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On commutativity of prime rings with skew derivations

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Abstract. Let $\mathscr{R}$ be a prime ring of $\mathrm{Char}(\mathscr{R}) \neq 2$ and $m \neq 1$ be a positive integer. If S is a nonzero skew derivation with an associated automorphism $\mathscr{T}$ of $\mathscr{R}$ such that $([S([a,b]),[a,b]])^m = [S([a,b]),[a,b]]$ for all $a, b \in \mathscr{R}$, then $\mathscr{R}$ is commutative.

1 Introduction

In all that follows, unless specifically stated otherwise, $\mathscr{R}$ will be an associative ring, $Z(\mathscr{R})$ the center of $\mathscr{R}$, $\mathscr{Q}$ its Martindale quotient ring and U its Utumi quotient ring. The center $\mathscr{C}$ of $\mathscr{Q}$ or U, called the extended centroid of $\mathscr{R}$, is a field (see [3] for further details). For any $a, b \in \mathscr{R}$, the symbol $[a,b]$ denotes the Lie product $ab - ba$. Recall that a ring $\mathscr{R}$ is prime if for any $a, b \in \mathscr{R}$, $a\mathscr{R}b = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in \mathscr{R}$, $a\mathscr{R}a = (0)$ implies $a = 0$. An additive subgroup $\mathscr{L}$ of $\mathscr{R}$ is said to be a Lie ideal of $\mathscr{R}$ if $[l, r] \in \mathscr{L}$ for all $l \in \mathscr{L}$ and $r \in \mathscr{R}$. By a derivation of $\mathscr{R}$, we mean an additive map $d : \mathscr{R} \longrightarrow \mathscr{R}$ such that $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in \mathscr{R}$. An additive map $F : \mathscr{R} \longrightarrow \mathscr{R}$ is called a generalized derivation if there exists a derivation $d : \mathscr{R} \longrightarrow \mathscr{R}$ such that $F(ab) = F(a)b + ad(b)$ holds for all $a, b \in \mathscr{R}$, and d is called the associated derivation of F. The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^\tau X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^\tau$ is the sign of a permutation τ of the symmetric group of degree 4.

It is well known that any automorphism of $\mathscr{R}$ can be uniquely extended to an automorphism of $\mathscr{Q}$. An automorphism $\mathscr{T}$ of $\mathscr{R}$ is called $\mathscr{Q}$-inner if there exists an invertible

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element $\alpha \in \mathscr{Q}$ such that $\mathscr{T}(a) = \alpha a \alpha^{-1}$ for every $a \in \mathscr{R}$. Otherwise, $\mathscr{T}$ is called $\mathscr{Q}$-outer. Following [10], an additive map $S : \mathscr{R} \to \mathscr{R}$ is said to be a skew derivation if there exists an automorphism $\mathscr{T}$ of $\mathscr{R}$ such that $S(ab) = S(a)b + \mathscr{T}(a)S(b)$ holds for every $a, b \in \mathscr{R}$. It is easy to see that if $\mathscr{T} = 1_{\mathscr{R}}$, where $1_{\mathscr{R}}$ the identity map on $\mathscr{R}$, then a skew derivation is just a usual derivation. If $\mathscr{T} \neq 1_{\mathscr{R}}$, then $\mathscr{T} - 1_{\mathscr{R}}$ is a skew derivation. Given any $b \in \mathscr{Q}$, obviously the map $S : a \in \mathscr{R} \to ba - \mathscr{T}(a)b$ defines a skew derivation of $\mathscr{R}$, called $\mathscr{Q}$-inner skew derivation. If a skew derivation S is not $\mathscr{Q}$-inner, then it is called $\mathscr{Q}$-outer. Hence the concept of skew derivations unites the notions of derivations and automorphisms, which have been examined many algebraists from diverse points of view (see [8], [19] and [20]).

A classical result of Divinsky [14] states that if $\mathscr{R}$ is a simple Artinian ring, σ a non-identity automorphism such that $[\sigma(a), a] = 0$ for all $a \in \mathscr{R}$, then $\mathscr{R}$ must be commutative. Many authors have recently investigated and demonstrated commutativity of prime and semiprime rings using derivations, automorphisms, skew derivations, and other techniques that satisfy specific polynomial criteria (see [1], [9], [22], [23], [24] and references therein). Carini and De Filippis [4], showed if a 2-torsion free semiprime ring $\mathscr{R}$ admits a nonzero derivation d such that $[d([a, b]), [a, b]]^n = 0$ for all $a, b \in \mathscr{R}$, then there exists a central idempotent element $e \subseteq U$ such that on the direct sum decomposition $eU \bigoplus (1 - e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative. In [15], Scudo and Ansari studied the identity $[G(u), u]^n = [G(u), u]$ involving a nonzero generalized derivation G on a noncentral Lie ideal of a prime ring $\mathscr{R}$ and they described the structure of $\mathscr{R}$. Wang [25] obtained that if $\mathscr{R}$ is a prime ring, $\mathscr{L}$ a non-central Lie ideal of $\mathscr{R}$ such $[\sigma(a), a]^n = 0$ for all $a \in \mathscr{L}$, and if either $char(\mathscr{R}) > n$ or $char(\mathscr{R}) = 0$, then $\mathscr{R}$ satisfies s_4. Replaced the automorphism σ by a skew derivation d, it is proved in [12] the following result: Let $\mathscr{R}$ be a prime ring of characteristic different from 2 and 3, $\mathscr{L}$ a non-central Lie ideal of $\mathscr{R}$, d a nonzero skew derivation of $\mathscr{R}$, n is a fixed positive integer. If $[d(a), a]^n = 0$ for all $a \in \mathscr{L}$, then $\mathscr{R}$ satisfies s_4.

Motivated by the previous cited results, our aim here is to examine what happens if a prime ring $\mathscr{R}$ admits a nonzero skew derivation S such that

$$([S([a, b]), [a, b]])^m = [S([a, b]), [a, b]] \quad \text{for all } a, b \in \mathscr{R}.$$

2 Notation and Preliminaries

First, we mention some important well-known facts which are needed in the proof of our results.

Fact 1 ([2, Lemma 7.1]). *Let V_D be a vector space over a division ring D with $dimV_D \geq 2$ and $\phi \in End(V)$. If r and ϕr are D-dependent for every $r \in V$, then there exists $\lambda \in D$ such that $\phi r = \lambda r$ for every $r \in V$.*

Fact 2 ([6, Theorem 1]). *Let $\mathscr{R}$ be a prime ring and I be a two-sided ideal of $\mathscr{R}$. Then I, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identities (GPIs) with automorphisms.*

Fact 3 ([11, Fact 4]). *Let $\mathscr{R}$ be a domain and $\mathscr{T}$ be an automorphism of $\mathscr{R}$ which is outer. If $\mathscr{R}$ satisfies a GPI $\Xi(r_i, \mathscr{T}(r_i))$, then $\mathscr{R}$ also satisfies the nontrivial GPI $\Xi(r_i, s_i)$, where r_i and s_i are distinct indeterminates.*

Lemma 2.1. *Let $\mathscr{R}$ be a dense subring of the ring of linear transformations of a vector space V over a division ring D and $m \neq 1$ a positive integer. If $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ is an automorphism of $\mathscr{R}$ and $\vartheta \in \mathscr{R}$ such that*

$$([\vartheta[a,b] - \mathscr{T}([a,b])\vartheta, [a,b]])^m = [\vartheta[a,b] - \mathscr{T}([a,b])\vartheta, [a,b]],$$

for every $a, b \in \mathscr{R}$, then $dim_D V = 1$.

Proof. We have

$$([\vartheta[a,b] - \mathscr{T}([a,b])\vartheta, [a,b]])^m = [\vartheta[a,b] - \mathscr{T}([a,b])\vartheta, [a,b]],$$

for every $a, b \in \mathscr{R}$. As $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same GPIs with automorphisms by Fact 2, and hence it is a GPI for $\mathscr{Q}$. We prove it by contradiction. We assume that $dim_D V \geq 2$. There exists a semi-linear automorphism $\Phi \in End(V)$, by [17, p.79], such that $\mathscr{T}(a) = \Phi a \Phi^{-1}$ $\forall a \in \mathscr{Q}$. Hence, $\mathscr{Q}$ satisfies

$$([\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]])^m = [\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]].$$

Suppose that $\Phi u \notin span_D\{u, \Phi^{-1}\vartheta u\}$, then $\{u, \Phi u, \Phi^{-1}\vartheta u\}$ is linearly D-independent. By density theorem for $\mathscr{R}$, there exists $a, b \in \mathscr{R}$ such that

$$\begin{array}{lll} au = 0 & a\Phi^{-1}\vartheta u = 2u & a\Phi u = u \\ bu = -u & b\Phi^{-1}\vartheta u = 0 & b\Phi u = 0. \end{array}$$

The above relation gives $[a,b]u = 0$, $[a,b]\Phi^{-1}\vartheta u = 2u$ and $[a,b]\Phi u = u$. This implies that

$$(2^m - 2)u = \big(([\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]])^m - [\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]]\big)\, u = 0,$$

a contradiction.

Now, we assume that $\Phi u \in Span_D\{u, \Phi^{-1}\vartheta u\}$, then $\Phi u = u\zeta + \Phi^{-1}\vartheta u\theta$ for some $\zeta, \theta \in D$. We see that $\theta \neq 0$ otherwise if $\theta = 0$, then we get $\Phi u = u\zeta$ and hence this gives that $u = \Phi^{-1}u\zeta$. Again by density theorem for $\mathscr{R}$, $\exists a, b \in \mathscr{R}$, we have

$$\begin{array}{ll} au = 0 & a\Phi^{-1}u = 2u \\ bu = -u & b\Phi^{-1}u = 0. \end{array}$$

The above expression again gives that a contradiction

$$(2^m\theta^m - 2\theta)u = \big(([\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]])^m - [\vartheta[a,b] - \Phi[a,b]\Phi^{-1}\vartheta, [a,b]]\big)\, u = 0.$$

For $u \in V$, the set $\{u, \Phi^{-1}\vartheta u\}$ is D-dependent. By Fact 1, $\exists \Delta \in D$ such that $\Phi^{-1}\vartheta u = u\Delta$, $\forall u \in V$ and hence we have

$$\mathscr{T}(a)\vartheta u = (\Phi a \Phi^{-1})\vartheta u = \Phi a u \Delta$$

and

$$(\mathscr{T}(a)\vartheta - \vartheta a)u = \Phi(au\Delta) - \vartheta a u = \Phi(\Phi^{-1}\vartheta a u) - \vartheta a u = 0.$$

The last expression forces that $(\mathscr{T}(a)\vartheta - \vartheta a)V = (0)$ $\forall a \in \mathscr{R}$, and hence $\mathscr{T}(a)V = (0)$ $\forall a \in \mathscr{R}$ and as V is faithful, it yields that $\mathscr{T}(a) = 0$ $\forall a \in \mathscr{R}$. This is a contradiction. $\square$

3 Main Results

Proposition 3.1. *Let $m \neq 1$ be a positive integer, $\mathscr{R}$ be a prime ring of char$(\mathscr{R}) \neq 2$ and $\vartheta \in \mathscr{Q}$ such that*

$$([\mathscr{T}([a,b])\vartheta, [a,b]])^m = [\mathscr{T}([a,b])\vartheta, [a,b]].$$

Then $\vartheta \in \mathscr{C}$.

Proof. First we assume that $\mathscr{T}$ is an identity automorphism of $\mathscr{R}$. Then we have that $([[a,b]\vartheta, [a,b]])^m = [[a,b]\vartheta, [a,b]]$ is a GPI of $\mathscr{R}$. On contrary we assume that $\vartheta \notin \mathscr{C}$. Since the identity $([[a,b]\vartheta, [a,b]])^m = [[a,b]\vartheta, [a,b]]$ is satisfied by $\mathscr{Q}$ (Fact 2). As $\vartheta \notin \mathscr{C}$, then the above identity is an non-trivial GPI for $\mathscr{Q}$. By Martindale's theorem in [21], $\mathscr{Q}$ is primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over $\mathscr{C}$.

Assume that $\dim\mathscr{C}(V) = l$, where $1 < l \in \mathbb{Z}^+$. For this situation, we take $\mathscr{Q} = M_l(\mathscr{C})$ as a ring of $l \times l$ matrices over the field $\mathscr{C}$ such that $([[a,b]\vartheta, [a,b]])^m = [[a,b]\vartheta, [a,b]]$ for all $a, b \in M_l(\mathscr{C})$.

Let e_{ij} be the usual unit matrix with 1 in (i,j)-entry and zero elsewhere. First, we claim that ϑ is a diagonal matrix. Say $\vartheta = \sum_{ij} e_{ij}\vartheta_{ij}$, where $\vartheta_{ij} \in \mathscr{C}$. Choose $a = e_{ij}, b = e_{jj}$. Then by the hypothesis, we have $([e_{ij}\vartheta, e_{ij}])^m = [e_{ij}\vartheta, e_{ij}]$, i.e, $e_{ij}\vartheta_{ij} = 0$ and so $\vartheta_{ji} = 0$, for any $i \neq j$ and hence ϑ is a diagonal matrix.

Since $\xi \in Aut_{\mathscr{C}}(\mathscr{Q})$, the expression

$$([[a,b]\xi(\vartheta), [a,b]])^m = [[a,b]\xi(\vartheta), [a,b]]$$

is also a GPI for $\mathscr{Q}$, therefore $\xi(\vartheta)$ is also diagonal. The automorphism, in particular $\xi(\vartheta) = (1 + e_{ij})\vartheta(1 - e_{ij})$, for any $i \neq j$ and say $\vartheta^{\xi} = \sum_{ij} e_{ij}\vartheta'_{ij}$, where $\vartheta'_{ij} \in \mathscr{C}$. Since $\vartheta'_{ij} = 0$, then we get $0 = \vartheta'_{ij} = \vartheta_{jj} - \vartheta_{ii}$, by easy computation. So that $\vartheta_{jj} = \vartheta_{ii}$ hold for any $i \neq j$, and we get a contradiction that $\vartheta \in \mathscr{C}$.

Assume that $dim_{\mathscr{C}}V = \infty$.

$$([[a,b]\vartheta, [a,b]])^m = [[a,b]\vartheta, [a,b]], \text{ for all } a, b \in \mathscr{Q}. \tag{1}$$

By Martindale's theorem [21], it observes that $Soc(\mathscr{Q}) = F \neq (0)$ and eFe is finite dimensional simple central algebra over $\mathscr{C}$, for any minimal idempotent element $e \in F$. We can also suppose that F is non-commutative, because else $\mathscr{Q}$ must be commutative. Clearly, F satisfies $([[a,b]\vartheta, [a,b]])^m = [[a,b]\vartheta, [a,b]]$ (see, for example, the proof of [18, Theorem 1]). As F is a simple ring, either F does not contain any non-trivial idempotent element or F is generated by its idempotents. In this last case, assume that F contains two minimal orthogonal idempotent elements e and f. Using the assumption, one can see that, for $[a,b] = [ea, f] = eaf$, we have

$$eaf\vartheta eaf = 0, \tag{2}$$

in this case we get $f\vartheta eaf\vartheta eaf\vartheta e = 0$, and primeness of $\mathscr{R}$, we get $f\vartheta e = 0$ for any rank 1 orthogonal idempotent element e and f. Notably, for any rank 1 idempotent element e, we have $e\vartheta(1-e)=0$ and $(1-e)\vartheta e=0$, that is, $e\vartheta=e\vartheta e=\vartheta e$. Hence, $[\vartheta,e]=0$ gives that F is commutative or $\vartheta\in\mathscr{C}$. We get a contradiction, in this case.

Now, we consider when F cannot contain two minimal orthogonal idempotent elements and so, $F=D$ for suitable finite dimensional division ring D over its center which implies that $\mathscr{Q}=F$ and $\vartheta\in F$. By [17, Theorem 2.3.29] (see also [18, Lemma 2]), there exists a field $\mathbb{K}$ such that $F\subseteq M_n(\mathbb{K})$ and $M_n(\mathbb{K})$ satisfies $([[a,b]\vartheta,[a,b]])^m=[[a,b]\vartheta,[a,b]]$. If $n=1$ then $F\subseteq\mathbb{K}$ and we have also a contradiction. Moreover, as we have just seen, if $n\geq 2$, then $\vartheta\in Z(M_n(\mathbb{K}))$.

Finally, if F does not contain any non-trivial idempotent element, then F is finite dimensional division algebra over $\mathscr{C}$ and $\vartheta\in F=\mathscr{R}\mathscr{C}=\mathscr{Q}$. If $\mathscr{C}$ is finite, then F is finite division ring, that is, F is a commutative field and so $\mathscr{R}$ is commutative too. If $\mathscr{C}$ is infinite, then $F\bigotimes_{\mathscr{C}}\mathbb{K}\cong M_n(\mathbb{K})$, where $\mathbb{K}$ is a splitting field of F. We get the conclusion.

Henceforward, $\mathscr{T}$ is non-identity automorphism of $\mathscr{R}$. Now, we have two cases:

Case I: If $\mathscr{T}$ is $\mathscr{Q}-$inner, then there exists an invertible element α of $\mathscr{Q}$ such that $\mathscr{T}(a)=\alpha a\alpha^{-1}$ for every $a\in\mathscr{R}$. Thus, $([\alpha[a,b]\alpha^{-1}\vartheta,[a,b]])^m=[\alpha[a,b]\alpha^{-1}\vartheta,[a,b]]$ for every $a,b\in\mathscr{R}$. If $\alpha^{-1}\vartheta\in\mathscr{C}$, then $\mathscr{R}$ satisfies $([\alpha[a,b],[a,b]])^m=[\vartheta[a,b],[a,b]]$ and we get the conclusion as above. Now we assume that $\alpha^{-1}\vartheta\notin\mathscr{C}$, therefore

$$([\alpha[a,b]\alpha^{-1}\vartheta,[a,b]])^m=[\alpha[a,b],[a,b]]$$

is a non-trivial GPI for $\mathscr{R}$ and hence for $\mathscr{Q}$ by Fact 2. In light of “Martindale's theorem [21], $\mathscr{Q}$ is isomorphic to a dense subring of linear transformations of a vector space V over D, where D is a finite dimensional division ring over $\mathscr{C}$”. By Lemma 2.1, the result follows.

Case II: If $\mathscr{T}$ is $\mathscr{Q}$-outer, and $\mathscr{Q}$ satisfies $([\mathscr{T}([a,b])\vartheta,[a,b]])^m=[\mathscr{T}([a,b])\vartheta,[a,b]]$, then by Lemma 2.1 we get $dim_DV=1$, that is $\mathscr{Q}$ is a domain. By Fact 3, $\mathscr{Q}$ satisfies $[[r,s]\vartheta,[a,b]]^m=[[r,s],[a,b]]$ and in particular, for $r=a$ and $s=b$, we have $[[a,b]\vartheta,[a,b]]^m=[[a,b]\vartheta,[a,b]]$ for every $a,b\in\mathscr{Q}$. Hence, using the same technique as above we get the required conclusion. □

Theorem 3.2. *Let $\mathscr{R}$ be a prime ring of $Char(\mathscr{R})\neq 2$ and $m\neq 1$ be a positive integer. If S is a nonzero skew derivation with an associated automorphism $\mathscr{T}$ of $\mathscr{R}$ such that $([S([a,b]),[a,b]])^m=[S([a,b]),[a,b]]$ for all $a,b\in\mathscr{R}$, then $\mathscr{R}$ is commutative.*

Proof. We have

$$([S([a,b]),[a,b]])^m=[S([a,b]),[a,b]] \text{ for every} a,b\in\mathscr{R}.$$

Firstly, we assume that S is $\mathscr{Q}$-inner, that is, $S(a)=\vartheta a-\mathscr{T}(a)\vartheta$ with $0\neq\vartheta\in\mathscr{Q}$. Thus, $\forall a,b\in\mathscr{R}$, we have

$$[\vartheta[a,b]-\mathscr{T}([a,b])\vartheta,[a,b]])^m=[\vartheta[a,b]-\mathscr{T}([a,b])\vartheta,[a,b]].$$

If $\vartheta\in\mathscr{C}$, then $\mathscr{R}$ satisfies the GPI $([\mathscr{T}([a,b])\vartheta,[a,b]])^m=[\mathscr{T}([a,b])\vartheta,[a,b]]$. We get the desired conclusion, by Proposition 3.1. Therefore $\vartheta\notin\mathscr{C}$, and so

$$[\vartheta[a,b]-\mathscr{T}([a,b])\vartheta,[a,b]])^m=[\vartheta[a,b]-\mathscr{T}([a,b])\vartheta,[a,b]]$$

is nontrivial GPI for $\mathscr{R}$. Thus, Lemma 2.1 yields the required result.

Finally, when S is $\mathscr{Q}$-outer, then the above identity can be rewritten as

$$[S(a)b+\mathscr{T}(a)S(b)-S(b)a\mathscr{T}(b)S(a),[a,b]]^m=[S(a)b+\mathscr{T}(a)S(b)-S(b)a-\mathscr{T}(b)S(a),[a,b]],$$

and hence $\mathscr{R}$ satisfies

$$([\vartheta b+\mathscr{T}(a)s-sa-\mathscr{T}(b)r,[a,b]])^m=[rb+\mathscr{T}(a)s-sa-\mathscr{T}(b)r,[a,b]].$$

In particular $\mathscr{R}$ satisfies $([\mathscr{T}(a)s-sa,[a,b]])^m=[\mathscr{T}(a)s-sa,[a,b]]$. We divide it into two cases. First, $\mathscr{T}$ be an identity map of $\mathscr{R}$. Then $([[r,s],[a,b]])^m=[[r,s],[a,b]]$ for every $a,b,r,s\in\mathscr{R}$, that is, $\mathscr{R}$ is a polynomial identity ring. Thus, $\mathscr{R}$ and $M_n(\mathbb{K})$ satisfy the same polynomial identities [18, Lemma 1], i.e.,

$$([[r,s],[a,b]])^m=[[r,s],[a,b]] \quad \text{for each } a,b,r,s\in M_n(\mathbb{K}),$$

Let $n\geq 2$ and e_{ij} be the usual unit matrix. Then $r=b=e_{12}$, $s=e_{21}$ and $a=e_{11}$, we get a contradiction $2e_{12}=0$. Thus, $n=1$ and we are done.

Now consider $\mathscr{T}$ is not the identity map. Therefore,

$$([\mathscr{T}(a)s-sa,[a,b]])^m=[\mathscr{T}(a)s-sa,[a,b]]$$

is a non-trivial GPI for $\mathscr{R}$, by Main Theorem in [5]. Moreover, by Fact 2, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same GPIs with automorphisms and hence $([\mathscr{T}(a)s-sa,[a,b]])^m=[\mathscr{T}(a)s-sa,[a,b]]$ is also an identity for $\mathscr{Q}$. Since $\mathscr{R}$ is a GPI-ring, by [21] "$\mathscr{Q}$ is a primitive ring, which is isomorphic to a dense subring of the ring of linear transformations of a vector space V over a division ring D". If $\mathscr{Q}$ is a domain, then by Fact 3, we have that $\mathscr{Q}$ satisfies the equation $([ts-sa,[a,b]])^m=[ts-sa,[a,b]]$. In particular, $([[a,z],[a,b]])^m=[[a,z],[a,b]]$ for all $a,b,z\in\mathscr{Q}$, which yields that $\mathscr{Q}$ is commutative (by using the same above argument) and hence $\mathscr{R}$. Henceforth, $\mathscr{Q}$ is not a domain. We have $\mathscr{T}(a)=hah^{-1}\ \forall a\in\mathscr{Q}$, as mentioned above. Thus, $([hah^{-1}z-za,[a,b]])^m=[hah^{-1}z-za,[a,b]]$ Hence, we may consider that $dim\ D_V\geq 2$. By [17, p. 79], there exists a semi-linear automorphism $h\in End(V)$ such that $\mathscr{T}(a)=hah^{-1}\ \forall a\in\mathscr{Q}$. Hence, $\mathscr{Q}$ satisfies $([hah^{-1}z-za,[a,b]])^m=[hah^{-1}z-za,[a,b]]$.

If for any $v\in V\ \exists\ \Theta_v\in D$ such that $h^{-1}v=v\Theta_v$, then, it follows that there exists a unique $\Theta\in D$ such that $h^{-1}v=v\Theta$, $\forall v\in V$ (see for example Lemma 1 in [7]). In this case $\mathscr{T}(a)v=(hah^{-1})v=hav\Theta$ and

$$(\mathscr{T}(a)-a)v=h(av\Theta)-av=h(h^{-1}av)-av=0,$$

since V is faithful, which is a contradiction that $\mathscr{T}$ is the identity map. Thus, $\exists\ v\in V$ such that $\{v,h^{-1}v\}$ is linearly D-independent. In this case, first we assume that $dim\ V_D\geq 3$. Thus, $\exists\ u\in V$ such that $\{u,v,h^{-1}v\}$ is linearly D-independent. Hence, the density theorem for $\mathscr{Q}$, $\exists\ a,b,z\in\mathscr{Q}$ such that

$$\begin{aligned} zv&=0 & zh^{-1}v&=h^{-1}v\\ bv&=0 & bh^{-1}v&=0\\ av&=h^{-1}v & bu&=-2v\\ ah^{-1}v&=u. \end{aligned}$$

The above relation gives that

$$0 = (([hah^{-1}z - za, [a, b]])^m - [hah^{-1}z - za, [a, b]])v = (2^m - 2)v \neq 0$$

again a contradiction.

Now, the case when $dim\ V_D = 2$ that is, $\mathscr{Q} = M_2(\mathbb{K})$. Thus

$$([\mathscr{T}(a)z - za, [a, b]])^2 = [\mathscr{T}(a)z - za, [a, b]] \quad \text{for all } a, b, z \in \mathscr{Q}.$$

Since $\mathscr{T}(a)$-word of degree 2 and Char$(\mathscr{R}) > 3$ by [6, Theorem 3],

$$([tz - za, [a, b]])^2 - [tz - za, [a, b]] = 0 \quad \text{for every } t, z, a, b \in \mathscr{Q}.$$

Using the same technique as above its shows that $\mathscr{Q}$ is commutative and hence $\mathscr{R}$ is commutative. □

The following corollary is an immediate consequence of our result.

Corollary 3.3. *[13, Theorem 2.3] Let $\mathscr{R}$ be a prime ring of characteristic not two and d be a nonzero derivation of $\mathscr{R}$ such that $([d([a, b]), [a, b]])^m = [d([a, b]), [a, b]]$ for all $a, b \in \mathscr{R}$. Then $\mathscr{R}$ is commutative.*

Theorem 3.4. *Let $\mathscr{R}$ be a prime ring of Char$(\mathscr{R}) \neq 2$, $m \neq 1$ be a positive integer and $\mathscr{L}$ a Lie ideal of $\mathscr{R}$. If S is a nonzero skew derivation with an associated automorphism $\mathscr{T}$ of $\mathscr{R}$ such that $([S(v), v])^m = [S(v), v]$ for all $v \in \mathscr{L}$, then L contained in the center of $\mathscr{R}$.*

Proof. Suppose that $\mathscr{L} \nsubseteq Z(\mathscr{R})$ is a Lie ideal of $\mathscr{R}$. Then by [16], there exists an ideal I of $\mathscr{R}$ such that $0 \neq [I, \mathscr{R}] \subseteq \mathscr{L}$ and $[\mathscr{L}, \mathscr{L}] \neq (0)$. Also, $\mathscr{R} \nsubseteq Z(\mathscr{R})$ as $\mathscr{L}$ is a noncentral Lie ideal of $\mathscr{R}$. Therefore by the given hypothesis, I as well as $\mathscr{R}$ (Fact 2) satisfy $[S([a, b]), [a, b]])^m = [S([a, b]), [a, b]]$. By Theorem 3.2, we get the required result. □

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