC 2020: 20D60
Keywords: finite groups, conjugacy class
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In this article, the following theorem is proved.

Theorem 0.2. *Let Ω be a set of integers and $\Gamma(\Omega\setminus\{1\})$ be disconnected, and n be a positive integer such that $\gcd(n,\alpha)=1$ for each $\alpha\in\Omega\setminus\{1\}$. Let G be a finite group such that $N(G)=\Omega\times\{1,n\}$. Then $G\simeq A\times B$, where $N(A)=\Omega$, $N(B)=\{1,n\}$ and n is a prime power.*

1 Preliminaries

We fix the following notation: for an integer k, denote by $\pi(k)$ the set of prime divisors of k. If Ω is a set of integers, denote $\pi(\Omega)=\bigcup_{\alpha\in\Omega}\pi(\alpha)$. For a prime number r, denote by k_r the highest power of r dividing k. For integers $m_1,\dots,m_s$, write $\gcd(m_1,m_2,\dots,m_s)$ to denote their greatest common divisor, and write $\mathrm{lcm}(m_1,m_2,\dots,m_s)$ for their least common multiple.

Let Ω be a set of integers, and order it by the relation of divisibility. The subset of maximal elements is denoted by $\mu(\Omega)$ and the set of minimal elements is denoted by $\nu(\Omega)$.

Definition 1.1. We say that the set Ω is separated if, for each $\alpha\in\Omega$, there exists $\beta\in\mu(\Omega)$ such that α does not divide β.

Let G be a group and take $a\in G$. We denote by a^G the conjugacy class of G containing a. If N is a subgroup of G, then $\mathrm{Ind}(N,a)=|N|/|C_N(a)|$. Note that $\mathrm{Ind}(G,a)=|a^G|$. Denote by $|G||_p$ the highest power p^n of p such that $N(G)$ contains multiples of p^n while avoiding multiples of p^{n+1}. For $\pi\subseteq\pi(G)$ put $|G||_\pi=\prod_{p\in\pi}|G||_p$. For brevity, write $|G||$ to mean $|G||_{\pi(G)}$. Observe that $|G||_p$ divides $|G|_p$ for each $p\in\pi(G)$. In general, $|G||_p$ is less than $|G|_p$.

Definition 1.2. We say that a group G satisfies the condition $R(p)$, or that G is an $R(p)$-group, if there exists an integer $\alpha>0$ such that $a_p\in\{1,p^\alpha\}$ for each $a\in N(G)$. In that case, we write $G\in R(p)$.

The set of $R(p)$-groups can be seen as the disjoints of the two subsets $R(p)^*$ and $R(p)^{**}$:

a) $G\in R(p)^*$ if $G\in R(p)$ and contains a p-element h such that $\mathrm{Ind}(G,h)_p>1$;

b) $G\in R(p)^{**}$ if $G\in R(p)$ and $\mathrm{Ind}(G,h)_p=1$ for each p-element $h\in G$.

Lemma 1.3 ([9, Main theorem]). *If $G\in R(p)^*$, then G has a normal p-complement.*

Lemma 1.4 ([9, Corollary]). *If $G\in R(p)^*$ and $P\in Syl_p(G)$, then $Z(P)\le Z(G)$.*

Lemma 1.5 ([8, Lemma 1.4]). *For a finite group G, take $K\trianglelefteq G$ and put $\overline{G}=G/K$. Take $x\in G$ and $\overline{x}=xK\in G/K$. The following claims hold:*

(i) $|x^K|$ and $|\overline{x}^{\overline{G}}|$ divide $|x^G|$.

(ii) *For neighboring members L and M of a composition series of G, with $L < M$, take $x \in M$ and the image $\widetilde{x} = xL$ of x. Then $|\widetilde{x}^S|$ divides $|x^G|$, where $S = M/L$.*

(iii) *If $y \in G$ with $xy = yx$ and $(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

(iv) *If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

(v) *$\overline{C_G(x)} \leq C_{\overline{G}}(\overline{x})$.*

Lemma 1.6 ([10, Lemma 4])**.** *Let $g \in G$. If each conjugacy class of G contains an element h such that $g \in C_G(h)$ then $g \in Z(G)$.*

Lemma 1.7 ([3, Theorem A])**.** *Let G be a finite group, and let p and q be distinct primes. Then some Sylow p-subgroup of G commutes with some Sylow q-subgroup of G if and only if the class sizes of the q-elements of G are not divisible by p and the class sizes of the p-elements of G are not divisible by q.*

We call a p-element x of G p-central if $x \in Z(P)$ for some Sylow p-subgroup P of G.

Lemma 1.8 ([12, Theorem B])**.** *Let G be a finite group and p a prime. Suppose that every p-element of G is p-central. Then*

$$O^{p'}(G/O_{p'}(G)) = S_1 \times \cdots \times S_r \times H,$$

where H has an abelian Sylow p-subgroup, $r \geq 0$, and S_i is a non-abelian simple group with either

(i) *$p = 3$ and: $S_i \simeq Ru$, or J_4, or $S_i \simeq {}^2F_4(q_i)', 9 \not| (q_i + 1)$; or*

(ii) *$p = 5$ and $S_i \simeq Th$ for all i.*

Lemma 1.9. *If $G \in R^{**}(p)$, then the Sylow p-subgroups of G are abelian.*

Proof. Note that $R^{**}(p)$-groups satisfy the condition of Lemma 1.8. Hence, if a Sylow p-subgroup is non-abelian, then $p \in \{3, 5\}$ and $O^{p'}(G/O_{p'}(G)) = S_1 \times \cdots \times S_r \times H$, where S_i is isomorphic to one of the groups $Ru, J_4, {}^2F_4(q_i)', Th$. Note that if $r > 1$, then the group G is not an $R^{**}(p)$ group. It follows from the description of conjugacy class sizes in [17] and [15] that S contains a p'-element g_1 such that $1 < \mathrm{Ind}(S, g_1)_p < |S|_p$ and a p'-element g_2 such that $\mathrm{Ind}(S, g_2)_p = |S|_p$. Since p and $|O_{p'}(G)|$ are relatively prime, there exists $g_1' \in G$ such that $g_1' O_{p'}(G) = g_1$ and $\mathrm{Ind}(G, g_1')_p = \mathrm{Ind}(G/O_{p'}(G), g_1)_p$. Let $g_2' \in G$ be such that $g_2' O_{p'}(G) = g_2$. We have $C_G(g_2')O_{p'}/O_{p'} \leq C_{G/O_{p'}(G)}(g_2)$. In particular $\mathrm{Ind}(G, g_2')_p \geq \mathrm{Ind}(G/O_{p'}(G), g_2)_p > \mathrm{Ind}(G, g_1')_p$, contradicting the definition of $R^{**}(p)$-groups. □

Lemma 1.10. *Any $R^{**}(p)$-group contains at most one non-abelian composition factor whose order is divisible by p.*

Proof. Let G be an $R^{**}(p)$-group. Lemma 1.9 implies that the Sylow p-subgroup of G is abelian. Let $1 < G_1 < \cdots < G_k = G$ be the chief series. Assume that $G_i/G_{i-1} = H$ is a non-solvable group and the order of H is divisible by p. Lemma 1.5 implies that the conjugacy class sizes of the group H divide the corresponding conjugacy class sizes of G. We have $H = S_1 \times S_2 \times \cdots \times S_t$, where the S_i are isomorphic non-abelian finite simple groups, for $1 \leq i \leq t$.

Assume that $|G_{i-1}|$ is divisible by p. Let $P \leq G_{i-1}$ be a Sylow p-subgroup of G_{i-1}. From Frattini's argument, it follows that $N_{G_i}(P)/N_{G_{i-1}}(P) \simeq G_i/G_{i-1}$. Let $\widehat{H} \leq N_{G_i}(P)$ be a subgroup generated by all Sylow p-subgroups of $N_{G_i}(P)$. Since any Sylow p-subgroup of G is abelian and H is generated by p-elements, we infer that $\widehat{H}G_{i-1}/G_{i-1} = H$ and $\widehat{H}$ centralizes some Sylow p-subgroup of the group G_{i-1}.

Assume that $g \in G/G_{i-1}$ is a p-element acting on H as an outer automorphism. The fact that the Sylow p-subgroups of G are abelian implies that $S_j^g = S_j$ for any $1 \leq j \leq t$. Assume that g acts non-trivially on S_j. Since the Sylow 2-subgroup of a simple alternating group of degree greater than 5 is non-abelian and the outer automorphism group of an alternating group is a 2-group, we obtain that S_j cannot be isomorphic to any of the alternating groups. It follows from [17] and Lemma 1.8 that S_j cannot be isomorphic to any of the sporadic groups, and therefore S_j is a group of Lie type. In [14, Theorem 1] and in [16] it is described when a Sylow p-subgroup of a simple group of Lie type is abelian. We can show that g acts on S_j as a field automorphism. It follows from the description of the centralizers of field automorphisms (see [6, Theorem 4.9.1]) that the Sylow p-subgroup of $S_j.\langle g\rangle$ is non-abelian, and hence the Sylow p-subgroup of G is non-abelian, which is a contradiction. Therefore, it can be considered that H contains a Sylow p-subgroup of G/G_{i-1}.

Assume that $t > 1$. For each $j \in \{1, \ldots, t\}$, there is an element $h_j \in S_j$ such that $\mathrm{Ind}(S_j, h_j)_p = |S_j|_p$. Let $g = h_1 \cdots h_t$ and $\widehat{g} \in \widehat{H}$ be some pre-image of the element g. Since $\widehat{H}$ centralizes a Sylow p-subgroup of G_{i-1} and $\mathrm{Ind}(H, g)$ divides $\mathrm{Ind}(G, \widehat{g})$, we infer that $\mathrm{Ind}(G, \widehat{g})_p = (\mathrm{Ind}(H, g))_p = |H|_p$. If $t > 1$, then $\widehat{H}$ contains an element $\widehat{h_1}$, which is the pre-image of the element h_1 such that $1 < \mathrm{Ind}(G, \widehat{h_1})_p < |H|_p$. This contradicts the definition of an $R(p)$-group. □

Lemma 1.11 ([5, Theorem 5.2.3])**.** *Let A be a $\pi(G)'$-group of automorphisms of an abelian group G. Then $G = C_G(A) \times [G, A]$.*

Lemma 1.12. *Let $P \lhd G$ be a Sylow p-subgroup of G. If $P = A \times B$ with A, B normal subgroups of G, then $C_G(ab) = C_G(a) \cap C_G(b)$ for any $a \in A$ and $b \in B$.*

Proof. The assertion of the lemma follows from the fact that any p-element x is uniquely represented as $x = x_a x_b$ where $x_a \in A$ and $x_b \in B$. □

2 Proof of the Main Theorem

Let G be as in the hypothesis of the theorem. We divide the proof of the theorem into 3 propositions. In the preliminary lemma and in Propositions 2.2 and 2.4, we only use the separation property of the set Ω. The disconnection of the graph $\Gamma(\Omega \setminus \{1\})$ is used only in the proof of Proposition 2.

Note that G has the property $R(p)$ for any $p \in \pi(n)$. In Propositions 2.2 and 2.3 we prove that $G \notin R^{**}(p)$. In Proposition 2.4 we analyze the case $G \in R^{*}(p)$ and thus complete the proof of the Main Theorem.

Assume that $G \in R^{**}(p)$ for any $p \in \pi(n)$. In this case, Lemma 1.9 implies that a Sylow p-subgroup of G is abelian. It follows from Lemma 1.7 that a Hall $\pi(n)$-subgroup exists and is abelian. It follows from the well-known Wielandt theorem that all Hall $\pi(n)$-subgroups are conjugate.

Lemma 2.1. *The order of any non-abelian composition factor of G is not divisible by p.*

Proof. Lemma 1.10 implies that G contains at most one non-abelian composition factor S whose order is divisible by p. Let $R \lhd G$ be such that $S \leq G/R$. Let $g \in G$ be a p-element such that its image $gR \in S$ is not trivial. Let $x \in G$ be an element of minimal order such that $\mathrm{Ind}(G,x) = n$. Since n is minimal with respect to divisibility in $N(G)$, we infer that $|x| = r^{\alpha}$ is a power of a prime r. We have that x centralizes Sylow t-subgroups for any $t \in \pi(\Omega)$ and, in particular, x centralizes Sylow t-subgroups for any $t \in \pi(\mathrm{Ind}(G,g))$. Put $C = C_G(x)$. Since S is the unique non-abelian composition factor whose order is divisible by p, we infer that S is a normal subgroup of G/R. Note that CR/R contains Sylow t-subgroups of G/R for any $t \in \pi(\mathrm{Ind}(S,\overline{g}))$. Let T be a Sylow t-subgroup of G/R for some prime $t \in \pi(G/R)$. Since S is a normal subgroup of G/R we infer that $T \cap S$ is a Sylow t-subgroup of S. From the fact that finite simple groups do not have Hall p'-subgroups for each prime divisor p of its order, we get that group S is generated by its Sylow t-subgroups, where $t \in \pi(\mathrm{Ind}(S,\overline{g}))$. Hence $S \leq CR/R$. In particular, C contains a pre-image of the group S. Therefore, C contains an r'-element y such that $\mathrm{Ind}(C,y)_p > 1$. Thus, $\mathrm{Ind}(G,xy)_p > \mathrm{Ind}(G,x)_p$, which is a contradiction. □

Let $O = O_{\pi(n)'}(G)$. Lemma 2.1 implies that G/O contains a normal p-subgroup $\overline{P}$, for some $p \in \pi(n)$. Let $T = O_{\pi(n)}(G/O)$. Assume that T is not a Hall $\pi(n)$-subgroup of G/O. Since a Hall $\pi(n)$-subgroup of G is abelian, we have T is abelian. The centralizer of R in G/O is a normal subgroup of G/O for each Sylow subgroup R of T. For any $g \in G/O$ it follows from the inequality $\mathrm{Ind}(G/O,g)_p > 1$ that $\mathrm{Ind}(G/O,g)_{\pi(n)} = n$. Using these facts it is easy to obtain a contradiction. Therefore, G/O contains a normal Hall $\pi(n)$-subgroup $\overline{H}$. In particular, we can assume that $\overline{P}$ is a Sylow p-subgroup of G/O. Let $x \in G$ be an element of minimal order such that $\mathrm{Ind}(G,x) = n$. Since n is minimal by divisibility number of $N(G)$, we infer that x is an element of order t^{α}, where t is some prime and $t \notin \pi(n)$.

Proposition 2.2. *The image $\overline{x} \in G/O$ of x is trivial.*

Proof. Assume that $\overline{x}$ is not trivial. Lemma 1.11 implies that $\overline{P} = [\overline{x}, \overline{P}] \times C_{\overline{P}}(\overline{x})$. Let $\widetilde{x} \in G/OH$ denote the image of x. Since $\pi(G/OH)$ does not contain numbers from the set $\pi(n) = \pi(\mathrm{Ind}(G, x))$, and $\mathrm{Ind}(G/OH, \widetilde{x})$ divides $\mathrm{Ind}(G, x)$, we infer that $\mathrm{Ind}(G/OH, \widetilde{x})$ is equal to 1. Hence $\widetilde{x} \in Z(G/OH)$. Thus the subgroup $C_{\overline{P}}(\overline{x})$ is a normal subgroup of G/O. Since $p \notin \pi(G/OH)$, it follows from Maschke's theorem that $C_{\overline{P}}(\overline{x})$ has compliment in $\overline{P}$. In particular $[\overline{x}, \overline{P}]$ is a normal subgroup of G/O.

Let P be a Sylow p-subgroup of G, and let $P_1, P_2 \leq P$ be such that $P_1.O/O = [\overline{x}, \overline{P}]$ and $P_2O/O = C_{\overline{P}}(\overline{x})$. Since $\mathrm{lcm}(\mathrm{Ind}(O, x), O) = 1$, we have $x \in C_G(O)$. The group $C_G(O)$ is a normal subgroup of G. We have $C_G(O)O/O \trianglelefteq \overline{G}$ and $\overline{x} \in C_G(O)O/O$. From the fact that $\overline{x}$ acts without fixed points on $[\overline{x}, \overline{P}]$ and $[\overline{x}, \overline{P}] \trianglelefteq \overline{G}$ it follows that $[\overline{x}, \overline{P}]$ is the minimal normal subgroup of $\overline{G}$ which includes $\overline{x}$. In particular $P_1 < C_G(O)$.

The fact that the number $\mathrm{Ind}(G, x)_p$ is maximal implies that centralizer of any t'-element of $C_G(x)$ contains some Sylow p-subgroup of the group $C_G(x)$. Since $O.P_2 \trianglelefteq G$, we infer that the centralizer of any t'-element from O contains a subgroup conjugate to P_2 in $O.P_2$. Suppose there is a t-element $y \in O$ such that $\mathrm{Ind}(O.P_2, y)_p > 1$. Since $\mathrm{Ind}(G, x)_t = 1$, we infer that $C_G(x)$ contains some Sylow t-subgroup of G. In particular, one can assume that $y \in C_G(x)$. Consider $C_G(xy)$. Let R be a Sylow p-subgroup of G such that $\widetilde{R} = R \cap C_G(xy)$ is a Sylow p-subgroup of $C_G(xy)$. Since $P_1 \leq C_G(O)$, we have $P_1 \leq R$ and $P_1 \cap C_G(xy) = 1$. It follows from the fact that $\mathrm{Ind}(G, x)_p = \mathrm{Ind}(G, xy)_p = |P_1|$ and the fact that R is an abelian group that $R = P_1 \times \widetilde{R}$. Note that $\widetilde{R} < C_G(x)$, and hence $\widetilde{R}$ is conjugate to P_2 in $C_G(x)$. In particular, $\widetilde{R}$ is conjugate to P_2 in $O.P_2$. Therefore y centralizes $\widetilde{R}$ and $\mathrm{Ind}(O.P_2, y)_p = 1$, which is a contradiction. Thus any element of O centralizes some Sylow p-subgroup. Lemma 1.6 implies that $P_2 < C_G(O)$. Thus G contains a normal abelian Hall $\pi(n)$-subgroup N.

We have that P_2 is a Sylow p-subgroup of $C_G(x)$ and $P_2 \trianglelefteq C_G(x)$. From the fact that $\mathrm{Ind}(G, x)_p$ is maximal it follows that any t'-element centralizes P_2. Therefore, we have that $\pi(\mathrm{Ind}(C_G(x), h)) \subseteq \{t\}$ for any $h \in P_2$. Since $C_G(x)$ contains Sylow r-subgroups of G for any $r \in \pi(\Omega)$ and a Hall $\pi(n)$-subgroup of G is abelian, we infer that $\pi(\mathrm{Ind}(G, h)) \subseteq \{t\}$ for any $h \in P_2$. Let $g \in C_G(x)$ be some t'-element. Then g acts on P_1, and

$$\mathrm{Ind}(G, g)_p = \mathrm{Ind}(P_1, g)_p.$$

Since $\mathrm{Ind}(P_1, g)_p \in \{1, |P_1|\}$, we see that g acts on P_1 either trivially or without fixed points. Note that $x^G = x^N$. Thus, $\mathrm{Ind}(G, a)_{t'} = \mathrm{Ind}(G, b)_{t'}$ for any $a, b \in P_1$ and $\pi(\mathrm{Ind}(G, c)) \subseteq \{t\}$ for any $c \in P_2$. It follows from Lemma 1.12 that, for any p-element a, there exists k such that $\mathrm{Ind}(G, a)_{t'} \in \{1, k\}$. Thus, Ω contains a number α dividing the index of any p-element. Let $h_1 \in P_1$ be such that $\mathrm{Ind}(G, h_1)$ is minimal among $\{\mathrm{Ind}(G, g) | g \in P_1\}$, and let $h_2 \in P_2$ be such that $\mathrm{Ind}(G, h_2)$ is minimal among $\{\mathrm{Ind}(G, g) | g \in P_2\}$.

Assume that $\mathrm{Ind}(G, h_2)_t \leq \mathrm{Ind}(G, h_1)_t$. Then $\mathrm{Ind}(G, h_2)$ divides $\mathrm{Ind}(G, g)$ for any p-element g. Since Ω is separated, we obtain that $\mu(\Omega)$ contains an element β that is not divisible by $\mathrm{Ind}(G, h_2)$. Let $l \in G$ be such that $\mathrm{Ind}(G, l) = \beta$. Since $\mathrm{Ind}(G, h_2)$ does not divide β, we infer that $C_G(l)$ does not contain p-elements. But β is not divisible by p and hence $C_G(h)$ contains some Sylow p-subgroup, therefore we have a contradiction.

Thus, $\mathrm{Ind}(G,h_2)_t > \mathrm{Ind}(G,h_1)_t$. Since Ω is separated, we infer that $\mu(\Omega)$ contains an element β that is not divisible by $\mathrm{Ind}(G,h_2)$. Let $l \in G$ be such that $\mathrm{Ind}(G,l) = \beta$. Since $\mathrm{Ind}(G,h_2)$ does not divide β, we see that $|l|$ is divisible by p. Further, we have $l = ab$, where a is a p-element and b is a p'-element. We have that $\mathrm{Ind}(G,a)$ divides β. From Lemma 1.12 and the fact that β is not divisible by numbers in $\{\mathrm{Ind}(G,g) | g \in P_2\}$, it follows that $a \in P_1$. It follows from Lemma 1.12 that $\mathrm{Ind}(G,abh_2)$ is divisible by β and $\mathrm{Ind}(G,h_2)$ contradicting the fact that β is maximal in Ω. □

Proposition 2.3. *The element $x \notin O$.*

Proof. Assume that $x \in O$. Since $\mathrm{Ind}(G,x)$ is relatively prime to $|O|$, we have $O \leq C_G(x)$. Let $X = \langle x^G \rangle$. The fact that O is a normal subgroup of G implies that $O \leq C_G(X)$. Hence, X is an abelian t-subgroup of the group O. Let P be a Sylow p-subgroup of G such that $P_1 = P \cap C_G(x)$ is a Sylow p-subgroup of $C_G(x)$. The fact that $x^G = x^{O.H}$ implies that $P_1 < C_G(X)$. Thus, any t'-element of O centralizes some subgroup conjugate to P_1.

Consider X as a $\widetilde{P} = P/P_1$-module. It follows from Lemma 1.11 that the group X can be represented as $[X,\widetilde{P}] \times C_X(\widetilde{P})$. Since $\widetilde{P}$ acts non-trivially on X, we see that $[X,\widetilde{P}]$ is non-trivial. Since for any element $y \in [X,\widetilde{P}]$ we have $\widetilde{P} \cap C_{O.\widetilde{P}}(y) = 1$, we infer that $\widetilde{P}$ acts without fixed points on $[X,\widetilde{P}]$. Hence $\widetilde{P}$ is a cyclic group.

Assume that P_1 contains an element f such that $\mathrm{Ind}(G,f) > 1$. We will use the fact that graph $\Gamma(\Omega \setminus \{1\})$ is disconnected. Let Γ_1 be a connected component of the graph $\Gamma(\Omega \setminus \{1\})$ such that $\mathrm{Ind}(G,f) \in \Gamma_1$. Since any t'-element centralizes some element from f^G, we infer that $\mathrm{Ind}(G,g)_{\pi(n)'} \in \Gamma_1 \cup \{1\}$ for any $\{p,t\}'$-element g.

Denote by Γ_2 some connected component of the graph $\Gamma(\Omega \setminus \{1\})$ different from Γ_1. Let $y \in G$ be such that $\mathrm{Ind}(G,y) \in \Gamma_2$. We have that y is a $\{p,t\}$-element. Assume that y is a t-element. Since $\mathrm{Ind}(G,y)_p = 0$, we infer that $C_G(y)$ contains a subgroup conjugate to P_1, and hence $\mathrm{Ind}(G,y) \in \Gamma_1 \cup \{1\}$, deriving a contradiction.

Therefore, if $\mathrm{Ind}(G,g)_p = 1$, then g is the product of a p-element and an element from the center of G. In particular, if $\mathrm{Ind}(G,g) \in \Gamma_2$, where g is an element of primary order, then $\pi(g) = \{p\}$. It also follows from here that $\pi(n) = \{p\}$.

Since Γ_2 is an arbitrary connected component of $\Gamma(\Omega \setminus \{1\})$ different from Γ_1, then we can assume that there exists $z \in P$ such that $\mathrm{Ind}(G,z) \in \Gamma_2$. Then $\mathrm{Ind}(G,y) \in \Gamma_2 \cup \{1\}$ for any $y \in \langle z \rangle$. This means that $\langle z \rangle \cap P_1 \leq Z(G)$. Let $g \in P$ be such that $z \in \langle g \rangle$. Since $\mathrm{Ind}(G,z)$ divides $\mathrm{Ind}(G,g)$ it follows that $\mathrm{Ind}(G,g) \in \Gamma_2$. Since P/P_1 is a cyclic group and P is an abelian group, we can write $P = \langle z, P_1 \rangle$.

We have $\mathrm{Ind}(G,g) \in \Gamma_1$ for any non-central $\{p,t\}'$-element g. Therefore for any h such that $\mathrm{Ind}(G,h) \in \Gamma_2$ it is true that $C_G(h)/Z(G)$ is a $\{p,t\}$-group. In particular, $\mathrm{Ind}(G,h)_{\{p,t\}'} = |G||_{\{p,t\}'}$. Assume that there exists $z' \in P \setminus P_1$ such that $\mathrm{Ind}(G,z') \in \Gamma_1$. If $C_G(z')$ does not contain non-central $\{p,t\}'$-elements, then $\mathrm{Ind}(G,z)$ is connected to $\mathrm{Ind}(G,z')$ in $\Gamma(\Omega \setminus \{1\})$ and hence $\mathrm{Ind}(G,z') \in \Gamma_2$, contradicting $\mathrm{Ind}(G,z') \in \Gamma_1$. Let $s \in C_G(z')$ be a $\{p,t\}'$-element and $E \in Syl_p(C_G(z'))$. Since $C_G(x)$ contains some Sylow p-subgroup of $C_G(s)$, we can assume that $C_G(x^g)$ contains E for some g. But $C_G(x^g) \cap P = P_1$, and we have a contradiction.

Let $y \in C_G(z) \setminus (Z(G) \cup P)$. As noted above, y is a t-element. We can assume that $C_G(y) \cap P \in Syl_p(C_G(y))$. Hence $C_G(xy) \cap P \in Syl_p(C_G(xy))$. Obviously, z and P_1 do not lie in $C_G(xy)$. Let $zg \in C_G(xy)$, where $g \in P_1 \setminus Z(G)$. Let $\widetilde{\ } : G \to G/O$ be a natural homomorphism. Note $\widetilde{(xy)} = \widetilde{y}$. Hence $\widetilde{z} \in C_{\widetilde{G}}(\widetilde{xy})$, and thus $\widetilde{g} \in C_{\widetilde{G}}(\widetilde{xy})$. Since $|O|$ is coprime to $|g|$, then $\widetilde{C_G(g)} = C_{\widetilde{G}}(\widetilde{g})$. Hence $C_G(g)$ contains the group $O.\langle\widetilde{y}\rangle$, and therefore $y \in C_G(g)$ contradicting the fact that $C_G(y) \cap P < Z(G)$. Thus, it is proved that $P_1 < Z(G)$.

Since z acts without fixed points on x^G, $\mathrm{Ind}(G,z) > 1$. Denote by Γ' the connected component of $\Gamma(\Omega \setminus \{1\})$ containing $\mathrm{Ind}(G,z)$. Note that $\mathrm{Ind}(G,g)_{p'} \in \Gamma' \cup \{1\}$ for any $g \in C_G(z)$. Assume that there exists $h \notin \langle z\rangle$ such that $\mathrm{Ind}(G,h) \in \Omega \setminus (\Gamma' \cup \{1\})$. Then h centralizes some Sylow p-subgroup and, therefore, we can assume that $h \in C_G(z)$. Thus $\mathrm{Ind}(G,h) \in \Gamma'$, contradicting the hypotesis on h. We have $|\Gamma'| = 1$ and $C_G(z)/Z(G) = \langle z\rangle$. Therefore $\Omega = N(\langle z\rangle)$, and in particular $\Gamma(\Omega)$ is connected, which is a contradiction. □

It follows from the proposition 2.2 and 2.3 that $G \in R(p)^*$.

Proposition 2.4. *If $G \in R(p)^*$ for some $p \in \pi(n)$ then $G = A \times B$, where $N(A) = \Omega$ and $N(B) = \{1, p^\alpha\}$. In particular, n is a p-number.*

Proof. Lemma 1.3 implies that $G = N \rtimes P$ where P is a Sylow p-subgroup of G. Lemma 1.4 implies that $Z(P) \leq Z(G)$.

Assume that there is $z \in P$ such that $\mathrm{Ind}(G,z)_{p'} > 1$. The separation of Ω implies that there exists $k \in \mu(\Omega)$ such that k is not divisible by $\mathrm{Ind}(G,z)_{p'}$. Let $g \in G$ be such that $\mathrm{Ind}(G,g) = k$. We have $g = g_1g_2$, where g_1 is a p'-element and g_2 is a p-element. Since $C_G(g) = C_G(g_1) \cap C_G(g_2)$ and $\mathrm{Ind}(G,g)_p = 1$, it follows that $\mathrm{Ind}(G,g_2)_p = 1$. Hence $g_2 \in Z(G)$. Thus, $\mathrm{Ind}(G,g) = \mathrm{Ind}(G,g_1)$. We have that $C_G(g_1)$ contains some Sylow p-subgroup of G, and therefore there is $z' \in C_G(g_1) \cap z^G$, deriving a contradiction.

Thus any p-element centralizes N and hence $G \simeq N \times P$. Therefore for each $g \in P$ we have $\pi(\mathrm{Ind}(G,g)) = \{p\}$. In particular n is a $\{p\}$-number. □

The assertion of the theorem follows from Propositions 2.2, 2.3 and 2.4.

Acknowledgment

The work was supported by the grant of the President of the Russian Federation for young scientists (MD-1264.2022.1.1).

References

[1] Beltran A., Felipe M.J.: Some class size conditions implying solvability of finite groups. *J. Group Theory* 9 (2006) 787–797.

[2] Beltran A., Felipe M.J.: Variations on a theorem by Alan Camina on conjugacy class sizes. *J. Algebra* 296 (1) (2006) 253–266.

[3] Beltran A., Felipe M.J., Malle G., Moreto A., Navarro G., Sanus L., Solomon R., Tiep P.H.: Nilpotent and abelian Hall subgroups in finite groups. *Trans. Amer. Math. Soc.* 368 (4) (2016) 2497–2513.

[4] Camina A.R.: Arithmetical conditions on the conjugacy class numbers of a finite group. *J. London Math. Soc.* 5 (2) (1972) 127–132.

[5] Gorenstein D.: *Finite groups*. New York-London (1968).

[6] Gorenstein D., Lyons R., Solomon R.: *The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K-groups*. American Mathematical Society, Providence, RI (1998).

[7] Gorshkov I.B.: Towards Thompson's conjecture for alternating and symmetric groups. *J. Group Theory* 19 (2) (2016) 331–336.

[8] Gorshkov I.B.: On Thompson's conjecture for alternating and symmetric groups of degree more then 1361. *Proceedings of the Steklov Institute of Mathematics* 293 (1) (2016) 58–65.

[9] Gorshkov I.B.: On existence of normal p-complement of finite groups with restrictions on the conjugacy class sizes. *Communications in Mathematics* 30 (1) (2022) .

[10] Gorshkov I.B.: On a finite group with restriction on set of conjugacy classes size. *Bull. Malays. Math. Sci. Soc.* 43 (4) (2020) 2995–3005.

[11] Gorshkov I.B.: On characterisation of a finite group by the set of conjugacy class sizes. *Journal of Algebra and Its Applications* 21 (11) (2022) .

[12] Navarro G., Solomon R., Tiep P.H.: Abelian Sylow subgroups in a finite group, II. *J. Algebra* 421 (2015) 3–11.

[13] Shao C., Jiang Q.: Determining group structure by set of conjugacy class sizes. *Comm. Algebra* 48 (4) (2020) 1626–1631.

[14] Rulin S., Yuanyang Z.: Finite simple groups with some abelian Sylow subgroups. *Kuwait J. Sci* 43 (2) (2016) 1–15.

[15] Shinoda K.: The conjugacy classes of the finite Ree groups of type F_4. *J. Fac. Sci. Univ. Tokyo Sect. I A Math.* 22 (1975) 1–15.

[16] Walter J.: The Characterzation of Finite Groups with Abelian Sylow 2-Subgroup. *Ann. Math.* 89 (1969) 405–514.

[17] ATLAS of Finite Group Representations - Version 3: http://brauer.maths.qmul.ac.uk/Atlas/v3.

Received: June 22, 2022
Accepted for publication: January 11, 2023
Communicated by: Ivan Kaygorodov