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General terms of all almost balancing numbers of first and second type

Ahmet Tekcan and Alper Erdem

Abstract. In this work, we determined the general terms of all almost balancing numbers of first and second type in terms of balancing numbers and conversely we determined the general terms of all balancing numbers in terms of all almost balancing numbers of first and second type. We also set a correspondence between all almost balancing numbers of first and second type and Pell numbers.

1 Introduction

Behera and Panda ([2]) defined that a positive integer n is called a balancing number if the Diophantine equation

$$1 + 2 + \dots + (n-1) = (n+1) + (n+2) + \dots + (n+r)$$
(1)

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1) they get

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}.$$
(2)

So from (2), they noted that n is a balancing number if and only if $8n^2 + 1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2), they noted that $8(0)^2 + 1 = 1$ and $8(1)^2 + 1 = 3^2$ are perfect squares. So they accepted that 0 and 1 to be balancing numbers. Let B_n denote the n^{th} balancing number. Then $B_0 = 0, B_1 = 1, B_2 = 6$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 2$.

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Affiliation:

Ahmet Tekcan and Alper Erdem– Bursa Uludag University, Faculty of Science, Department of Mathematics, Bursa, Turkiye

E-mail: tekcan@uludag.edu.tr, alper.erdem@outlook.com

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \dots + n = (n+1) + (n+2) + \dots + (n+r)$$
(3)

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (3) they get

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}.$$
(4)

So from (4), they noted that n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, they accepted 0 to be a cobalancing number just like Behera and Panda accepted 0 and 1 to be balancing numbers. Let b_n denote the nth cobalancing number. Then $b_0 = b_1 = 0, b_2 = 2$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \ge 2$.

It is clear from (1) and (3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n = r_{n+1}$ and $R_n = b_n$ for $n \ge 1$, where R_n is the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n = b_n$, we get from (1) that

$$b_n = \frac{-2B_n - 1 + \sqrt{8B_n^2 + 1}}{2} \quad \text{and} \quad B_n = \frac{b_n + 1 + \sqrt{8b_n^2 + 8b_n + 1}}{2}.$$
 (5)

Thus from (5), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. So

$$C_n = \sqrt{8B_n^2 + 1}$$
 and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ (6)

are integers which are called the n^{th} Lucas-balancing number and n^{th} Lucas-cobalancing number, respectively.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of the characteristic equation for Pell numbers P_n . Then Binet formulas for balancing numbers, cobalancing numbers, Lucasbalancing numbers and Lucas-cobalancing numbers are

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2} \text{ and } c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$$

for $n \ge 1$, respectively (see also [4], [10], [11], [17], [20]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [8], Liptai proved that there is no Fibonacci balancing number except 1 and in [9] he proved that there is no Lucas balancing number. In [19], Szalay considered the same problem and obtained some nice results by a different method. In [6], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a, b)balancing numbers defined as follows: Let a > 0 and $b \ge 0$ be coprime integers. If

$$(a+b) + \dots + (a(n-1)+b) = (a(n+1)+b) + \dots + (a(n+r)+b)$$

for some positive integers n and r, then an + b is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \ge 1$. In [7], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ such that $y \ge 4$. Then a positive integer x with $x \le y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \dots + (x-1)^k = (x+1)^l + \dots + (y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For positive integers k, x, let

$$\Pi_k(x) = x(x+1)\dots(x+k-1)$$

Then it was proved in [6] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \ge 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [24] Tengely considered the case

$$B_m = x(x+1)(x+2)(x+3)(x+4)$$

for k = 5 and proved that this Diophantine equation has no solution for $m \ge 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [16], Patel, Irmak and Ray considered incomplete balancing and Lucas-balancing numbers and in [18], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [21], Tekcan and Erdem considered the *t*-cobalancing numbers and *t*-cobalancers, in [22], Tekcan and Aydın considered the *t*-balancers, *t*-balancing numbers and Lucas *t*-balancing numbers and in [23], Tekcan and Yıldız considered the balcobalancing numbers and balcobalancers.

2 Results

In this section we determine the general terms of almost balancing numbers, almost cobalancing numbers, almost Lucas-balancing numbers and almost Lucas-cobalancing numbers of first and second type. Almost balancing numbers first defined by Panda and Panda in [13]. A positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \dots + (n+r)] - [1+2+\dots + (n-1)]| = 1$$
(7)

holds for some positive integer r which is called the almost balancer.

From (7), they have two cases: If $[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]=1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 9}}{2}.$$
(8)

If $[(n + 1) + (n + 2) + \dots + (n + r)] - [1 + 2 + \dots + (n - 1)] = -1$, then *n* is called an almost balancing number of second type and *r* is called an almost balancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 7}}{2}.$$
(9)

Let B_n^* and B_n^{**} denote the n^{th} almost balancing number of first type and of second type, respectively. Then from (8), B_n^* is an almost balancing number of first type if and only if $8(B_n^*)^2 + 9$ is a perfect square and from (9), B_n^{**} is an almost balancing number of second type if and only if $8(B_n^{**})^2 - 7$ is a perfect square. Thus

$$C_n^* = \sqrt{8(B_n^*)^2 + 9}$$
 and $C_n^{**} = \sqrt{8(B_n^{**})^2 - 7}$ (10)

are integers which are called the n^{th} almost Lucas-balancing number of first type and of second type, respectively.

Later in [15], Panda defined that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \dots + (n+r)] - (1+2+\dots+n)| = 1$$
(11)

holds for some positive integer r which is called an almost cobalancer.

From (11), they have two cases: If $[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)=1$, then n is called an almost cobalancing number of first type and r is called an almost cobalancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 9}}{2}.$$
(12)

If $[(n + 1) + (n + 2) + \dots + (n + r)] - (1 + 2 + \dots + n) = -1$, then *n* is called an almost cobalancing number of second type and *r* is called an almost cobalancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n - 7}}{2}.$$
(13)

Let b_n^* and b_n^{**} denote the n^{th} almost cobalancing number of first type and of second type, respectively. Then from (12), b_n^* is an almost cobalancing number of first type if and only if $8(b_n^*)^2 + 8b_n^* + 9$ is a perfect square and from (13), b_n^{**} is an almost cobalancing number of second type if and only if $8(b_n^{**})^2 + 8b_n^{**} - 7$ is a perfect square. Thus

$$c_n^* = \sqrt{8(b_n^*)^2 + 8b_n^* + 9}$$
 and $c_n^{**} = \sqrt{8(b_n^{**})^2 + 8b_n^{**} - 7}$ (14)

are integers which are called the n^{th} almost Lucas-cobalancing number of first type and of second type, respectively.

Like in balancing numbers, we notice that every almost balancing number is an almost cobalancer and every almost cobalancing number is an almost balancer, that is, $B_n^* = r_{n+1}^*$, $B_n^{**} = r_n^*, b_n^* = R_{n+2}^*$ and $b_n^{**} = R_n^*$ for $n \ge 1$, where R_n^* is the n^{th} almost balancer of first type, R_n^{**} is the n^{th} almost balancer of second type, r_n^* is the n^{th} almost cobalancer of first type and r_n^{**} is the n^{th} almost cobalancer of second type.

2.1 Almost Balancing and Almost Lucas-Balancing Numbers of First and Second Type.

We see in (8) that x is an almost balancing number of first type if and only if $8x^2 + 9$ is a perfect square and in (9), x is an almost balancing number of second type if and only if $8x^2 - 7$ is a perfect square. Let $8x^2 + 9 = y^2$ and let $8x^2 - 7 = w^2$ for some positive integers y and w. Then we get the Pell equations ([1], [5])

$$8x^2 - y^2 = -9$$
 and $8x^2 - w^2 = 7.$ (15)

For the set of all (positive) integer solutions of (15), we need some notations: Let Δ be a non-square discriminant. Then the Δ -order O_{Δ} is defined to be the ring

$$O_{\Delta} = \{ x + y\rho_{\Delta} : x, y \in \mathbb{Z} \}$$

where $\rho_{\Delta} = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 \pmod{4}$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 \pmod{4}$. So O_{Δ} is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_{Δ}^{u} is defined to be the group of units of the ring O_{Δ} . We can rewrite an integral indefinite quadratic form ([3]) $F(x, y) = ax^2 + bxy + cy^2$ of discriminant Δ to be

$$F(x,y) = \frac{(xa+y\frac{b+\sqrt{\Delta}}{2})(xa+y\frac{b-\sqrt{\Delta}}{2})}{a}$$

So the module M_F of F is $M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta})$. Therefore we get $(u + v\rho_{\Delta})(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{cases} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 \pmod{4} \\ \begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 \pmod{4}. \end{cases}$$
(16)

Let *m* be any integer and let Ω denote the set of all integer solutions of F(x, y) = m, that is, $\Omega = \{(x, y) : F(x, y) = m\}$. Then there is a bijection $\Psi : \Omega \to \{\gamma \in M_F : N(\gamma) = am\}$. The action of $O_{\Delta,1}^u = \{\alpha \in O_{\Delta}^u : N(\alpha) = 1\}$ on the set Ω is most interesting when Δ is a positive non-square since $O_{\Delta,1}^u$ is infinite. Therefore the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O_{\Delta,1}^u$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_{Δ} be the smallest unit of O_{Δ} that is grater than 1 and let $\tau_{\Delta} = \varepsilon_{\Delta}$ if $N(\varepsilon_{\Delta}) = 1$ or ε_{Δ}^2 if $N(\varepsilon_{\Delta}) = -1$. Then every $O_{\Delta,1}^u$ orbit of integral solutions of F(x, y) = m contains a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq y \leq U$, where $U = \left|\frac{am\tau_{\Delta}}{\Delta}\right|^{\frac{1}{2}} (1 - \frac{1}{\tau_{\Delta}})$ if am > 0 or $U = \left|\frac{am\tau_{\Delta}}{\Delta}\right|^{\frac{1}{2}} (1 + \frac{1}{\tau_{\Delta}})$ if am < 0. So for finding a set of representatives of the $O_{\Delta,1}^u$ orbits of integral solutions of F(x, y) = m, we must find for each integer y_0 in the range $0 \leq y_0 \leq U$, whether $\Delta y_0^2 + 4am$ is a perfect square or not since

$$ax_0^2 + bx_0y_0 + cy_0^2 = m \Leftrightarrow \Delta y_0^2 + 4am = (2ax_0 + by_0)^2.$$

If $\Delta y_0^2 + 4am$ is a perfect square, then $x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a}$. So there is a set of representatives Rep = { $[x_0 \ y_0]$ }. Thus for the matrix M defined in (16), the set of all integer solutions of F(x, y) = m is $\Omega = \{\pm (x, y) : [x \ y] = [x_0 \ y_0]M^n, n \in \mathbb{Z}\}$.

For the set of all integer solutions of (15), we can can give the following theorem.

Theorem 2.1. The set of all integer solutions of $8x^2 - y^2 = -9$ is $\Omega = \{(3B_n, 3C_n) : n \ge 1\}$, and the set of all integer solutions of $8x^2 - w^2 = 7$ is

$$\Omega = \{ (B_{n-1} + C_{n-1}, 8B_{n-1} + C_{n-1}) : n \ge 1 \} \cup \{ (-B_n + C_n, 8B_n - C_n) : n \ge 1 \}.$$

Proof. For the Pell equation $8x^2 - y^2 = -9$, we have $F(x, y) = 8x^2 - y^2$ of discriminant $\Delta = 32$. So we get $\tau_{32} = 3 + \sqrt{8}$. Thus the set of representatives is Rep = {[0 3]} and $M = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$ by (16). Here we notice that [0 3] M^n generates all integer solutions (x_n, y_n) of $8x^2 - y^2 = -9$ for $n \ge 1$. It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 8B_n \\ B_n & C_n \end{bmatrix}$$

for $n \ge 1$. Thus the set of all integer solutions is $\Omega = \{(3B_n, 3C_n) : n \ge 1\}.$

For the second Pell equation $8x^2 - w^2 = 7$, we get $\tau_{32} = 3 + \sqrt{8}$. So the set of representatives is Rep = { $[\pm 1 \ 1]$ } and in this case $\begin{bmatrix} 1 \ 1\end{bmatrix}M^{n-1}$ generates all integer solutions (x_{2n-1}, w_{2n-1}) and $\begin{bmatrix} 1 \ -1\end{bmatrix}M^n$ generates all integer solutions (x_{2n}, w_{2n}) for $n \ge 1$. Thus the result is obvious.

From Theorem 2.1, we can give the following theorem.

Theorem 2.2. The general terms of almost balancing and almost Lucas-balancing numbers of first type are

$$B_n^* = 3B_n, \ C_n^* = 3C_n$$

for $n \ge 1$, and the general terms of almost balancing and almost Lucas-balancing numbers of second type are

$$B_{2n-1}^{**} = B_{n-1} + C_{n-1}, \ B_{2n}^{**} = -B_n + C_n$$
$$C_{2n-1}^{**} = 8B_{n-1} + C_{n-1}, \ C_{2n}^{**} = 8B_n - C_n$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of $8x^2 - y^2 = -9$ is $\Omega = \{(3B_n, 3C_n) : n \ge 1\}$. Since $x = B_n^*$, we get $B_n^* = 3B_n$. So from (10), we deduce that

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} = \sqrt{8(3B_n)^2 + 9} = 3\sqrt{8B_n^2 + 1} = 3C_n$$

Similarly since the set of all integer solutions of $8x^2 - w^2 = 7$ is

$$\Omega = \{ (B_{n-1} + C_{n-1}, 8B_{n-1} + C_{n-1}) : n \ge 1 \} \cup \{ (-B_n + C_n, 8B_n - C_n) : n \ge 1 \},\$$

we get $B_{2n-1}^{**} = B_{n-1} + C_{n-1}, C_{2n-1}^{**} = 8B_{n-1} + C_{n-1}, B_{2n}^{**} = -B_n + C_n$ and $C_{2n}^{**} = 8B_n - C_n$ for $n \ge 1$.

Here we note that $B_0^* = 0$, $C_0^* = 3$, $B_0^{**} = 1$, $C_0^{**} = -1$. Also since $8(1)^2 - 7 = 1$ and $8(2)^2 - 7 = 5^2$ are perfect squares by (9), we accept 1 and 2 be almost balancing numbers of second type.

2.2 Almost Cobalancing and Almost Lucas-Cobalancing Numbers of First and Second Type.

In this subsection, we will determine the general terms of almost cobalancing and almost Lucas-cobalancing numbers of first and second type. Since n is an almost cobalancing number of first type if and only if $8n^2 + 8n + 9$ is a perfect square by (12) and n is an almost cobalancing number of second type if and only if $8n^2 + 8n - 7$ is a perfect square by (13), we set $8n^2 + 8n + 9 = y^2$ and $8n^2 + 8n - 7 = w^2$ for some positive integers y and w. Then we get the equations $2(2n + 1)^2 - y^2 = -7$ and $2(2n + 1)^2 - w^2 = 9$. Taking 2n + 1 = x, we get the Pell equations

$$2x^2 - y^2 = -7$$
 and $2x^2 - w^2 = 9.$ (17)

For the set of all integer solutions of (17), we can can give the following theorem.

Theorem 2.3. The set of all integer solutions of $2x^2 - y^2 = -7$ is

$$\Omega = \{ (6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \ge 1 \} \cup \{ (6B_n - C_n, -4B_n + 3C_n) : n \ge 1 \}$$

and the set of all integer solutions of $2x^2 - w^2 = 9$ is

$$\Omega = \{ (6B_{n-1} + 3C_{n-1}, 12B_{n-1} + 3C_{n-1}) : n \ge 1 \}.$$

Proof. For the Pell equation $2x^2 - y^2 = -7$, we get $F(x, y) = 2x^2 - y^2$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$ and hence the set of representatives is Rep = { $[\pm 1 \ 3]$ } and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here $\begin{bmatrix} 1 & 3 \end{bmatrix} M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) and $\begin{bmatrix} -1 & 3 \end{bmatrix} M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. Since the nth power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \ge 1$, we deduce that the set of all integer solutions is

$$\Omega = \{ (6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \ge 1 \} \cup \{ (6B_n - C_n, -4B_n + 3C_n) : n \ge 1 \}$$

For the second Pell equation $2x^2 - w^2 = 9$, we get $\tau_8 = 3 + 2\sqrt{2}$ and the set of representatives is Rep = {[± 3 3]}. In this case [3 3] M^{n-1} generates all integer solutions (x_n, w_n) for $n \ge 1$. Thus the result is obvious.

From Theorem 2.3, we can give the following theorem.

Theorem 2.4. The general terms of almost cobalancing and almost Lucas-cobalancing numbers of first type are

$$b_{2n}^* = 2b_{n+1} - b_n, \ b_{2n-1}^* = 4b_n - b_{n-1} + 1,$$

 $c_{2n}^* = c_{n+2} - 4c_{n+1}, \ c_{2n-1}^* = c_{n+1} - 2c_n$

for $n \geq 1$, and the general terms of almost cobalancing and almost Lucas-cobalancing numbers of second type are

$$b_n^{**} = 3b_n + 1, \ c_n^{**} = 3c_n$$

for $n \geq 1$.

Proof. We proved in Theorem 2.3 that the set of all integer solutions of $2x^2 - y^2 = -7$ is $\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \ge 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \ge 1\}$. Since x = 2n + 1, we get

$$b_{2n}^{*} = \frac{6B_n + C_n - 1}{2}$$

$$= \frac{6(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}) + \frac{\alpha^{2n} + \beta^{2n}}{2} - 1}{2}$$

$$= \frac{\alpha^{2n}(2\alpha - \alpha^{-1}) + \beta^{2n}(-2\beta + \beta^{-1})}{4\sqrt{2}} - \frac{1}{2}$$

$$= 2(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}) - (\frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2})$$

$$= 2b_{n+1} - b_n.$$

Thus from (14), we get

$$\begin{split} c_{2n}^* &= \sqrt{8(b_{2n}^*)^2 + 8b_{2n}^* + 9} \\ &= \sqrt{8(2b_{n+1} - b_n)^2 + 8(2b_{n+1} - b_n) + 9} \\ &= \sqrt{\alpha^{4n} \left(\frac{11 + 6\sqrt{2}}{4}\right) + \beta^{4n} \left(\frac{11 - 6\sqrt{2}}{4}\right) + \frac{7}{2}} \\ &= \sqrt{\left(\frac{\alpha^{2n+3} + \beta^{2n+3}}{2}\right)^2 - 4\left(\frac{\alpha^{2n+3} + \beta^{2n+3}}{2}\right)(\alpha^{2n+1} + \beta^{2n+1}) + 4(\alpha^{2n+1} + \beta^{2n+1})^2} \\ &= \sqrt{\left[\left(\frac{\alpha^{2n+3} + \beta^{2n+3}}{2} - 4\left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right)\right)\right]^2} \\ &= \frac{\alpha^{2n+3} + \beta^{2n+3}}{2} - 4\left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right) \\ &= c_{n+2} - 4c_{n+1}. \end{split}$$

The others can be proved similarly.

Here we note that $b_0^* = 0$, $c_0^* = 3$, $b_0^{**} = 1$ and $c_0^{**} = 3$. Also since $8(1)^2 + 8(1) - 7 = 3^2$ is a perfect square by (13), we accept 1 to be an almost cobalancing number of second type.

In Theorems 2.2 and 2.4, we deduce the general terms of all almost balancing numbers of first and second type in terms of balancing numbers. Conversely, we can deduce the general terms of all balancing numbers in terms of all almost balancing numbers of first and second type as follows:

Theorem 2.5. The general terms of all balancing numbers are

$$B_n = \frac{B_n^*}{3}, \ b_n = \frac{b_{2n-1}^* - b_{2n-2}^* - 1}{2}, \ C_n = \frac{C_n^*}{3}, \ c_n = \frac{c_{2n-1}^* - c_{2n-2}^*}{2}$$

for $n \geq 1$, or

$$B_n = \frac{B_{2n+1}^{**} - B_{2n}^{**}}{2}, \ b_n = \frac{b_n^{**} - 1}{3}, \ C_n = \frac{C_{2n+1}^{**} - C_{2n}^{**}}{2}, \ c_n = \frac{c_n^{**}}{3}$$

for $n \geq 1$.

Proof. The result is obvious from Theorems 2.2 and 2.4.

Thus we construct a one-to-one correspondence between all balancing numbers and all almost balancing numbers of first and second type. Moreover, the general terms of all almost balancing numbers of first type can be given in terms of all almost balancing numbers of second type and conversely the general terms of all almost balancing numbers of second type can be given in terms of all almost balancing numbers of first type as follows.

Theorem 2.6. The general terms of all almost balancing numbers of first type are

$$B_n^* = \frac{3B_{2n+1}^{**} - 3B_{2n}^{**}}{2}, \ C_n^* = \frac{3C_{2n+1}^{**} - 3C_{2n}^{**}}{2},$$

$$b_{2n-1}^* = \frac{4b_n^{**} - b_{n-1}^{**}}{3}, \ b_{2n}^* = \frac{2b_{n+1}^{**} - b_n^{**} - 1}{3},$$

$$c_{2n-1}^* = \frac{c_{n+1}^{**} - 2c_n^{**}}{3}, \ c_{2n}^* = \frac{c_{n+2}^{**} - 4c_{n+1}^{**}}{3}$$

for $n \geq 1$, and the general terms of all almost balancing numbers of second type are

$$B_{2n-1}^{**} = \frac{B_{n-1}^{*} + C_{n-1}^{*}}{3}, \quad B_{2n}^{**} = \frac{-B_{n}^{*} + C_{n}^{*}}{3},$$

$$b_{n}^{**} = \frac{3b_{2n-1}^{*} - 3b_{2n-2}^{*} - 1}{2}, \quad c_{n}^{**} = \frac{3c_{2n-1}^{*} - 3c_{2n-2}^{*}}{2},$$

$$C_{2n-1}^{**} = \frac{8B_{n-1}^{*} + C_{n-1}^{*}}{3}, \quad C_{2n}^{**} = \frac{8B_{n}^{*} - C_{n}^{*}}{3}$$

for $n \geq 1$.

Proof. Since $B_n^* = 3B_n$ and $B_n = \frac{B_{2n+1}^{**} - B_{2n}^{**}}{2}$ by Theorems 2.2 and 2.5, we deduce that $B_n^* = \frac{3B_{2n+1}^{**} - 3B_{2n}^{**}}{2}$. The others can be proved similarly.

Thus we construct a one-to-one correspondence between all almost balancing numbers of first type and all almost balancing numbers of second type.

3 Relationship with Pell Numbers.

In this section, we consider the relationship between all almost balancing numbers of first and second type and Pell numbers. It is known that the general terms of all balancing numbers can be given in terms of Pell numbers, namely

$$B_n = \frac{P_{2n}}{2}, b_n = \frac{P_{2n-1} - 1}{2}, C_n = P_{2n} + P_{2n-1}, c_n = P_{2n-1} + P_{2n-2}$$
(18)

for $n \ge 1$. Similarly we can give the following theorem.

Theorem 3.1. The general terms of all almost balancing numbers of first type are

$$B_n^* = \frac{3P_{2n}}{2}, \ b_{2n}^* = \frac{4P_{2n} + P_{2n-1} - 1}{2}, \ C_n^* = 3P_{2n} + 3P_{2n-1}, \\ c_{2n-1}^* = 5P_{2n-1} + P_{2n-2}, \ c_{2n}^* = 3P_{2n+1} - P_{2n}$$

for $n \geq 1$, and $b_{2n-1}^* = \frac{8P_{2n-2}+3P_{2n-3}-1}{2}$ for $n \geq 2$, and the general terms of all almost balancing numbers of second type are

$$B_{2n}^{**} = \frac{P_{2n} + 2P_{2n-1}}{2}, \ b_n^{**} = \frac{3P_{2n-1} - 1}{2}$$
$$C_{2n}^{**} = 3P_{2n} - P_{2n-1}, \ c_n^{**} = 3P_{2n-1} + 3P_{2n-2}$$

for $n \ge 1$, and $B_{2n-1}^{**} = \frac{3P_{2n-2}+2P_{2n-3}}{2}, C_{2n-1}^{**} = 5P_{2n-2} + P_{2n-3}$ for $n \ge 2$.

Proof. Note that $B_n^* = 3B_n$ and $B_n = \frac{P_{2n}}{2}$. So $B_n^* = \frac{3P_{2n}}{2}$. Since $B_{2n-1}^{**} = B_{n-1} + C_{n-1}$ by Theorem 2.2 and $B_n = \frac{P_{2n}}{2}$, $C_n = P_{2n} + P_{2n-1}$ by (18), we easily get

$$B_{2n-1}^{**} = \frac{P_{2n-2}}{2} + P_{2n-2} + P_{2n-3} = \frac{3P_{2n-2} + 2P_{2n-3}}{2}$$

for $n \ge 2$ as we wanted. The other cases can be proved similarly.

In Theorem 3.1, we can give the general terms of all almost balancing numbers of first and second type in terms of Pell numbers. Conversely, we can give the general terms of Pell numbers in terms of almost balancing numbers of first and second type as follows:

Theorem 3.2. The general terms of Pell numbers are $P_{2n} = \frac{2B_n^*}{3}$ and $P_{2n-1} = b_{2n-1}^* - b_{2n-2}^*$ for $n \ge 1$, or $P_{2n} = B_{2n+1}^{**} - B_{2n}^{**}$ and $P_{2n-1} = \frac{2b_n^{**}+1}{3}$ for $n \ge 1$.

Proof. It can be easily deduced from Theorem 3.1.

Thus we construct a one-to-one correspondence between all almost balancing numbers of first and second type and Pell numbers.

4 Concluding Remarks

For almost balancing and almost Lucas-balancing numbers of first and second type, in [13] Panda and Panda proved in Theorem 3.1 that the solutions of the Diophantine equation $8x^2 + 9 = y^2$ in positive integers are given by $x = 3B_n$ and $y = 3C_n$ for $n \ge 1$. Similarly they proved in Theorem 3.2 that the solutions of the Diophantine equation $8x^2 - 7 = y^2$ in positive integers constitute two classes: the first class is $(x, y) = (B_n - 2B_{n-1}, C_n - 2C_{n-1})$, and the second class is $(x, y) = (2B_n - B_{n-1}, 2C_n - C_{n-1})$, for $n \ge 1$. Since $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$ and $C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$, we easily deduce that

$$B_n - 2B_{n-1} = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} - 2\left(\frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}}\right)$$
$$= \frac{\alpha^{2n-2}(1+2\sqrt{2}) + \beta^{2n-2}(-1+2\sqrt{2})}{4\sqrt{2}}$$
$$= \frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}} + \frac{\alpha^{2n-2} + \beta^{2n-2}}{2}$$
$$= B_{n-1} + C_{n-1}$$
$$= B_{2n-1}^{**}$$

for $n \ge 1$. Similarly it can be shown that:

$$C_n - 2C_{n-1} = 8B_{n-1} + C_{n-1} = C_{2n-1}^{**}, \qquad 2B_n - B_{n-1} = -B_n + C_n = B_{2n}^{**}, 2C_n - C_{n-1} = 8B_n - C_n = C_{2n}^{**},$$

for $n \ge 1$, that is, we get same result in Theorem 2.4. Similarly for the almost cobalancing numbers of first and second type in [15], Panda proved in Theorem 4.3.1 that the values of x satisfying the Diophantine equation $8x^2 + 8x + 9 = y^2$ in positive integers partition in two classes. The first class is given by $U_n = \frac{3B_n + B_{n-1} - 1}{2}$ and the second class is $V_n = \frac{3B_n + B_{n+1} - 1}{2}$ for $n \ge 1$. Here we notice that

$$U_n = \frac{3B_n + B_{n-1} - 1}{2}$$

= $\frac{3(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}) + \frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}} - 1}{2}$
= $\frac{6(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}) - \frac{\alpha^{2n} + \beta^{2n}}{2} - 1}{2}$
= $\frac{6B_n - C_n - 1}{2}$
= $4b_n - b_{n-1} + 1$
= b_{2n-1}^*

and similarly it can be shown that $V_n = b_{2n}^*$ as we proved in Theorem 2.4. But he did not determine the general terms of almost Lucas-cobalancing numbers of first and second type. Apart from these in this paper,

- 1. we determined the general terms of almost Lucas-cobalancing numbers of first and second type in Theorem 2.4.
- 2. we can give the general terms of all balancing numbers in terms of all almost balancing numbers of first and second type in Theorem 2.5. Thus we construct a one-to-one correspondence between all balancing numbers and all almost balancing numbers of first and second type.
- 3. We can give the general terms of all almost balancing numbers of first type in terms of all almost balancing numbers of second type and conversely give the general terms of all almost balancing numbers of second type in terms of all almost balancing numbers of first type in Theorem 2.6. Thus, we construct a one-to-one correspondence between all almost balancing numbers of first type and of second type.
- 4. We can give the general terms of all almost balancing numbers of first and second type in terms of Pell numbers in Theorem 3.1 and conversely give the general terms of Pell numbers in terms of almost balancing numbers of first and second type in Theorem 3.2. Thus, we construct a one-to-one correspondence between all almost balancing numbers of first and second type and Pell numbers.

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