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Application of homotopy analysis method (HAM) to the non-linear KdV equations

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Abstract. In this work, approximate analytic solutions for different types of KdV equations are obtained using the homotopy analysis method (HAM). The convergence control parameter h helps us to adjust the convergence region of the approximate analytic solutions. The solutions are obtained in the form of power series. The obtained solutions and the exact solutions are shown graphically, highlighting the effects of non-linearity. We have compared the approximate analytical results which are determined by HAM, with the exact solutions and shown graphically with their absolute errors. By choosing an appropriate value of the convergence control parameter, we can obtain the solution in few iterations. All the computations have been performed using the software package MATHEMATICA.

1 Introduction

Many nonlinear physical phenomena arise in various fields of engineering and science such as fluid dynamics, nuclear reactor dynamics, plasma physics, biology, optical fibres and solid state physics. To describe these complex physical phenomena, nonlinear differential equations play a significant role. Therefore, obtaining the solutions of these nonlinear equations is a topic of great interest in the study of many fields of science. To better understand the working of the physical problem, mathematical model came into picture in the form of nonlinear PDEs. The solutions of partial differential equations give the detailed summary about the nature of phenomena involved. Many numerical and analytical methods have been derived to deal with these kind of scientific problems.

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We need to adopt an effective and powerful method to investigate such type of mathematical model which gives the solutions upholding to physical reality. In most of the analytic techniques, linearization of the system is the main topic to focus on, and also, it is assumed that the nonlinearities are relatively insignificant. Sometimes, these assumptions made a strong affect on the solutions in respect to the real physics of the phenomena involved. Thus, finding the solutions of nonlinear ODEs and PDEs is still a significant problem. For this, we need new techniques to develop approximate and exact solutions. In the recent years, many powerful methods such as differential transform method [7], Adomian decomposition method [1], [3], Homotopy perturbation method [6], [2], Variational iteration method [10], Modified variational iteration method [8], F-expansion method [9], Hirota bilinear method [28], G'/G-expansion method [11], tanh-coth method [13], Lie symmetry method [26],[30], [29], [32], B-spline collocation method [31] are used to determine the approximate or exact analytical solutions of nonlinear PDEs.

One of the powerful and important methods for solving nonlinear problems is the homotopy analysis method (HAM). To propose a general analytic method for nonlinear problems, Liao [14]) first proposed the idea of the homotopy in the field of topology, called as Homotopy Analysis Method (HAM) [4], [16], [14], [15]. Whether small parameters exist or not, the validity of the HAM is independent due to the homotopy. Therefore, the homotopy analysis method can overcome the limitations and restrictions of perturbation techniques (see [5], [17]). The HAM provides the valuable series solution with minimum number of calculations and avoids the discretization of domain and unrealistic assumptions. These series solution expressions always exist in terms of parameter h. The convergence region and the convergence region for every solution can be obtained easily by the parameter h which is called the convergence control parameter.

In this paper, our aim is to solve KdV equations of fifth-order by using HAM. The general form of KdV equation can be written as follows (see [20], [19], [18]):

$$u_t \pm u_{5r} = F(r, t, u, u^2, u_r, u_{2r}, u_{3r}). \tag{1}$$

where $u(r, t)$ is the function of r and t, which represent the space and time variables, respectively. The KdV equation generally occurs in the theory of shallow water waves [21] and magneto-acoustic waves [22] in a plasma. Researchers have studied the travelling wave solutions of fifth-order KdV equations over the last decades which do not vanish on infinity. The KdV equation has been the topic of extensive research in recent years [23], [20]. We have considered the following three well-known forms of the KdV equation (see [24]):

$$u_t + u_r + u^2 u_{2r} + u_r u_{2r} - 20u^2 u_{3r} + u_{5r} = 0, \tag{2}$$

$$u_t + u\, u_r - u u_{3r} + u_{5r} = 0, \tag{3}$$

$$u_t + u\, u_r + u_{3r} - u_{5r} = 0, \tag{4}$$

Eqs. (2) and (3) are known as KdV equations and Eq. (4) is known as Kawahara equation.

The research paper is arranged as follows: Basic idea of HAM is presented in brief in section 2. In section 3, the proposed method is applied to obtain the solutions of three different types of KdV equations of fifth-order with the initial conditions. In section 4, the convergence of the obtained solutions are discussed. A comparison is made between the exact solutions and obtained solutions. Absolute errors of the solutions are shown in the Tables 1, 2 and 3. In the last section, a short summary of results is presented.

2 Basics Concepts of Homotopy Analysis Method

We consider the nonlinear differential equation given as follows:

$$M[u(r,t)] = 0, \tag{5}$$

where M is a non-linear operator.

Let θ be a homotopy parameter and ξ be a function of θ, then we have

$$D_n(\xi) = \frac{1}{n!}\frac{\partial^n \xi(r,t;\theta)}{\partial \theta^n}\bigg|_{\theta=0}, \quad \text{where} \ \ n \geq 0, \tag{6}$$

where $D_n(\xi)$ denotes the n^{th}-order homotopy derivative of ξ (see [14]). Now, the deformation equation of zero-order is constructed as:

$$(1-\theta)\, N[\xi(r,t;\theta) - u_0(r,t)] = \theta\, \hbar\, H(r,t) M[\xi(r,t;\theta)], \tag{7}$$

where N is the auxiliary linear operator, $\xi(r,t;\theta)$ denotes an unknown function, $\hbar$ is a non-zero auxiliary parameter, θ is an embedding parameter whose value lies in the interval $[0,1]$, H is a non-zero auxiliary function and u_0 is the initial guess of u.
Now, the function $\xi(r,t;p)$ is expanded in the Taylor's series about $p=0$ in the following manner:

$$\xi(r,t;\theta) = u_0(r,t) + \sum_{n=1}^{\infty} u_n(r,t)\theta^n, \tag{8}$$

where

$$u_n(r,t) = D_n(\xi). \tag{9}$$

If we properly select the value of auxiliary parameter $\hbar$, initial guess u_0, non-zero auxiliary function H and auxiliary linear operator N, then the series (8) converges for $\theta = 1$. Now, after some manipulation, we get the n^{th}-order deformation equation as follows:

$$N[u_n(r,t) - \sigma_n\, u_{n-1}(r,t)] = \hbar\, H R_n(u_{n-1}, r, t), \tag{10}$$

where

$$R_n(u_{n-1}, r, t) = D_{n-1}(M[\xi(r,t;\theta)]) = \frac{1}{(n-1)!}\frac{\partial^{n-1}\xi(r,t;\theta)}{\partial \theta^{n-1}}\bigg|_{\theta=0} \tag{11}$$

and

$$\sigma_n = \begin{cases} 0, & n \leq 1; \\ 1, & n > 1. \end{cases}$$

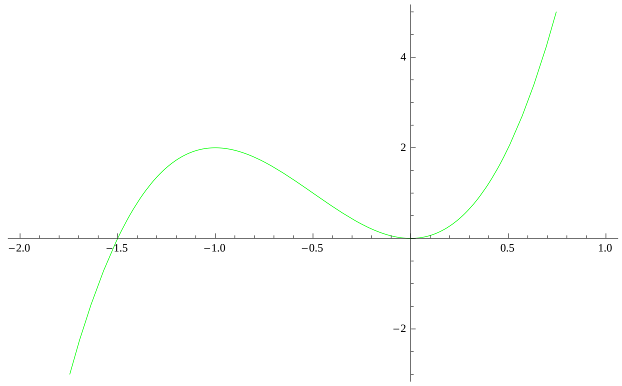


Figure 1: The h-curve for Eq. (12) obtained by the third order approximation.

Now, the components $u_n(r,t)$ for $n \geq 1$ can be calculated from Eq. (10) with the initial conditions of the original problem.

It is worth noting that we have a liberty to choose the value of the parameters $\hbar$ and M in the HAM, which provides a simple way to select and adjust the rate of convergence of the approximate analytic solution.

3 Applications of HAM

Example 3.1. We take the following fifth-order KdV equation:

$$u_t + u_r + u^2 u_{2r} + u_r u_{2r} - 20u^2 u_{3r} + u_{5r} = 0, \tag{12}$$

with initial condition

$$u(r,0) = \frac{1}{r}, \tag{13}$$

The exact solution of the Eq. (12) is as follows:

$$u(r,t) = \frac{1}{(r-t)}. \tag{14}$$

Now, for applying HAM to the equation Eq. (12), we define the following linear operator:

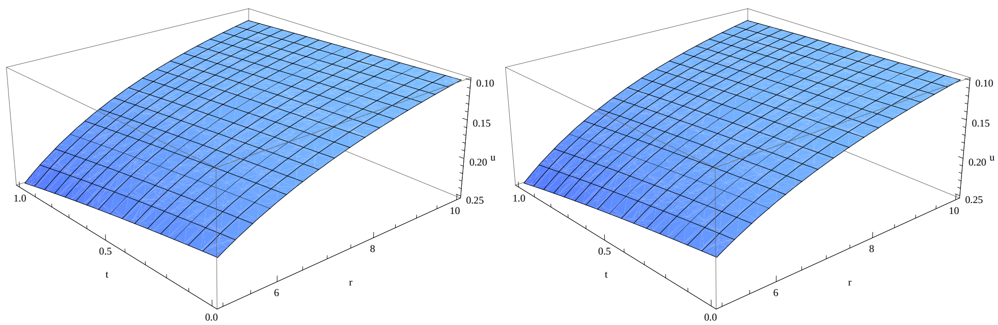
$$N[\xi(r,t;\theta)] = \frac{\partial \xi(r,t;\theta)}{\partial t}, \tag{15}$$

having the property $N[c] = 0$,($c =$ CONSTANT). With the help of Eq. (12), we define non-linear operator M such that

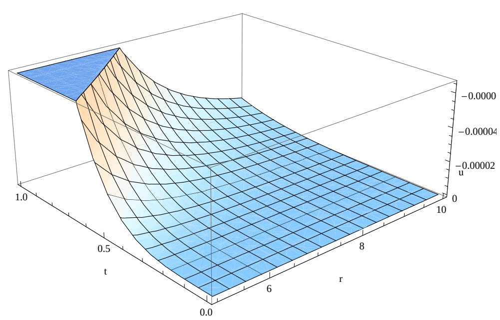
$$M[\xi(r,t;\theta)] = \xi_t + \xi_r + \xi^2 \xi_{2r} + \xi_r \, \xi_{2r} - 20\xi^2 \xi_{3r} + \xi_{5r}. \tag{16}$$

With the help of above equation, we construct the zero-order deformation equation as follows:

$$(1-\theta)\, N[\xi(r,t;\theta) - u_0(r,t)] = \theta \, \hbar \, H(r,t) M[\xi(r,t;\theta)]. \tag{17}$$



(a) Approximate Solution (b) Exact Solution



(c) Absolute Error $|u_{eract} - u_{approrimate}|$

Figure 2: Solution profiles of the equation (12): (a) approximate solution (b) exact solution (c) absolute error $|u_{exact} - u_{approrimate}|$.

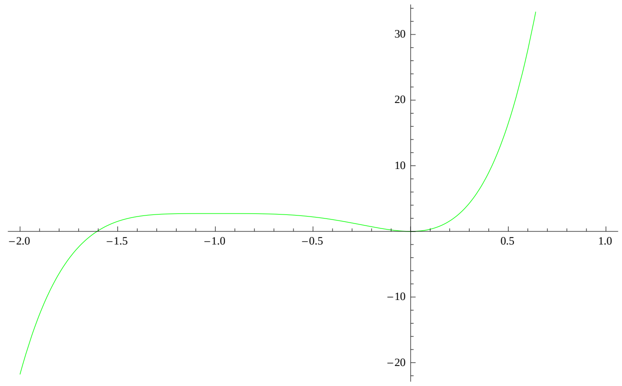


Figure 3: The h-curve for Eq. (21) obtained by the 5^{th} order approximation.

From the above equation, we can see that for $\theta = 0$ and $\theta = 1$, $\xi(r, t; 0) = u_0(r, t)$ and $\xi(r, t; 1) = u(r, t)$, respectively.

Differentiating the deformation Eq. (17) n-times with respect to θ, we obtain the n^{th}-order deformation equation as follows:

$$N[u_n(r,t) - \sigma_n\, u_{n-1}(r,t)] = \hbar\, H\, R_n(u_{n-1}), \tag{18}$$

with the initial condition $u_n(r, 0) = \frac{1}{r}$, where

$$\begin{aligned} R_n(u_{n-1}) =& \frac{\partial u_{n-1}}{\partial t} + \frac{\partial u_{n-1}}{\partial r} + \sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k} u_k u_j \frac{\partial^2 u_{n-1-k-j}}{\partial r^2} \\ &+ \sum_{k=0}^{n-1} \frac{\partial u_k}{\partial r}\frac{\partial^2 u_{n-1-k}}{\partial r^2} - 20\sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k} u_k u_j \frac{\partial^3 u_{n-1-k-j}}{\partial r^3} + \frac{\partial^5 u_{n-1}}{\partial r^5}. \end{aligned} \tag{19}$$

For convenience, we select the value of the auxiliary function $H = 1$ and parameter $\hbar = h$. Therefore, the solution of the deformation equation becomes

$$u_n(r,t) = \sigma_n u_{n-1}(r,t) + hN^{-1}R_n(u_{n-1}, r, t).$$

The third order approximate series solution of equation (12) is as follows:

$$\begin{aligned} u(r,t) &= \sum_{k=0}^{3} u_k(r,t) \\ &= \frac{1}{r} - \frac{h^3 t}{r^2} - \frac{3h^2 t}{r^2} - \frac{3ht}{r^2} + \frac{2h^3t^2}{r^3} + \frac{3h^2t^2}{r^3} - \frac{h^3t^3}{r^4}. \end{aligned} \tag{20}$$

Example 3.2. We take the following fifth-order KdV equation:

$$u_t + u\, u_r - uu_{3r} + u_{5r} = 0, \tag{21}$$

with initial condition

$$u(r,0) = e^{r}, \tag{22}$$

The exact solution of the Eq. (21) is given by

$$u(r,t) = e^{r-t}. \tag{23}$$

Now, for applying HAM to the differential equation Eq. (21), we define the following linear operator:

$$N[\xi(r,t;\theta)] = \frac{\partial \xi(r,t;\theta)}{\partial t}, \tag{24}$$

with the property $N[c] = 0$,($c = constant$). With the help of Eq. (21), we define non-linear operator M such that

$$M[\xi(r,t;\theta)] = \xi_t + \xi\xi_r - \xi\xi_{3r} + \xi_{5r}. \tag{25}$$

With the help of above equation, we construct the zero-order deformation equation as follows:

$$(1-\theta)\, N[\xi(r,t;\theta) - u_0(r,t)] = \theta\, \hbar\, H(r,t) M[\xi(r,t;\theta)]. \tag{26}$$

From the above equation, we can see that for $\theta = 0$ and $\theta = 1$, $\xi(r,t;0) = u_0(r,t)$ and $\xi(r,t;1) = u(r,t)$, respectively.

Differentiating the zero-order deformation Eq. (26) n-times with respect to θ, we obtain the n^{th}-order deformation equation as follows:

$$N[u_n(r,t) - \sigma_n\, u_{n-1}(r,t)] = \hbar\, H\, R_n(u_{n-1}), \tag{27}$$

with the initial condition $u_n(r,0) = e^{r}$, where

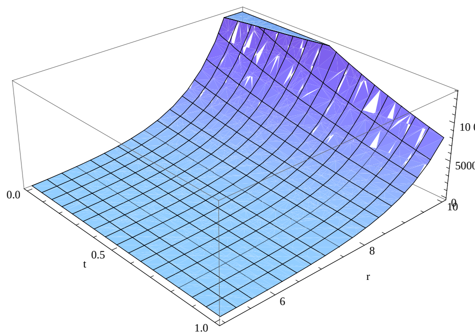
$$R_n(u_{n-1}) = \frac{\partial u_{n-1}}{\partial t} + \sum_{k=0}^{n-1} u_k \frac{\partial u_{n-1-k}}{\partial r} - \sum_{k=0}^{n-1} u_k \frac{\partial^3 u_{n-1-k}}{\partial r^3} + \frac{\partial^5 u_{n-1}}{\partial r^5}. \tag{28}$$

For convenience, we select the value of the auxiliary function $H = 1$ and parameter $\hbar = h$. Therefore, the solution of the deformation becomes

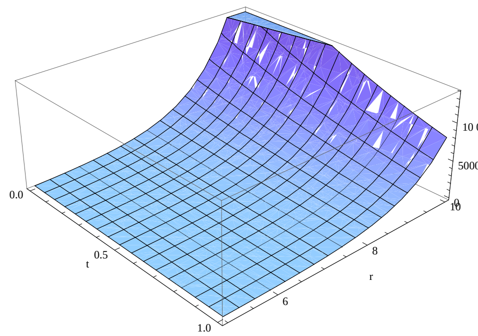
$$u_n(r,t) = \sigma_n u_{n-1}(r,t) + hN^{-1} R_n(u_{n-1}, r, t).$$

Hence, the 5^{th} order approximate series solution of equation (21) is obtained as

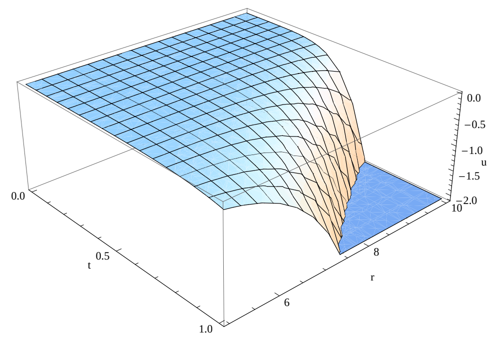
$$\begin{aligned} u(r,t) &= \sum_{k=0}^{5} u_k(r,t) \\ &= e^{r} + 5e^{r}ht + 10e^{r}h^2t + 10e^{r}h^3t + 5e^{r}h^4t + e^{r}h^5t + 5e^{r}h^2t^2 + 10e^{r}h^3t^2 \\ &+ \frac{15e^{r}h^4t^2}{2} + 2e^{r}h^5t^2 + \frac{e^{r}h^3t^3}{3} + \frac{5e^{r}h^4t^3}{2} + e^{r}h^5t^3 + \frac{5e^{r}h^4t^4}{24} + \frac{e^{r}h^5t^4}{6} + \frac{e^{r}h^5t^5}{120}. \end{aligned} \tag{29}$$



(a) Approximate Solution



(b) Exact Solution



(c) Absolute Error $|u_{eract} - u_{approrimate}|$

Figure 4: Solution profiles of the equation (21): (a) approximate solution (b) exact solution (c) absolute error $|u_{exact} - u_{approrimate}|$.

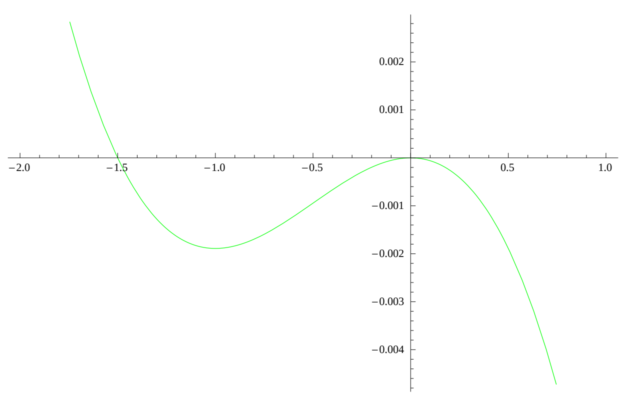


Figure 5: The h-curve for Eq. (30) obtained by the 3^{rd} order approximation.

Example 3.3. We consider the following fifth-order KdV equation

$$u_t + u\,u_r + u_{3r} - u_{5r} = 0, \tag{30}$$

with initial condition

$$u(r,0) = \frac{105}{169}\text{Sech}^4\left[\frac{1}{2\sqrt{13}}(r-a)\right], \tag{31}$$

The exact solution of the Eq. (30) is given as follows:

$$u(r,t) = \frac{105}{169}\text{Sech}^4\left[\frac{1}{2\sqrt{13}}(r-\frac{36t}{169}-a)\right]. \tag{32}$$

Now, for applying HAM to the equation Eq. (30), we define the following linear operator:

$$N[\xi(r,t;\theta)] = \frac{\partial\xi(r,t;\theta)}{\partial t}, \tag{33}$$

with the property $N[c] = 0$, ($c = constant$). With the help of Eq. (30), we define non-linear operator M such that

$$M[\xi(r,t;\theta)] = \xi_t + \xi\xi_r + \xi_{3r} - \xi_{5r}. \tag{34}$$

With the help of above equation, we construct the following zero-order deformation equation:

$$(1-\theta)\,N[\xi(r,t;\theta) - u_0(r,t)] = \theta\,\hbar\,H(r,t)M[\xi(r,t;\theta)]. \tag{35}$$

From the above equation, we can see that for $\theta = 0$ and $\theta = 1$, $\xi(r,t;0) = u_0(r,t)$ and $\xi(r,t;1) = u(r,t)$, respectively.

Differentiating the zero-order deformation Eq. (35) n-times with respect to θ, we obtain the n^{th}-order deformation equation as follows:

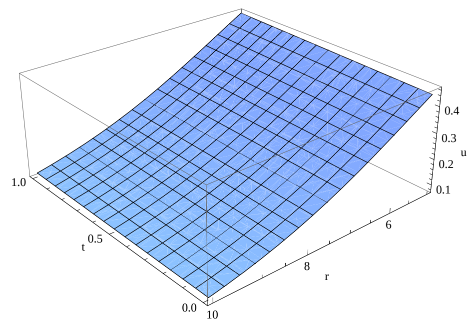
$$N[u_n(r,t) - \sigma_n\,u_{n-1}(r,t)] = \hbar\,H\,R_n(u_{n-1}), \tag{36}$$

with the initial condition $u_n(r,0) = \frac{105}{169}\text{Sech}^4\left[\frac{1}{2\sqrt{13}}(r-a)\right]$, where

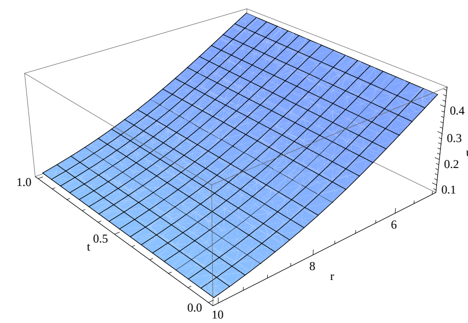
$$R_n(u_{n-1}) = \frac{\partial u_{n-1}}{\partial t} + \sum_{k=0}^{n-1} u_k\frac{\partial u_{n-1-k}}{\partial r} + \frac{\partial^3 u_{n-1}}{\partial r^3} - \frac{\partial^5 u_{n-1}}{\partial r^5}. \tag{37}$$

For convenience, we select the value of the auxiliary function $H = 1$ and parameter $\hbar = h$. Therefore, the solution of the deformation becomes

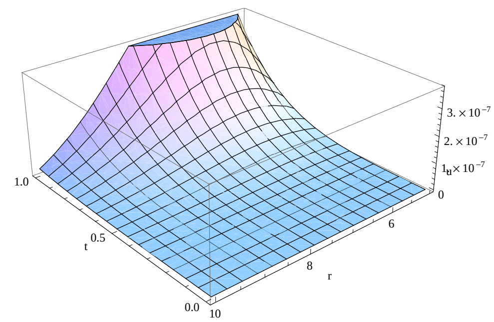
$$u_n(r,t) = \sigma_n u_{n-1}(r,t) + hN^{-1}R_n(u_{n-1},r,t).$$



(a) Approximate Solution



(b) Exact Solution



(c) Absolute Error $|u_{eract} - u_{approrimate}|$

Figure 6: Solution profiles of the equation (30): (a) approximate solution (b) exact solution (c) absolute error $|u_{exact} - u_{approrimate}|$.

Hence, the third order approximate series solution of equation (30) is obtained as

$$
\begin{aligned}
u(r,t) &= \sum_{k=0}^{3} u_k(r,t) \\
&= \frac{315}{676}\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] - \frac{408240}{62748517}h^2t^2\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] - \frac{272160}{62748517}h^3t^2\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] \\
&+ \frac{105}{676}\mathrm{Cosh}\left[\frac{3(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^7\left[\frac{r-2}{2\sqrt{13}}\right] + \frac{204120}{6748517}h^2t^2\mathrm{Cosh}\left[\frac{3(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^7\left[\frac{r-2}{2\sqrt{13}}\right] \\
&+ \frac{136080}{6748517}h^3t^2\mathrm{Cosh}\left[\frac{3(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^7\left[\frac{r-2}{2\sqrt{13}}\right] \\
&- \frac{11340}{28561\sqrt{13}}ht\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] \\
&- \frac{11340}{28561\sqrt{13}}h^2t\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] \\
&- \frac{3780}{28561\sqrt{13}}h^3t\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right]Sech^6\left[\frac{r-2}{2\sqrt{13}}\right] \\
&+ \frac{8981280}{10604499373\sqrt{13}}h^3t^3\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right] \\
&- \frac{11340}{28561\sqrt{13}}ht\mathrm{Cosh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right]\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right] \\
&- \frac{11340}{28561\sqrt{13}}h^2t\mathrm{Cosh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right]\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right] \\
&- \frac{3780}{28561\sqrt{13}}h^3t\mathrm{Cosh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right]\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right] \\
&- \frac{3265920}{10604499373\sqrt{13}}h^3t^3\mathrm{Cosh}\left[\frac{(r-2)}{2\sqrt{13}}\right]\mathrm{Sech}^6\left[\frac{r-2}{2\sqrt{13}}\right]\mathrm{Tanh}\left[\frac{(r-2)}{2\sqrt{13}}\right]. \qquad (38)
\end{aligned}
$$

4 Convergence Analysis and Numerical Solution

Here, we calculate the numerical results and absolute errors from third to fifth order approximations. The Absolute error is defined as:

$$E_n = |u_{exact} - \sum_{k=0}^{n} u_k|. \qquad (39)$$

The approximate series solutions of the Eqs. (12), (21) and (30) are given by Eqs. (20), (29) and (38), respectively. The convergence of these obtained series solutions by the HAM strongly depend upon the value of the parameter h, called h-curve. The h-curves obtained by the HAM for selecting the range of values of h, which is admissible to our problem, are

t	x	E_3	E_4	E_5
0.1	2	3.289474E-6	1.644737E-7	8.223684E-9
	4	1.001603E-6	2.504006E-9	6.260020E-11
	6	1.307805E-8	2.179675E-10	3.632789E-12
0.3	2	2.977941E-4	4.466912E-5	6.700368E-6
	4	8.551520E-6	6.413640E-7	4.810230E-8
	6	1.096491E-6	5.482456E-8	2.7412281E-9
0.5	2	2.604167E-3	6.510417E-4	1.627604E-4
	4	6.975446E-5	8.719308E-6	1.089913E-6
	6	8.768238E-6	7.306865E-7	6.0890541E-8

Table 1: Comparison of absolute errors between the exact solutions and approximate solutions of 3^{rd}, 4^{th} and 5^{th} order of equation (12) obtained by HAM for $h = -1$ and different values of t and r.

t	x	E_3	E_4	E_5
0.1	-2	5.528044E-7	1.109263E-8	2.652004E-12
	-4	7.481430E-8	1.500865E-9	3.588969E-13
	-6	1.012502E-8	2.031120E-10	4.857226E-14
0.3	-2	4.306649E-4	2.609172E-6	5.659489E-9
	-4	5.829181E-6	3.523471E-7	7.659286E-10
	-6	7.888936E-7	4.768494E-8	1.036572E-10
0.5	-2	3.199317E-4	3.250396E-5	1.973652E-7
	-4	4.332475E-5	4.372223E-6	2.671048E-8
	-6	5.863368E-6	5.917160E-7	3.614871E-10

Table 2: Comparison of absolute errors between the exact solutions and approximate solutions of 3^{rd}, 4^{th} and 5^{th} order of equation (21) obtained by HAM for $h = -1$ and different values of t and r.

t	x	E_3	E_4	E_5
0.1	2	1.103910E-10	5.551115E-16	4.440892E-16
	4	2.344447E-11	2.745581E-13	3.330669E-16
	6	5.741696E-11	3.574918E-14	2.220446E-16
0.3	2	8.941089E-9	6.286083E-13	6.286083E-13
	4	1.943460E-9	6.669609E-11	6.805667E-14
	6	4.644709E-9	8.9594445E-12	3.149486E-13
0.5	2	6.898125E-8	1.347233E-11	1.347233E-11
	4	1.553945E-8	8.582410E-10	1.398881E-12
	6	3.578978E-8	1.181518E-10	7.344014E-12

Table 3: Comparison of absolute errors between the exact solutions and approximate solutions of 3^{rd}, 4^{th} and 5^{th} order of equation (30) obtained by HAM for $h = -1$ and different values of t and r.

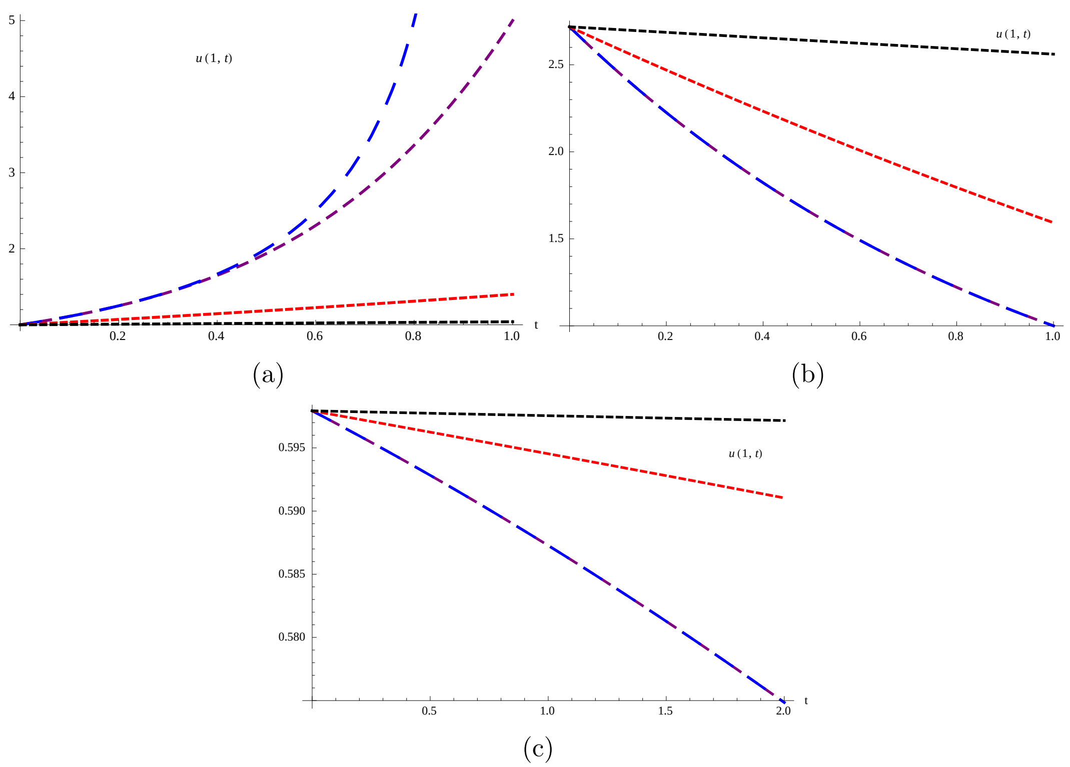


Figure 7: The results obtained by HAM for different values of h for Eps. (12), (21) and (30) as figure (a), (b) and (c), respectively; black line for $h = -0.1$, red line for $h = -0.01$, purple line for $h = -1$ and blue line for exact solution.

shown in Figures 1, 2 and 3. In Figures 1, 2 and 3, we have taken the convergence region for which the h-curve is parallel or almost parallel to the h-axis. So, we have taken $h = -1.0$ for our analysis as this is the most suitable value for our problems. We should note that if we choose an appropriate initial guess and linear operator, then by only few terms, we can get accurate approximations. However, if the the auxiliary linear operator and the initial guess are not good enough but reasonable, still we can get convergent solutions by properly selecting the value of parameter h. Figures 7(a)-7(c) show the results obtained by HAM for different values of parameter h. Here, blue line shows the exact solution while black, red and purple lines show the approximate solutions for $h = -0.1, -0.01, -1$, respectively. We can see that $h = -1$ is the best value of h as for this value, our solutions converge to the exact solutions.

For the analysis of our results, we have taken the value of constant $a = 2$ in Eq. (31). We have compared our obtained approximate results with the known exact results given by Eqs. (14), (23) and (32). The approximate solutions, exact solutions and absolute errors for our problems are shown graphically in Figures 4, 5 and 6. On the basis of comparison between obtained results and known results, absolute errors are shown by Tables $1, 2$, and 3. From error analysis, we can conclude that our results are in good agreements with the known results and have a very high accuracy. Also, it may be noted that as we increase the order of approximation, accuracy of the solutions increases.

5 Conclusion

In this work, the HAM is applied to obtain the approximate analytical solutions of three different types of KdV equations of fifth-order arising in many areas of science like mathematical physics, fluid dynamics etc. Approximate series solutions are obtained by the HAM. In Figures 4, 5 and 6, the approximate and exact solutions are shown together with absolute errors. By the error analysis (see Table $1, 2, 3$) which is made between known exact results and obtained approximate results, it can be concluded that the obtained results have a very high accuracy. On the basis of this study, we have found that the HAM is very powerful and effective method in the numerical methods for solving non-linear PDEs as it gives the liberty of controlling the rate of approximate series and the convergence region. This method does not depend upon linearization or any kind of physically unrealistic assumptions. The method is capable of reducing the volume of the computational work and maintaining the high accuracy of the numerical results. Thus, it can be used to solve other higher order nonlinear integer and fractional order equations. HAM has many applications in the field of engineering, mathematical science and physics for solving a large class of non-linear partial differential equations.

"Data sharing not applicable to this article as no data sets were generated or analysed during the current study."

Conflict of interest "The authors declare that they have no conflict of interest."

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