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On bi-variate poly-Bernoulli polynomials

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Abstract. We introduce poly-Bernoulli polynomials in two variables by using a generalization of Stirling numbers of the second kind that we studied in a previous work. We prove the bi-variate poly-Bernoulli polynomial version of some known results on standard Bernoulli polynomials, as the addition formula and the binomial formula. We also prove a result that allows us to obtain poly-Bernoulli polynomial identities from polynomial identities, and we use this result to obtain several identities involving products of poly-Bernoulli and/or standard Bernoulli polynomials. We prove two generalized recurrences for bi-variate poly-Bernoulli polynomials, and obtain some corollaries from them.

1 Introduction

Bernoulli numbers are one of the most important mathematical objects that have been studied by mathemathicians since they appeared in the 18-th century (see [13]). A recent important generalization of the Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ is about the so-called poly-Bernoulli numbers and poly-Bernoulli polynomials. Poly-Bernoulli numbers $B_n^{(k)}$, where k is a given positive integer, were introduced by M. Kaneko [10] in 1997, by means of the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

where $\mathrm{Li}_k(z) = \sum_{j=1}^{\infty} z^j/j^k$ is the polylogarithm function. The case $k=1$ corresponds to the standard Bernoulli numbers B_n (except the sign of $B_1^{(1)}$). Poly-Bernoulli polynomials

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$B_n^{(k)}(x)$ can be defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-tx}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!},$$

(see [5]). The case $k=1$ corresponds to $(-1)^nB_n(x)$, and the case $x=0$ corresponds to the poly-Bernoulli numbers $B_n^{(k)}$ mentioned before. Some slightly different definitions of poly-Bernoulli polynomials $B_n^{(k)}(x)$, with x replaced by $-x$, and/or with an additional factor $(-1)^n$, can be found in some related papers (see [1], [5], [6], [9]). In this work we use the following explicit formula

$$B_n^{(k)}(x)=\sum_{l=0}^{n}\frac{1}{(l+1)^k}\sum_{j=0}^{l}(-1)^j\binom{l}{j}(j+x)^n, \tag{1}$$

as our definition of poly-Bernoulli polynomials (see formula (1.8) in [1]). It is important to mention that the notation $B_n^{(k)}(x)$ is also used for a different kind of mathematical objects, namely, Bernoulli polynomials of k-th order (see [3]).

A different generalization of Bernoulli polynomials, studied in the past few years, is about considering Bernoulli polynomials in several variables $B_{p_1,\ldots,p_t}(x_1,\ldots,x_t)$, that is, polynomials of degree p_i in the variable x_i, with

$$B_{0,\ldots,p_i,\ldots,0}(x_1,\ldots,x_t)=B_{p_i}(x_i)\qquad\text{for each } i\in\{1,\ldots,t\},$$

seeking that reasonable generalizations of the known properties in the one-variable case, remain valid. This kind of work is done in [16], with a flavor of multivariable analysis and working with Jack polynomials. A different approach is presented in [17] (see also [2], [7]).

In this work we study poly-Bernoulli polynomials in two variables (bi-variate poly-Bernoulli polynomials). We define the bi-variate poly-Bernoulli polynomials by using a generalization of Stirling numbers of the second kind we studied in [14], and then we use the results in [14] to obtain results for the bi-variate poly-Bernoulli polynomials considered in this work.

We present now the definitions and results in [14] that we will use in the remaining sections.

The generalized Stirling numbers of the second kind (GSN, for short), denoted as $S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k)$, where $a_j,b_j\in\mathbb{C}$, $a_j\neq 0$, $j=1,2$, and p_1,p_2 non-negative integers, are defined by means of the expansion

$$(a_1n+b_1)^{p_1}(a_2n+b_2)^{p_2}=\sum_{k=0}^{p_1+p_2}k!S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)\binom{n}{k}, \tag{2}$$

($S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)=0$ if $k<0$ or $k>p_1+p_2$). An explicit formula for these numbers is

$$S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(a_1(k-j)+b_1)^{p_1}(a_2(k-j)+b_2)^{p_2}. \tag{3}$$

If $p_2 = 0$, we write the GSN $S_{a,b}^{a_2,b_2,0}(p,k)$ as $S_{a,b}(p,k)$. We have

$$S_{a,b}(p,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(a\left(k-j\right)+b\right)^p. \tag{4}$$

In the case $a = 1, b = 0$, the corresponding GSN $S_{1,0}(p,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^p$ are the known Stirling numbers of the second kind. We will refer to them as "standard Stirling numbers", and in this case we use the known notation $S(p,k)$.

From (3) it is clear that $S_{a,b}^{a,b,p_2}(p_1,k) = S_{a,b}(p_1+p_2,k)$. We can see directly from (4) that

$$\begin{aligned} S_{1,1}(p,k) &= S(p+1,k+1), && (5)\\ S_{1,2}(p,k) &= S(p+2,k+2) - S(p+1,k+2). && (6) \end{aligned}$$

In this work we will use GSN of the form $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$. Some important facts about the GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ are the following:

- Some values of the GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ are

$$\begin{aligned} S_{1,x_1}^{1,x_2,p_2}(p_1,0) &= x_1^{p_1}x_2^{p_2}, \\ S_{1,x_1}^{1,x_2,p_2}(p_1,1) &= (x_1+1)^{p_1}(x_2+1)^{p_2} - x_1^{p_1}x_2^{p_2}, \\ &\vdots \\ S_{1,x_1}^{1,x_2,p_2}(p_1,p_1+p_2) &= 1. \end{aligned} \tag{7}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ can be written in terms of the GSN $S_{1,y_1}^{1,y_2,p_2}(p_1,k)$ as follows

$$S_{1,x_1}^{1,x_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}S_{1,y_1}^{1,y_2,j_2}(j_1,k). \tag{8}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ can be written in terms of standard Stirling numbers as follows

$$\begin{aligned} k!S_{1,x_1}^{1,x_2,p_2}(p_1,k) &= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-m)^{p_1-j_1}(x_2-m)^{p_2-j_2} \\ &\quad\times\sum_{i=0}^{m}\binom{m}{i}(k+i)!S(j_1+j_2,k+i), \end{aligned} \tag{9}$$

where m is an arbitrary non-negative integer.

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ can be written in terms of standard Stirling numbers as follows

$$S_{1,x_1}^{1,x_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-n)^{p_1-j_1}(x_2-n)^{p_2-j_2} \times\sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,k+n), \quad (10)$$

  where n is an arbitrary positive integer, and $s(\cdot,\cdot)$ are the unsigned Stirling numbers of the first kind.

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ satisfy the identity

$$S_{1,x_1+1}^{1,x_2+1,p_2}(p_1,k) = S_{1,x_1}^{1,x_2,p_2}(p_1,k)+(k+1)S_{1,x_1}^{1,x_2,p_2}(p_1,k+1). \quad (11)$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ satisfy the recurrence

$$S_{1,x_1}^{1,x_2,p_2}(p_1,k) = S_{1,x_1}^{1,x_2,p_2}(p_1-1,k-1)+(k+x_1)S_{1,x_1}^{1,x_2,p_2}(p_1-1,k). \quad (12)$$

2 Definitions and preliminary results

The relation of Bernoulli (numbers and polynomials) with Stirling (numbers of the second kind) is an old story, that dates back to Worpitsky [18] (see also [8, p. 560] and [11, p. 5]). We have the following formula for Bernoulli numbers

$$B_p = \sum_{l=0}^{p} S(p,l)\frac{(-1)^l l!}{l+1}, \quad (13)$$

and in the case of Bernoulli polynomials we have

$$B_p(x) = \sum_{l=0}^{p}\sum_{j=0}^{p}\binom{p}{j}x^{p-j}S(j,l)\frac{(-1)^l l!}{l+1}. \quad (14)$$

An important observation of formula (1) is that poly-Bernoulli polynomial $B_p^{(k)}(x)$ can be written in terms of the GSN as

$$B_p^{(k)}(x) = \sum_{l=0}^{p} S_{1,x}(p,l)\frac{(-1)^l l!}{(l+1)^k}. \quad (15)$$

The generalization of (15) to the case of two variables comes through the GSN: we define poly-Bernoulli polynomial in the variables x_1, x_2, denoted by $B_{p_1,p_2}^{(k)}(x_1,x_2)$, as

$$B_{p_1,p_2}^{(k)}(x_1,x_2) = \sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,l)\frac{(-1)^l l!}{(l+1)^k}. \quad (16)$$

If $p_2 = 0$, formula (16) becomes (15). By using (9) we can write $B^{(k)}_{p_1,p_2}(x_1, x_2)$ in terms of standard Stirling numbers as

$$\begin{aligned} B^{(k)}_{p_1,p_2}(x_1, x_2) &= \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - m)^{p_1-j_1} (x_2 - m)^{p_2-j_2} \\ &\quad \times \sum_{i=0}^{m} \binom{m}{i} S(j_1 + j_2, l + i) \frac{(-1)^l (l + i)!}{(l + 1)^k}, \end{aligned} \tag{17}$$

where m is an arbitrary non-negative integer. Similarly, by using (10) we have that

$$\begin{aligned} B^{(k)}_{p_1,p_2}(x_1, x_2) &= \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - n)^{p_1-j_1} (x_2 - n)^{p_2-j_2} \\ &\quad \times \sum_{i=0}^{n-1} (-1)^i s(n, n - i) S(j_1 + j_2 + n - i, l + n) \frac{(-1)^l l!}{(l + 1)^k}, \end{aligned} \tag{18}$$

where n is an arbitrary positive integer.

The simplest cases of (17) and (18) are

$$B^{(k)}_{p_1,p_2}(x_1, x_2) = \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1 + j_2, l) \frac{(-1)^l l!}{(l + 1)^k}, \tag{19}$$

and

$$\begin{aligned} &B^{(k)}_{p_1,p_2}(x_1, x_2) \\ &= \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - 1)^{p_1-j_1} (x_2 - 1)^{p_2-j_2} S(j_1 + j_2 + 1, l + 1) \frac{(-1)^l l!}{(l + 1)^k}, \end{aligned} \tag{20}$$

respectively.

Two examples are the following

$$\begin{aligned} B^{(k)}_{1,1}(x_1, x_2) &= \frac{2}{3^k} - \frac{1}{2^k}(x_1 + x_2 + 1) + x_1 x_2, \\ B^{(k)}_{1,2}(x_1, x_2) &= \frac{1}{3^k}(2x_1 + 4x_2 + 6) - \frac{1}{2^k}\left(x_1(2x_2 + 1) + (x_2 + 1)^2\right) + x_1 x_2^2 - \frac{6}{4^k}. \end{aligned}$$

Clearly we have

$$B^{(k)}_{0,0}(x_1, x_2) = 1. \tag{21}$$

Observe also that

$$B^{(k)}_{p_1,p_2}(x, x) = B^{(k)}_{p_1+p_2}(x). \tag{22}$$

In particular, we have

$$B^{(k)}_{p_1,p_2}(0, 0) = B^{(k)}_{p_1+p_2}, \tag{23}$$

From (3) and (16) we have

$$B^{(k)}_{p_1,p_2}(x_1,x_2)=\sum_{l=0}^{p_1+p_2}\sum_{j=0}^{l}(-1)^j\binom{l}{j}(l-j+x_1)^{p_1}(l-j+x_2)^{p_2}\frac{(-1)^l}{(l+1)^k}, \tag{24}$$

from where we see that

$$\begin{aligned}\frac{\partial}{\partial x_1}B^{(k)}_{p_1,p_2}(x_1,x_2) &= p_1B^{(k)}_{p_1-1,p_2}(x_1,x_2),\\ \frac{\partial}{\partial x_2}B^{(k)}_{p_1,p_2}(x_1,x_2) &= p_2B^{(k)}_{p_1,p_2-1}(x_1,x_2).\end{aligned}$$

We can use (8) to write $B^{(k)}_{p_1,p_2}(x_1,x_2)$ in terms of $B^{(k)}_{j_1,j_2}(y_1,y_2)$, $0\le j_i\le p_i$, $i=1,2$, as

$$B^{(k)}_{p_1,p_2}(x_1,x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}B^{(k)}_{j_1,j_2}(y_1,y_2), \tag{25}$$

that generalizes the known addition formula $B^{(k)}_p(x)=\sum_{j=0}^{p}\binom{p}{j}(x-y)^{p-j}B^{(k)}_j(y)$ for one-variable poly-Bernoulli polynomials. In fact, we have

$$\begin{aligned}B^{(k)}_{p_1,p_2}(x_1,x_2)&=\sum_{l=0}^{p_1+p_2}S^{1,x_2,p_2}_{1,x_1}(p_1,l)\frac{(-1)^l l!}{(l+1)^k}\\ &=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}\sum_{l=0}^{j_1+j_2}S^{1,y_2,j_2}_{1,y_1}(j_1,l)\frac{(-1)^l l!}{(l+1)^k}\\ &=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}B^{(k)}_{j_1,j_2}(y_1,y_2),\end{aligned}$$

as claimed. In particular, if we set $y_1=y_2=y$ in (25) we obtain an expression for the bi-variate poly-Bernoulli polynomial $B^{(k)}_{p_1,p_2}(x_1,x_2)$ in terms of one-variable poly-Bernoulli polynomials $B^{(k)}_j(y)$, $0\le j\le p_1+p_2$, namely

$$B^{(k)}_{p_1,p_2}(x_1,x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y)^{p_1-j_1}(x_2-y)^{p_2-j_2}B^{(k)}_{j_1+j_2}(y), \tag{26}$$

and then we can write the bi-variate poly-Bernoulli polynomial $B^{(k)}_{p_1,p_2}(x_1,x_2)$ in terms of poly-Bernoulli numbers $B^{(k)}_j$, $0\le j\le p_1+p_2$, as

$$B^{(k)}_{p_1,p_2}(x_1,x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}x_1^{p_1-j_1}x_2^{p_2-j_2}B^{(k)}_{j_1+j_2}. \tag{27}$$

The cases $k=0$ and $k=-1$ of (27) are

$$B_{p_1,p_2}^{(0)}(x_1,x_2)=(x_1-1)^{p_1}(x_2-1)^{p_2}\,, \tag{28}$$

and

$$B_{p_1,p_2}^{(-1)}(x_1,x_2)=(x_1-2)^{p_1}(x_2-2)^{p_2}\,, \tag{29}$$

respectively. In fact, according to (16), formulas (28) and (29) are the particular cases $r=0$ and $r=1$ of the identity

$$\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,l)(-1)^l(l+r)!=r!\,(x_1-r-1)^{p_1}(x_2-r-1)^{p_2}\,, \tag{30}$$

where r is a non-negative integer. We leave the proof of (30) to the reader.

Observe also that (25) implies that

$$B_{p_1,p_2}^{(k)}(x_1+1,x_2+1)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{j_1,j_2}^{(k)}(x_1,x_2), \tag{31}$$

which generalizes the known binomial formula for standard Bernoulli polynomials

$$B_p(x+1)=\sum_{j=0}^{p}\binom{p}{j}B_j(x). \tag{32}$$

If we set $x_1=x_2=x$ in (25), we obtain a formula for the standard poly-Bernoulli polynomial $B_{p_1+p_2}^{(k)}(x)$ in terms of the bi-variate poly-Bernoulli polynomials $B_{j_1,j_2}^{(k)}(y_1,y_2)$, $0\le j_i\le p_i$, $i=1,2$, namely

$$B_{p_1+p_2}^{(k)}(x)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x-y_1)^{p_1-j_1}(x-y_2)^{p_2-j_2}B_{j_1,j_2}^{(k)}(y_1,y_2). \tag{33}$$

Some additional simple observations are the following

$$B_{p_1,0}^{(k)}(x_1,x_2) = B_{p_1}^{(k)}(x_1), \tag{34}$$

$$B_{0,p_2}^{(k)}(x_1,x_2) = B_{p_2}^{(k)}(x_2), \tag{35}$$

and

$$B_{p_1,p_2}^{(k)}(0,x_2) = \sum_{j_2=0}^{p_2}\binom{p_2}{j_2}x_2^{p_2-j_2}B_{p_1+j_2}^{(k)}, \tag{36}$$

$$B_{p_1,p_2}^{(k)}(x_1,0) = \sum_{j_1=0}^{p_1}\binom{p_1}{j_1}x_1^{p_1-j_1}B_{j_1+p_2}^{(k)}. \tag{37}$$

3 Some identities

In this section we obtain some identities involving poly-Bernoulli polynomials, by using the following result:

Theorem 3.1. *The polynomial identity*

$$\sum_{r=0}^{n} a_{n,r}(x+\alpha)^{r} = \sum_{r=0}^{n} b_{n,r}(x+\beta)^{r}. \tag{38}$$

implies the poly-Bernoulli polynomial identity

$$\sum_{r=0}^{n} a_{n,r}B_{r}^{(k)}(x+\alpha) = \sum_{r=0}^{n} b_{n,r}B_{r}^{(k)}(x+\beta). \tag{39}$$

Proof. Observe that the hypothesis of the polynomial identity (38), comes together with the identity of its derivatives:

$$\sum_{r=0}^{n}\binom{r}{j}a_{n,r}(x+\alpha)^{r-j} = \sum_{r=0}^{n}\binom{r}{j}b_{n,r}(x+\beta)^{r-j}$$

where j is a non-negative integer.

We have

$$\begin{aligned}
\sum_{r=0}^{n} a_{n,r}B_{r}^{(k)}(x+\alpha) &= \sum_{r=0}^{n} a_{n,r}\sum_{l=0}^{r} S_{1,x+\alpha}(r,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{r=0}^{n} a_{n,r}\sum_{l=0}^{r}\sum_{j=0}^{r}\binom{r}{j}(x+\alpha)^{r-j}S(j,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{l=0}^{n}\sum_{j=0}^{n}\left(\sum_{r=0}^{n}\binom{r}{j}a_{n,r}(x+\alpha)^{r-j}\right)S(j,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{l=0}^{n}\sum_{j=0}^{n}\left(\sum_{r=0}^{n}\binom{r}{j}b_{n,r}(x+\beta)^{r-j}\right)S(j,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{r=0}^{n} b_{n,r}\sum_{l=0}^{r}\sum_{j=0}^{r}\binom{r}{j}(x+\beta)^{r-j}S(j,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{r=0}^{n} b_{n,r}\sum_{l=0}^{r} S_{1,x+\beta}(r,l)\frac{(-1)^{l}l!}{(l+1)^{k}} \\
&= \sum_{r=0}^{n} b_{n,r}B_{r}^{(k)}(x+\beta),
\end{aligned}$$

as desired. □

Remark 3.2. The case $k = 1$ of Theorem 3.1 is an old result: based on [12], we obtained Theorem 3.1 in the case $k = 1$ and we used it to generate several identities in [15].

For example, by using Theorem 3.1 in the trivial identity $x^p = \sum_{j=0}^{p} \binom{p}{j}(x-y)^j y^{p-j}$ we obtain the addition formula for poly-Bernoulli polynomials

$$B_p^{(k)}(x) = \sum_{j=0}^{p} \binom{p}{j}(x-y)^{p-j} B_j^{(k)}(y), \tag{40}$$

that we can write as

$$\sum_{j=0}^{p} \binom{p}{j} x^{p-j} B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j}(x-y)^{p-j} B_j^{(k)}(y). \tag{41}$$

We can use again Theorem 3.1 to obtain from (41) that

$$\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x) B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x-y) B_j^{(k)}(y). \tag{42}$$

Set $y = x$ in (42) to get the identity

$$\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x) B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)} B_j^{(k)}(x). \tag{43}$$

If we set $k = 1$ in (43), and replace x by $x + 1$, we obtain

$$\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x+1) B_j = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)} B_j(x+1). \tag{44}$$

By using that $B_j(x+1) = B_j(x) + jx^{j-1}$ together with (43), we obtain from (44), after some elementary algebraic steps, the curious identity

$$\sum_{j=0}^{p} \binom{p}{j} \left(B_{p-j}^{(k_1)}(x+1) - B_{p-j}^{(k_1)}(x) \right) B_j = pB_{p-1}^{(k_1)}(x). \tag{45}$$

From (17), (18) and (19) we have that

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1}\binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1+j_2, l) \frac{(-1)^l l!}{(l+1)^{k_0}} \tag{46}$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1-m)^{p_1-j_1} (x_2-m)^{p_2-j_2}$$

$$\times \sum_{i=0}^{m}\binom{m}{i}S(j_1+j_2,l+i)\frac{(-1)^l(l+i)!}{(l+1)^{k_0}}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-n)^{p_1-j_1}(x_2-n)^{p_2-j_2}$$
$$\times \sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,l+n)\frac{(-1)^l l!}{(l+1)^{k_0}}.$$

where m, n are arbitrary integers, $m \geq 0$, $n > 0$. Now we use Theorem 3.1 in (46) and then set $x_1 = x_2 = m+n$, to obtain the identities

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(m+n)B_{p_2-j_2}^{(k_2)}(m+n)S(j_1+j_2,l)\frac{(-1)^l l!}{(l+1)^{k_0}} \tag{47}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(n)B_{p_2-j_2}^{(k_2)}(n)$$
$$\times \sum_{i=0}^{m}\binom{m}{i}S(j_1+j_2,l+i)\frac{(-1)^l(l+i)!}{(l+1)^{k_0}}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(m)B_{p_2-j_2}^{(k_2)}(m)$$
$$\times \sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,l+n)\frac{(-1)^l l!}{(l+1)^{k_0}}.$$

From (26) we see that

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y)^{p_1-j_1}(x_2-y)^{p_2-j_2}B_{j_1+j_2}^{(k_0)}(y) \tag{48}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-z)^{p_1-j_1}(x_2-z)^{p_2-j_2}B_{j_1+j_2}^{(k_0)}(z).$$

We can use Theorem 3.1 to get from (48), the identity

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x_1-y)B_{p_2-j_2}^{(k_2)}(x_2-y)B_{j_1+j_2}^{(k_0)}(y) \tag{49}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x_1-z)B_{p_2-j_2}^{(k_2)}(x_2-z)B_{j_1+j_2}^{(k_0)}(z).$$

Set $x_1 = x_2 = y = x$ to obtain from (49) that

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}B_{p_2-j_2}^{(k_2)}B_{j_1+j_2}^{(k_0)}(x) = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x-z)B_{p_2-j_2}^{(k_2)}(x-z)B_{j_1+j_2}^{(k_0)}(z). \tag{50}$$

With $z = 0$, $z = 1-(q-1)\,x$, and $z = qx-1$, where q is an arbitrary parameter, we obtain from (50) the identities

$$\begin{aligned}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}B_{p_2-j_2}^{(k_2)}B_{j_1+j_2}^{(k_0)}(x)
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x)B_{p_2-j_2}^{(k_2)}(x)B_{j_1+j_2}^{(k_0)}\\
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(qx-1)B_{p_2-j_2}^{(k_2)}(qx-1)B_{j_1+j_2}^{(k_0)}(1-(q-1)\,x)\\
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(1-(q-1)x)B_{p_2-j_2}^{(k_2)}(1-(q-1)x)B_{j_1+j_2}^{(k_0)}(qx-1).\end{aligned} \tag{51}$$

In the case $q = 2$, if some (or all) of the parameters k_0, k_1, k_2 are equal to 1, we can use the known property $B_r\,(1-x) = (-1)^r B_r(x)$ to simplify the corresponding expression in (51). For example, if $k_0 = k_1 = k_2 = 1$ and $q = 2$, we have the following identities involving standard Bernoulli numbers and polynomials

$$\begin{aligned}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}B_{p_2-j_2}B_{j_1+j_2}(x)
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}(x)B_{p_2-j_2}(x)B_{j_1+j_2}\\
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(-1)^{j_1+j_2}B_{p_1-j_1}(2x-1)B_{p_2-j_2}(2x-1)B_{j_1+j_2}(x)\\
&= (-1)^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(-1)^{j_1+j_2}B_{p_1-j_1}(x)B_{p_2-j_2}(x)B_{j_1+j_2}(2x-1).\end{aligned} \tag{52}$$

In particular, by setting $x = 1$ in (52), we see that if $p_1 + p_2$ is odd, then

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(-1)^{j_1+j_2}B_{p_1-j_1}B_{p_2-j_2}B_{j_1+j_2} = 0. \tag{53}$$

From (30) together with (9) (with $m=0$) and (10) (with $n=1$), we have

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}x_1^{p_1-j_1}x_2^{p_2-j_2}S(j_1+j_2,l)(-1)^l(l+r)! \tag{54}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-1)^{p_1-j_1}(x_2-1)^{p_2-j_2}S(j_1+j_2+1,l+1)(-1)^l(l+r)!$$
$$=r!(x_1-r-1)^{p_1}(x_2-r-1)^{p_2},$$

and then, applying Theorem 3.1 in (54) we get

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x_1)B_{p_2-j_2}^{(k_2)}(x_2)S(j_1+j_2,l)(-1)^l(l+r)! \tag{55}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}B_{p_1-j_1}^{(k_1)}(x_1-1)B_{p_2-j_2}^{(k_2)}(x_2-1)S(j_1+j_2+1,l+1)(-1)^l(l+r)!$$
$$=r!B_{p_1}^{(k_1)}(x_1-r-1)B_{p_2}^{(k_2)}(x_2-r-1),$$

where r is an arbitrary non-negative integer.

Now let us consider the difference

$$B_{p_1,p_2}^{(k_0)}(x_1+r,x_2+r)-B_{p_1,p_2}^{(k_0)}(x_1,x_2), \tag{56}$$

where r is an arbitrary positive integer. In the case $k_0=1$, we know that (56) is equal to (see [16])

$$p_1\sum_{t=0}^{r-1}(x_2+t)^{p_2}(x_1+t)^{p_1-1}+p_2\sum_{t=0}^{r-1}(x_1+t)^{p_1}(x_2+t)^{p_2-1}. \tag{57}$$

We can use (26) to write (56) as

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\left((x_1+r)^{p_1-j_1}(x_2+r)^{p_2-j_2}B_{j_1+j_2}^{(k_0)}-x_1^{p_1-j_1}x_2^{p_2-j_2}B_{j_1+j_2}^{(k_0)}\right) \tag{58}$$
$$=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\Big((x_1+r-y)^{p_1-j_1}(x_2+r-y)^{p_2-j_2}B_{j_1+j_2}^{(k_0)}(y)$$
$$-(x_1-z)^{p_1-j_1}(x_2-z)^{p_2-j_2}B_{j_1+j_2}^{(k_0)}(z)\Big),$$

where y and z are arbitrary parameters. If $k_0=1$, we have from (57) and (58) that

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\left((x_1+r)^{p_1-j_1}(x_2+r)^{p_2-j_2}-x_1^{p_1-j_1}x_2^{p_2-j_2}\right)B_{j_1+j_2} \tag{59}$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \big((x_1 + r - y)^{p_1 - j_1} (x_2 + r - y)^{p_2 - j_2} B_{j_1+j_2}(y) \\ - (x_1 - z)^{p_1 - j_1} (x_2 - z)^{p_2 - j_2} B_{j_1+j_2}(z) \big) \\ = p_1 \sum_{t=0}^{r-1} (x_2 + t)^{p_2} (x_1 + t)^{p_1 - 1} + p_2 \sum_{t=0}^{r-1} (x_1 + t)^{p_1} (x_2 + t)^{p_2 - 1}.$$

By using Theorem 3.1 in (59) we get

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left(B^{(k_1)}_{p_1-j_1}(x_1 + r) B^{(k_2)}_{p_2-j_2}(x_2 + r) - B^{(k_1)}_{p_1-j_1}(x_1) B^{(k_2)}_{p_2-j_2}(x_2) \right) B_{j_1+j_2} \tag{60}$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \Big(B^{(k_1)}_{p_1-j_1}(x_1 + r - y) B^{(k_2)}_{p_2-j_2}(x_2 + r - y) B_{j_1+j_2}(y) \\ - B^{(k_1)}_{p_1-j_1}(x_1 - z) B^{(k_2)}_{p_2-j_2}(x_2 - z) B_{j_1+j_2}(z) \Big) \\ = p_1 \sum_{t=0}^{r-1} B^{(k_1)}_{p_1-1}(x_1 + t) B^{(k_2)}_{p_2}(x_2 + t) + p_2 \sum_{t=0}^{r-1} B^{(k_1)}_{p_1}(x_1 + t) B^{(k_2)}_{p_2-1}(x_2 + t).$$

Set $(y, z) = (r, r), (r, 0), (0, -r), (2r, r)$ in (60), to obtain the following identities involving poly-Bernoulli polynomials, standard Bernoulli polynomials and Bernoulli numbers

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left(B^{(k_1)}_{p_1-j_1}(x_1 + r) B^{(k_2)}_{p_2-j_2}(x_2 + r) - B^{(k_1)}_{p_1-j_1}(x_1) B^{(k_2)}_{p_2-j_2}(x_2) \right) B_{j_1+j_2} \tag{61}$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left(B^{(k_1)}_{p_1-j_1}(x_1) B^{(k_2)}_{p_2-j_2}(x_2) - B^{(k_1)}_{p_1-j_1}(x_1 - r) B^{(k_2)}_{p_2-j_2}(x_2 - r) \right) B_{j_1+j_2}(r) \\ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(x_1) B^{(k_2)}_{p_2-j_2}(x_2) \left(B_{j_1+j_2}(r) - B_{j_1+j_2} \right) \\ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(x_1 + r) B^{(k_2)}_{p_2-j_2}(x_2 + r) \left(B_{j_1+j_2} - B_{j_1+j_2}(-r) \right) \\ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(x_1 - r) B^{(k_2)}_{p_2-j_2}(x_2 - r) \left(B_{j_1+j_2}(2r) - B_{j_1+j_2}(r) \right) \\ = p_1 \sum_{t=0}^{r-1} B^{(k_1)}_{p_1-1}(x_1 + t) B^{(k_2)}_{p_2}(x_2 + t) + p_2 \sum_{t=0}^{r-1} B^{(k_1)}_{p_1}(x_1 + t) B^{(k_2)}_{p_2-1}(x_2 + t).$$

4 Generalized recurrences

In this section we show two generalized recurrences for bi-variate poly-Bernoulli polynomials, and obtain some consequences of them.

Proposition 4.1. *We have*

$$\begin{aligned}\sum_{l=0}^{q}\binom{q}{l}(-x_1)^{q-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2) &= \sum_{l=0}^{q}\binom{q}{l}(-x_2)^{q-l}B^{(k)}_{p_1,p_2+l}(x_1,x_2) \\ &= -\sum_{l=0}^{p_1+p_2}S^{1,x_2,p_2}_{1,x_1}(p_1,l)\frac{(-1)^l l!\mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.\end{aligned} \tag{62}$$

where $\mathcal{R}_{-1,k}(y)=-1$, and for $\mu\geq 0$ the functions $\mathcal{R}_{\mu,k}(y)$ are defined recursively by

$$\mathcal{R}_{\mu,k}(y)=y(y+\mu+2)^k\mathcal{R}_{\mu-1,k}(y)-(y+1)^{k+1}\mathcal{R}_{\mu-1,k}(y+1).$$

Proof. We prove that

$$\sum_{l=0}^{q}\binom{q}{k}(-x_1)^{q-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2)=-\sum_{l=0}^{p_1+p_2}S^{1,x_2,p_2}_{1,x_1}(p_1,k)\frac{(-1)^l l!\mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k} \tag{63}$$

by induction on q. The case $q=0$ of (63) is the definition (16). Let us suppose formula (63) is true for a given $q\in\mathbb{N}$. Then

$$\begin{aligned}&\sum_{l=0}^{q+1}\binom{q+1}{l}(-x_1)^{q+1-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2) \\ &\quad= -x_1\sum_{l=0}^{q}\binom{q}{l}(-x_1)^{q-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2)+\sum_{l=0}^{q}\binom{q}{l}(-x_1)^{q-l}B^{(k)}_{p_1+1+l,p_2}(x_1,x_2) \\ &\quad= x_1\sum_{l=0}^{p_1}S^{1,x_2,p_2}_{1,x_1}(p_1,l)\frac{(-1)^l l!\mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}-\sum_{l=0}^{p_1+1}S^{1,x_2,p_2}_{1,x_1}(p_1+1,l)\frac{(-1)^l l!\mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.\end{aligned}$$

Now we use the recurrence (12) to write

$$\begin{aligned}&\sum_{l=0}^{q+1}\binom{q+1}{l}(-x_1)^{q+1-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2) \\ &\qquad= x_1\sum_{l=0}^{p_1}S^{1,x_2,p_2}_{1,x_1}(p_1,l)\frac{(-1)^l l!\mathcal{R}_{q-1}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k} \\ &\qquad\quad-\sum_{l=0}^{p_1+1}\left(S^{1,x_2,p_2}_{1,x_1}(p_1,l-1)+(l+x_1)S^{1,x_2,p_2}_{1,x_1}(p_1,l)\right)\frac{(-1)^l l!\mathcal{R}_{q-1}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.\end{aligned}$$

Some further simplifications give us

$$\sum_{l=0}^{q+1}\binom{q+1}{l}(-x_1)^{q+1-l}B^{(k)}_{p_1+l,p_2}(x_1,x_2)$$

$$= -\sum_{l=0}^{p_1+1} \left(S_{1,x_1}^{1,x_2,p_2}(p_1, l-1) + l S_{1,x_1}^{1,x_2,p_2}(p_1, l)\right) \frac{(-1)^l l! \mathcal{R}_{q-1}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}$$
$$= \sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) \frac{(-1)^l (l+1)! \mathcal{R}_{q-1}(l+1)}{\left(\prod_{i=1}^{q+1}(l+i+1)\right)^k} - \sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) \frac{(-1)^l l! l \mathcal{R}_{q-1}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}$$
$$= -\sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) \left(l(l+q+2)^k \mathcal{R}_{q-1}(l) - (l+1)^{k+1} \mathcal{R}_{q-1}(l+1)\right) \frac{(-1)^l l!}{\left(\prod_{i=1}^{q+2}(l+i)\right)^k}$$
$$= -\sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) \frac{(-1)^l l! \mathcal{R}_q(l)}{\left(\prod_{i=1}^{q+2}(l+i)\right)^k},$$

as desired. The proof of

$$\sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} B_{p_1,p_2+l}^{(k)}(x_1, x_2) = -\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1, l) \frac{(-1)^l l! \mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k},$$

is similar. □

For example, we have

$$\mathcal{R}_{0,k}(y) = (y+1)^{k+1} - y(y+2)^k,$$
$$\mathcal{R}_{1,k}(y) = (2y+1)(y+1)^{k+1}(y+3)^k - y^2(y+2)^k(y+3)^k - (y+1)^{k+1}(y+2)^{k+1}, \quad (64)$$

and then, formula (62) with $q = 1$ is

$$\begin{aligned} -x_1 B_{p_1,p_2}^{(k)}(x_1, x_2) + B_{p_1+1,p_2}^{(k)}(x_1, x_2) &= -x_2 B_{p_1,p_2}^{(k)}(x_1, x_2) + B_{p_1,p_2+1}^{(k)}(x_1, x_2) \\ &= -\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1, k)(-1)^l l! \left(\frac{l+1}{(l+2)^k} - \frac{l}{(l+1)^k}\right), \end{aligned} \quad (65)$$

and with $q = 2$ is

$$\begin{aligned} & x_1^2 B_{p_1,p_2}^{(k)}(x_1, x_2) - 2x_1 B_{p_1+1,p_2}^{(k)}(x_1, x_2) + B_{p_1+2,p_2}^{(k)}(x_1, x_2) \\ & \quad = x_2^2 B_{p_1,p_2}^{(k)}(x_1, x_2) - 2x_2 B_{p_1,p_2+1}^{(k)}(x_1, x_2) + B_{p_1,p_2+2}^{(k)}(x_1, x_2) \\ & \quad = -\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1, l)(-1)^l l! \left(\frac{(2l+1)(l+1)}{(l+2)^k} - \frac{l^2}{(l+1)^k} - \frac{(l+1)(l+2)}{(l+3)^k}\right). \end{aligned} \quad (66)$$

We can write (62) by using (26) as

$$\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_1=0}^{p_1+l} \sum_{j_2=0}^{p_2} \binom{p_1+l}{j_1} \binom{p_2}{j_2} (x_1-y)^{p_1+l-j_1} (x_2-y)^{p_2-j_2} B_{j_1+j_2}^{(k)}(y) \quad (67)$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2+l} \binom{p_1}{j_1} \binom{p_2+l}{j_2} (x_1-z)^{p_1-j_1} (x_2-z)^{p_2+l-j_2} B_{j_1+j_2}^{(k)}(z)$$

$$= -\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,l) \frac{(-1)^l l! \mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.$$

By setting $y, z = x_1, x_2$ in (67), we get the identities

$$\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (x_2-x_1)^{p_2-j_2} B_{p_1+l+j_2}^{(k)}(x_1) \tag{68}$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_1=0}^{p_1+l} \binom{p_1+l}{j_1} (x_1-x_2)^{p_1+l-j_1} B_{j_1+p_2}^{(k)}(x_2)$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} \sum_{j_2=0}^{p_2+l} \binom{p_2+l}{j_2} (x_2-x_1)^{p_2+l-j_2} B_{p_1+j_2}^{(k)}(x_1)$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (x_1-x_2)^{p_1-j_1} B_{j_1+p_2+l}^{(k)}(x_2)$$

$$= -\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1} \binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1+j_2, l) \frac{(-1)^l l! \mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.$$

If we set $x_2 = 0$ in (68), we get

$$\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (-x_1)^{p_2-j_2} B_{p_1+l+j_2}^{(k)}(x_1) \tag{69}$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_1=0}^{p_1+l} \binom{p_1+l}{j_1} x_1^{p_1+l-j_1} B_{j_1+p_2}^{(k)}$$

$$= \sum_{j_2=0}^{p_2+q} \binom{p_2+q}{j_2} (-x_1)^{p_2+q-j_2} B_{p_1+j_2}^{(k)}(x_1)$$

$$= \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} x_1^{p_1-j_1} B_{j_1+p_2+q}^{(k)}$$

$$= -\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,0,p_2}(p_1,l) \frac{(-1)^l l! \mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}.$$

The case $q = 0, x_1 = 1, k = 1$ of (69) reduces to

$$(-1)^{p_1+p_2} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} B_{p_1+j_2} = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} B_{j_1+p_2} \tag{70}$$

$$= \sum_{j_1=0}^{p_1}\sum_{l=0}^{j_1+p_2}\binom{p_1}{j_1}S(j_1+p_2,l)\frac{(-1)^l l!}{l+1}.$$

Formula (70) is the famous Carlitz identity [4]. In terms of bi-variate Bernoulli polynomials, Carlitz identity is written as

$$B_{p_1,p_2}(1,0) = (-1)^{p_1+p_2}B_{p_1,p_2}(0,1). \tag{71}$$

For example, we can use (71) to write the following version of (62) in the case $k=1$, when $x_1=0, x_2=1$

$$\begin{aligned}\sum_{l=0}^{q}\binom{q}{l}(-1)^{q-l}B_{p_1+l,p_2}(1,0) &= (-1)^{p_1+p_2+q}\sum_{l=0}^{q}\binom{q}{l}B_{p_1+l,p_2}(0,1)\\ &= B_{p_1,p_2+q}(1,0) = (-1)^{p_1+p_2+q}B_{p_1,p_2+q}(0,1)\\ &= -\sum_{l=0}^{p_1+p_2}S_{1,1}^{1,0,p_2}(p_1,l)\,\frac{(-1)^l l!\mathcal{R}_{q-1,1}(l)}{\prod_{i=1}^{q+1}(l+i)},\end{aligned}$$

or, explicitly

$$\begin{aligned}&\sum_{l=0}^{q}\binom{q}{l}(-1)^{q-l}\sum_{j_1=0}^{p_1+l}\binom{p_1+l}{j_1}B_{j_1+p_2}\\ &\quad= (-1)^{p_1+p_2+q}\sum_{l=0}^{q}\binom{q}{l}\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}B_{p_1+l+j_2}\\ &\quad= \sum_{j_1=0}^{p_1}\binom{p_1}{j_1}B_{j_1+p_2+q} = (-1)^{p_1+p_2+q}\sum_{j_2=0}^{p_2+q}\binom{p_2+q}{j_2}B_{p_1+j_2}\\ &\quad= -\sum_{l=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}S(j_1+p_2,l)\frac{(-1)^l l!\mathcal{R}_{q-1,1}(l)}{\prod_{i=1}^{q+1}(l+i)}.\end{aligned}$$

It is easy to check that in the case $k=0$, the functions $\mathcal{R}_{\mu,k}(y)$ of Proposition 4.1 are $\mathcal{R}_{\mu,0}(y) = (-1)^\mu$. Thus, the case $k=0$ of (62) is the case $r=0$ of (30). Also, in the case $k=-1$, we have that $\mathcal{R}_{\mu,-1}(y) = \frac{(-1)^\mu 2^{\mu+1}}{\prod_{i=2}^{\mu+2}(y+i)}$, and then the case $k=-1$ of (62) is the case $r=1$ of (30).

Proposition 4.2. *For non-negative integers p_1, p_2, q we have*

$$\begin{aligned}&\sum_{l=0}^{q}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i) \\ &\qquad= \sum_{l=0}^{q}(-1)^l B_{p_1,p_2+l}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_2^l}\prod_{i=0}^{q-1}(x_2+i)\end{aligned} \tag{72}$$

$$= \sum_{l=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}.$$

Proof. We prove that

$$\sum_{l=0}^{q}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i) = \sum_{l=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}, \qquad (73)$$

by induction on q. The case $q=0$ of (73) is the definition (16). If we suppose that (73) is true for a given $q \in \mathbb{N}$, then

$$\begin{aligned}
&\sum_{l=0}^{q+1}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q}(x_1+i)\\
&\quad= \sum_{l=0}^{q+1}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\left((x_1+q)\prod_{i=0}^{q-1}(x_1+i)\right)\\
&\quad= \sum_{l=0}^{q+1}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\left((x_1+q)\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i)+l\frac{d^{l-1}}{dx_1^{l-1}}\prod_{i=0}^{q-1}(x_1+i)\right)\\
&\quad= (x_1+q)\sum_{l=0}^{q}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i)\\
&\qquad+\sum_{l=1}^{q+1}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{(l-1)!}\frac{d^{l-1}}{dx_1^{l-1}}\prod_{i=0}^{q-1}(x_1+i)\\
&\quad= (x_1+q)\sum_{l=0}^{q}(-1)^l B_{p_1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i)\\
&\qquad-\sum_{l=0}^{q}(-1)^l B_{p_1+1+l,p_2}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i)\\
&\quad= (x_1+q)\sum_{l=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}\\
&\qquad-\sum_{l=0}^{p_1+p_2+1} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1+1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}.
\end{aligned}$$

Now we use the recurrence (12) and formula (11) to write

$$\sum_{l=0}^{q+1}(-1)^l B_{p_1+l,p_2}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q}(x_1+i)$$

$$= (x_1+q)\sum_{l=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}$$

$$-\sum_{l=0}^{p_1+p_2+1}\left(S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l-1)+(l+x_1+q)S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\right)\frac{(-1)^l(l+q)!}{(l+q+1)^k}$$

$$= -\sum_{l=0}^{p_1+p_2+1}\left(S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l-1)+lS_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\right)\frac{(-1)^l(l+q)!}{(l+q+1)^k}$$

$$= -\sum_{l=1}^{p_1+p_2+1} S_{1,x_1+q+1}^{1,x_2+q+1,p_2}(p_1,l-1)\frac{(-1)^l(l+q)!}{(l+q+1)^k}$$

$$= \sum_{l=0}^{p_1+p_2} S_{1,x_1+q+1}^{1,x_2+q+1,p_2}(p_1,l)\frac{(-1)^l(l+q+1)!}{(l+q+2)^k},$$

as desired. The proof of

$$\sum_{l=0}^{q}(-1)^l B_{p_1,p_2+l}^{(k)}(x_1,x_2)\frac{1}{l!}\frac{d^l}{dx_2^l}\prod_{i=0}^{q-1}(x_2+i) = \sum_{l=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k},$$

is similar. □

Formula (72) with $x_1=0, x_2=1$ looks as

$$\begin{aligned}\sum_{l=0}^{q}(-1)^l s(q,l)B_{p_1+l,p_2}^{(k)}(0,1) &= \sum_{l=0}^{q}(-1)^l s(q+1,l+1)B_{p_1,p_2+l}^{(k)}(0,1) \\ &= \sum_{l=0}^{p_1+p_2} S_{1,q}^{1,q+1,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k},\end{aligned} \tag{74}$$

or, explicitly

$$\begin{aligned}\sum_{l=0}^{q}(-1)^l s(q,l)\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}B_{p_1+l+j_2}^{(k)} &= \sum_{l=0}^{q}(-1)^l s(q+1,l+1)\sum_{j_2=0}^{p_2+l}\binom{p_2+l}{j_2}B_{p_1+j_2}^{(k)} \\ &= \sum_{l=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}q^{p_1-j_1}(q+1)^{p_2-j_2}S(j_1+j_2,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}.\end{aligned} \tag{75}$$

If we set $k=1$ in formula (74), we can use Carlitz identity (71) to write the following enriched version of (74)

$$\sum_{l=0}^{q}(-1)^l s(q,l)B_{p_1+l,p_2}(0,1) = (-1)^{p_1+p_2}\sum_{l=0}^{q}s(q,l)B_{p_1+l,p_2}(1,0) \tag{76}$$

$$
\begin{aligned}
&= \sum_{l=0}^{q}(-1)^l s(q+1,l+1)B_{p_1,p_2+l}(0,1) \\
&= (-1)^{p_1+p_2}\sum_{l=0}^{q} s(q+1,l+1)B_{p_1,p_2+l}(1,0) \\
&= \sum_{l=0}^{p_1+p_2} S_{1,q}^{1,q+1,p_2}(p_1,l)\frac{(-1)^l(l+q)!}{l+q+1},
\end{aligned}
$$

or, explicitly

$$
\begin{aligned}
&\sum_{l=0}^{q}(-1)^l s(q,l)\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}B_{p_1+l+j_2} \qquad (77)\\
&= (-1)^{p_1+p_2}\sum_{l=0}^{q} s(q,l)\sum_{j_1=0}^{p_1+l}\binom{p_1+l}{j_1}B_{j_1+p_2} \\
&= \sum_{l=0}^{q}(-1)^l s(q+1,l+1)\sum_{j_2=0}^{p_2+l}\binom{p_2+l}{j_2}B_{p_1+j_2} \\
&= (-1)^{p_1+p_2}\sum_{l=0}^{q} s(q+1,l+1)\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}B_{j_1+p_2+l} \\
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{l=0}^{j_1+j_2}\binom{p_1}{j_1}\binom{p_2}{j_2}q^{p_1-j_1}(q+1)^{p_2-j_2}S(j_1+j_2,l)\frac{(-1)^l(l+q)!}{l+q+1}.
\end{aligned}
$$

To end this section, let us consider the case $q=1$ of the first two lines of (72). That is

$$
B_{p_1,p_2+1}^{(k)}(x_1,x_2) - B_{p_1+1,p_2}^{(k)}(x_1,x_2) = (x_2-x_1)B_{p_1,p_2}^{(k)}(x_1,x_2). \qquad (78)
$$

Formula (78) is the first step of two results contained in the following proposition.

Proposition 4.3. *We have the following identities:*

a)

$$
\sum_{j=0}^{q}\binom{q}{j}(-1)^j B_{p_1+j,p_2+q-j}^{(k)}(x_1,x_2) = (x_2-x_1)^q B_{p_1,p_2}^{(k)}(x_1,x_2). \qquad (79)
$$

b)

$$
\sum_{j=0}^{q}\binom{q}{j}(x_2-x_1)^j B_{p_1+q-j,p_2}^{(k)}(x_1,x_2) = B_{p_1,p_2+q}^{(k)}(x_1,x_2). \qquad (80)
$$

Proof. Let us prove (79) by induction on q. The case $q=0$ is a trivial identity. Let us suppose that (79) is true for a given $q\in\mathbb{N}$. Then

$$
\sum_{j=0}^{q+1}\binom{q+1}{j}(-1)^j B_{p_1+j,p_2+q+1-j}^{(k)}(x_1,x_2)
$$

$$= \sum_{j=0}^{q+1} \left(\binom{q}{j} + \binom{q}{j-1} \right) (-1)^j B^{(k)}_{p_1+j, p_2+q+1-j}(x_1, x_2)$$
$$= \sum_{j=0}^{q} \binom{q}{j} (-1)^j B^{(k)}_{p_1+j, p_2+q+1-j}(x_1, x_2) + \sum_{j=0}^{q} \binom{q}{j} (-1)^{j+1} B^{(k)}_{p_1+1+j, p_2+q-j}(x_1, x_2)$$
$$= (x_2 - x_1)^q B^{(k)}_{p_1, p_2+1}(x_1, x_2) - (x_2 - x_1)^q B^{(k)}_{p_1+1, p_2}(x_1, x_2)$$
$$= (x_2 - x_1)^q \left(B^{(k)}_{p_1, p_2+1}(x_1, x_2) - B^{(k)}_{p_1+1, p_2}(x_1, x_2) \right)$$
$$= (x_2 - x_1)^{q+1} B^{(k)}_{p_1, p_2}(x_1, x_2),$$

as desired. In the last step we used (78).

Now let us prove (80). Again we proceed by induction on q. The case $q = 0$ is a trivial identity. Let us suppose that (80) is true for a given $q \in \mathbb{N}$. Then

$$\sum_{j=0}^{q+1} \binom{q+1}{j} (x_2 - x_1)^j B^{(k)}_{p_1+q+1-j, p_2}(x_1, x_2)$$
$$= \sum_{j=0}^{q+1} \left(\binom{q}{j} + \binom{q}{j-1} \right) (x_2 - x_1)^j B^{(k)}_{p_1+q+1-j, p_2}(x_1, x_2)$$
$$= \sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^j B^{(k)}_{p_1+q+1-j, p_2}(x_1, x_2) + \sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^{j+1} B^{(k)}_{p_1+q-j, p_2}(x_1, x_2)$$
$$= B^{(k)}_{p_1+1, p_2+q}(x_1, x_2) + (x_2 - x_1) B^{(k)}_{p_1, p_2+q}(x_1, x_2)$$
$$= B^{(k)}_{p_1, p_2+q+1}(x_1, x_2),$$

as desired. We used (78) (with p_2 replaced by $p_2 + q$) in the last step. □

The case $p_1 = 0$ of (80) is

$$\sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^j B^{(k)}_{q-j, p_2}(x_1, x_2) = B^{(k)}_{p_2+q}(x_2). \tag{81}$$

The case $p_2 = 0$ of (81) is the addition formula for standard poly-Bernoulli polynomials, namely

$$\sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^{q-j} B^{(k)}_j(x_1) = B^{(k)}_q(x_2).$$

The case $p_1 = p_2 = 0$ of (79) is

$$\sum_{j=0}^{q} \binom{q}{j} (-1)^j B^{(k)}_{j, q-j}(x_1, x_2) = (x_2 - x_1)^q. \tag{82}$$

As a final comment, we mention that by considering the GSN $S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,k)$ involved in the expansion

$$(m+x_1)^{p_1}\cdots(m+x_n)^{p_n}=\sum_{l=0}^{p_1+\cdots+p_n} l!S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,l)\binom{m}{l}, \tag{83}$$

where $p_1,p_2,\ldots,p_n$ are non-negative integers given, one can define poly-Bernoulli polynomials in n variables $x_1,\ldots,x_n$, denoted as $B_{p_1,\ldots,p_n}^{(k)}(x_1,\ldots,x_n)$, as

$$B_{p_1,\ldots,p_n}^{(k)}(x_1,\ldots,x_n)=\sum_{l=0}^{p_1+\cdots+p_n} S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,l)\frac{(-1)^l l!}{(l+1)^k},$$

or explicitly as

$$\begin{aligned}
&B_{p_1,\ldots,p_n}^{(k)}(x_1,\ldots,x_n)\\
&\quad=\sum_{l=0}^{p_1+\cdots+p_n}\sum_{j_1=0}^{p_1}\cdots\sum_{j_n=0}^{p_n}\binom{p_1}{j_1}\cdots\binom{p_n}{j_n}x_1^{p_1-j_1}\cdots x_n^{p_n-j_n}S(j_1+\cdots+j_n,l)\frac{(-1)^l l!}{(l+1)^k}\\
&\quad=\sum_{j_1=0}^{p_1}\cdots\sum_{j_n=0}^{p_n}\binom{p_1}{j_1}\cdots\binom{p_n}{j_n}x_1^{p_1-j_1}\cdots x_n^{p_n-j_n}B_{j_1+\cdots+j_n}^{(k)}.
\end{aligned}$$

In this more general setting we have natural generalizations of results (62), (72), (79) and (80). We show the corresponding results in the case of poly-Bernoulli polynomials in 3 variables:

$$B_{p_1,p_2,p_3}^{(k)}(x_1,x_2,x_3)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\binom{p_1}{j_1}\binom{p_2}{j_2}\binom{p_3}{j_3}x_1^{p_1-j_1}x_2^{p_2-j_2}x_3^{p_3-j_3}B_{j_1+j_2+j_3}^{(k)},$$

(a) (See (62)) We have the generalized recurrences

$$\begin{aligned}
&\sum_{l=0}^{q}\binom{q}{l}(-x_1)^{q-l}B_{p_1+l,p_2,p_3}^{(k)}(x_1,x_2,x_3)\\
&\quad=\sum_{l=0}^{q}\binom{q}{l}(-x_2)^{q-l}B_{p_1,p_2+l,p_3}^{(k)}(x_1,x_2,x_3)\\
&\quad=\sum_{l=0}^{q}\binom{q}{l}(-x_3)^{q-l}B_{p_1,p_2,p_3+l}^{(k)}(x_1,x_2,x_3)\\
&\quad=-\sum_{l=0}^{p_1+p_2+p_3}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\binom{p_1}{j_1}\binom{p_2}{j_2}\binom{p_3}{j_3}x_1^{p_1-j_1}x_2^{p_2-j_2}x_3^{p_3-j_3}\right.\\
&\qquad\qquad\left.\times S(j_1+j_2+j_3,l)\frac{(-1)^l l!\mathcal{R}_{q-1,k}(l)}{\left(\prod_{i=1}^{q+1}(l+i)\right)^k}\right),
\end{aligned}$$

where $\mathcal{R}_{q-1,k}(l)$ is defined in Proposition 4.1.

(b) (See (72)) We have the generalized recurrences

$$\begin{aligned}
&\sum_{l=0}^{q}(-1)^l B^{(k)}_{p_1+l,p_2,p_3}(x_1,x_2,x_3)\frac{1}{l!}\frac{d^l}{dx_1^l}\prod_{i=0}^{q-1}(x_1+i)\\
&\quad=\sum_{l=0}^{q}(-1)^l B^{(k)}_{p_1,p_2+l,p_3}(x_1,x_2,x_3)\frac{1}{l!}\frac{d^l}{dx_2^l}\prod_{i=0}^{q-1}(x_2+i)\\
&\quad=\sum_{l=0}^{q}(-1)^l B^{(k)}_{p_1,p_2,p_3+l}(x_1,x_2,x_3)\frac{1}{l!}\frac{d^l}{dx_3^l}\prod_{i=0}^{q-1}(x_3+i)\\
&\quad=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\binom{p_1}{j_1}\binom{p_2}{j_2}\binom{p_3}{j_3}(x_1+q)^{p_1-j_1}(x_2+q)^{p_2-j_2}(x_3+q)^{p_3-j_3}\right.\\
&\qquad\qquad\left.\times S(j_1+j_2+j_3,l)\frac{(-1)^l(l+q)!}{(l+q+1)^k}\right).
\end{aligned}$$

(c) (See (79)) We have the identities

$$\begin{aligned}
\sum_{j=0}^{q}\binom{q}{j}(-1)^j B^{(k)}_{p_1,p_2+j,p_3+q-j}(x_1,x_2,x_3) &= (x_3-x_2)^q B^{(k)}_{p_1,p_2,p_3}(x_1,x_2,x_3),\\
\sum_{j=0}^{q}\binom{q}{j}(-1)^j B^{(k)}_{p_1+j,p_2,p_3+q-j}(x_1,x_2,x_3) &= (x_3-x_1)^q B^{(k)}_{p_1,p_2,p_3}(x_1,x_2,x_3),\\
\sum_{j=0}^{q}\binom{q}{j}(-1)^j B^{(k)}_{p_1+j,p_2+q-j,p_3}(x_1,x_2,x_3) &= (x_2-x_1)^q B^{(k)}_{p_1,p_2,p_3}(x_1,x_2,x_3).
\end{aligned}$$

(d) (See (80)) We have the identities

$$\begin{aligned}
&\sum_{j=0}^{q}\binom{q}{j}(x_1-x_2)^j B^{(k)}_{p_1,p_2+q-j,p_3}(x_1,x_2,x_3)\\
&\qquad=\sum_{j=0}^{q}\binom{q}{j}(x_1-x_3)^j B^{(k)}_{p_1,p_2,p_3+q-j}(x_1,x_2,x_3)=B^{(k)}_{p_1+q,p_2,p_3}(x_1,x_2,x_3).
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{q}\binom{q}{j}(x_2-x_1)^j B^{(k)}_{p_1+q-j,p_2,p_3}(x_1,x_2,x_3)\\
&\qquad=\sum_{j=0}^{q}\binom{q}{j}(x_2-x_3)^j B^{(k)}_{p_1,p_2,p_3+q-j}(x_1,x_2,x_3)=B^{(k)}_{p_1,p_2+q,p_3}(x_1,x_2,x_3).
\end{aligned}$$

$$\sum_{j=0}^{q}\binom{q}{j}(x_3-x_1)^j B^{(k)}_{p_1+q-j,p_2,p_3}(x_1,x_2,x_3)$$
$$=\sum_{j=0}^{q}\binom{q}{j}(x_3-x_2)^j B^{(k)}_{p_1,p_2+q-j,p_3}(x_1,x_2,x_3)=B^{(k)}_{p_1,p_2,p_3+q}(x_1,x_2,x_3)\,.$$

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