Communications in Mathematics **31** (2023), no. 1, 273–291
DOI: https://doi.org/10.46298/cm.10404

The Markovian Bernoulli queues with operational server vacation, Bernoulli's weak and strong disasters, and linear impatient customers.

Zahia Ahmedi Ezzourgui, Hafida Saggou, Megdouda Ourbih-Tari and El Bey Bourennane

Abstract. This paper studies the stationary analysis of a Markovian queuing system with Bernoulli feedback, interruption vacation, linear impatient customers, strong and weak disaster with the server's repair during the server's operational vacation period. Each customer has its own impatience time and abandons the system as soon as that time ends. When the queue is not empty, the server's operational vacation can be interrupted if the service is completed and the server starts a busy period with a probability $\bar{q}$ or continues the operational vacation with a probability q. A strong disaster forces simultaneously all present customers (waiting and served) to abandon the system permanently with a probability p but a weak disaster is that all customers decide to be patient by staying in the system, and wait during the repair time with a probability $\bar{p}$, where arrival of a new customer can occur. As soon as the repair process of the server is completed, the server remains providing service in the operational vacation period. We analyze this proposed model and derive the probabilities generating functions of the number of customers present in the system together with explicit expressions of some performance measures such as the mean and the variance of the number of customers in the different states, together with the mean sojourn time. Finally, numerical results are presented to show the influence of the system parameters on some studied performance measures.

MSC 2020: 68M20, 05D40, 05A15
Keywords: single server,working vacation,strong and weak disaster,server's repair, linear impatient customer, interruption vacation.
Affiliation:
Zahia Ahmedi Ezzourgui –Laboratory of RIIMA, Faculty of Mathematics, University of USTHB, 16111, Algiers, Algeria,
E-mail: zahiaahmedi1992@gmail.com
Hafida Saggou –Laboratory of RIIMA, Faculty of Mathematics, University of USTHB, 16111, Algiers, Algeria,
E-mail: saggou_hafida@yahoo.fr
Megdouda Ourbih-Tari –Institut des Sciences, Centre Universitaire Morsli Abdellah de Tipaza, 42020, Tipaza, Algeria, and, laboratory of Applied Mathematics, Faculty of Exact Sciences, University of Bejaia, 06000, Bejaia, Algeria,
E-mail: ourbihmeg@gmail.com

1 Introduction

Queuing systems with server vacations accommodate the real-world situations more closely, becoming more and more important with the development of telephone systems and communications. Nowadays, there is growing interest in the analysis of queuing models with impatient customers. This is due to their potential applications and their increasing demands in network systems and industrial scenario, communications, transportation, planning, production and road-traffic areas. It is also very important in the issue where authors expect a potential response from their submitted papers, and in the problem of hospital emergency rooms where critical patients are treated.

Along with the intensity and frequency of different random events, a disaster is one of the most common problems in the economic world. When all customers are cleared from the system and lost, the impact of the disaster changes from some systems to others depending on its severity and risk, for example, infectious viruses as corona virus which raises concerns all mankind. Other examples are the lack of financial liquidity or also the scarcity of products, especially the basics in times of crisis. Repairing the server can reduce the risk of disaster.

In fact, since the paper of Altman and Yechiali [1], various aspects of server vacations with impatient customers for single server queuing systems have interested several authors for instance; Yue et al. [12] analyzed a queuing system with impatient customers and working vacations; Yue et al. [8] studied customer impatience in a single server queuing system with working vacations; Yue et al. [9] analyzed an M/M/1 queue with vacations and impatience timers which depend on the server's states; Bouchentouf et al. [2] were interested by the study of the economic analysis of Markovian Bernoulli feedback queuing system with waiting server, vacations and impatient arrivals.

In some systems, an interruption may occur during the server vacation when a customer has completed his service and then, the server starts the busy period. To name just few of the numerous contributions on the analysis of queues with vacation interruptions, Goswami [5] was interested in studying impatient customers in queues with two different types of vacation, and, Li and Tian [7] have studied the single server queues with working vacations where the server moves to another state after the interruption of the working vacation.

Most of the analysis of queuing systems, here, have focused on one type of disaster, only on busy period with repair state, for instance: the analysis of a single-server queue with disasters and repairs under Bernoulli vacation schedule [10]; the study of queues with impatient customers and disasters when the system is down [11]; the analysis of a disaster queue problem with Markovian and impatient arrivals [3]; and the Markovian single queue with disasters and geometric reneging [4].

For most economic problems, in all of the disaster related studies, generally all customers abandon the working system, but in the last three years the catastrophe make a big

El Bey Bourennane –ImVia laboratory, University of Burgundy, Dijon, 21000, France,
E-mail: ebourenn@u-bourgogne.fr

and dominant influence in the healthy and practical life, so, we have to study the challenge in the current type of disasters.

This paper considers a new model of Markovian queuing system with one server, vacation interruptions and impatient customers and two types of disaster during the server's operational vacation which begins when the queue is empty. The first disaster is a strong Bernoulli with a probability p (a strong Bernoulli disaster forced all customers to abandon the system), so the server remains in the operational vacation period because the system becomes empty. The second type is a weak disaster (when all customers don't abandon the system), in this case the server begins immediately the repair period with a probability $\bar{p}$. Once the repair time is completed, the customer already in service continuous his service in the server's operational vacation. During this repair period, a new customer can enter the system. During the server's operational vacation, one of the customers abandon the system when his time of impatience is expired. Once the server's operational vacation is completed, the server stays in this state with a probability q or interrupts this state and starts a normal service in the busy period with a probability $\bar{q}$ if the queue is not empty.

The rest of the article is structured as follows: After introducing the concept of the server's operational vacation and the types of disasters, we obtain a description of the model in section 2, and the explicit expressions of the steady-state probability generation functions of the system in different cases of services are given in section 3. While, important performance measures of the system as the mathematical expectation of the number of customers in the server vacation, busy period and repair state, the mathematical expectation of the number of customers in the system, the variance of the number of customers in each state, and the sojourn time are all described in section 4. Section 5 reports interesting numerical examples with their illustrations. We finish this paper with a conclusion.

2 Model description

We study a queuing system with one server having three periods of service; the first is the busy period with normal service, the second is the operational vacation period and the last is the repair period. During all times of services, the customers arrive according to a Poisson process with a rate λ and First Come First Serve discipline. The service times of customers are independently and identically distributed random variables following in the busy period, an exponential distribution with a rate μ, and in the server's operational vacation period, an exponential distribution with a rate η. It is worth mentioning that during the repair time, there is no service. The vacation time has an exponential distribution with a rate ϕ which begins when the queue is empty. A disaster occurs according to a Poisson process of a rate γ when the server is in an operational vacation period.

In the operational vacation period and when the system is not empty, the strong disaster will forces all current customers to leave the system with a p probability, otherwise, the weak disaster forced all customers to stay in the system and a repair period begins immediately with a probability of $\bar{p}$. The repair time follows an exponential distribution of rate α. When the repair time is completed, the customer in service continuous his service in the operational vacation period. One of the customers in the system is assumed to be impatient during the server's operational vacations, the customer operates an impatience time Γ considered as an exponential distribution with a rate ξ, Once his time of impatience is completed, he abandons the system.

If the server ends a service while he is in an operational vacation period, he will start a busy period with probability $\bar{q}$ if the queue is not empty of course, otherwise he remains in the same period with a probability q. The inter-arrival times, vacation times, disaster times, service times, and repair times are all considered mutually independent variables.

3 Analysis of the steady-state probabilities

We model a two-dimensional Markov chain by $\{(N(t), S(t)); t \geq 0\}$ to describe the system states. At time t, we denote by $N(t)$ the number of customers in the system and $S(t)$ the state of the server. The space of states of the Markov process is

$$\Omega = \{(0,0) \cup (i,j), i = 1,2,\dots \quad \text{and} \quad j = 0,1,2\}$$

and the server's states are

$$S(t) = \begin{cases} 0 & \text{if the server is in operational vacation period,} \\ 1 & \text{if the server is in busy period,} \\ 2 & \text{if the server is in repair period.} \end{cases}$$

If the steady-state distribution exists, we can define limiting probabilities by the followings

$$\begin{aligned}
\pi_{n,0} &= \lim_{t \longmapsto +\infty} P(N(t) = n, S(t) = 0), \qquad n \geq 0,\\
\pi_{n,1} &= \lim_{t \longmapsto +\infty} P(N(t) = n, S(t) = 1), \qquad n \geq 1,\\
\pi_{n,2} &= \lim_{t \longmapsto +\infty} P(N(t) = n, S(t) = 2), \qquad n \geq 1.
\end{aligned}$$

These stationary probabilities satisfy the following balance equations

$$\lambda \pi_{0,0} = \mu \pi_{1,1} + (\eta + \xi)\pi_{1,0} + \gamma p \sum_{n=1}^{\infty} \pi_{n,0} \tag{1}$$

$$(\eta + \lambda + \gamma + \phi + n\xi)\pi_{n,0} = \lambda \pi_{n-1,0} + \alpha \pi_{n,2} + (q\eta + (n+1)\xi)\pi_{n+1,0} \qquad n \geqslant 1 \tag{2}$$

$$(\lambda + \mu)\pi_{1,1} = \phi \pi_{1,0} + \bar{q}\eta \pi_{2,0} + \mu \pi_{2,1} \tag{3}$$

$$(\lambda + \mu)\pi_{n,1} = \phi \pi_{n,0} + \bar{q}\eta \pi_{n+1,0} + \mu \pi_{n+1,1} + \lambda \pi_{n-1,1} \qquad n \geqslant 2 \tag{4}$$

$$(\alpha + \lambda)\pi_{1,2} = \gamma \bar{p} \pi_{1,0} \tag{5}$$

$$(\alpha + \lambda)\pi_{n,2} = \gamma \bar{p} \pi_{n,0} + \lambda \pi_{n-1,2} \qquad n \geqslant 2 \tag{6}$$

where the normalization condition is

$$\sum_{i=0}^{+\infty}\pi_{i,0}+\sum_{i=1}^{+\infty}\pi_{i,1}+\sum_{i=1}^{+\infty}\pi_{i,2}=1.$$

Let

$$\mathbf{G}_j(z)=\begin{cases}\sum_{i=0}^{\infty}\pi_{i,j}z^i & \text{j} = 0\ ,\\ \sum_{i=1}^{\infty}\pi_{i,j}z^i & \text{j} = 1,\ 2.\end{cases}$$

The following theorem describes the steady-state distribution of the system in terms of probabilities generating functions.

Theorem 3.1. *In the steady state, for $|z|<1$, we have*

$$\boldsymbol{G}_0(z)=\frac{A_2K_2(z)-(\gamma p\boldsymbol{G}_0(1)+A_1)K_1(z)-A_3K_3(z)}{\xi(1-z)^{\frac{1}{\xi}(\beta-q\eta-\lambda)}z^{\frac{q\eta}{\xi}}\left[\frac{(\alpha+\lambda)(1-z)}{\lambda(1-z)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}e^{-\frac{\lambda}{\xi}z}},\tag{7}$$

$$\boldsymbol{G}_1(z)=\frac{(\phi z+\bar{q}\eta)\boldsymbol{G}_0(z)-(\phi+\bar{q}\eta)\boldsymbol{G}_0(1)z-\bar{q}\eta(1-z)\pi_{0,0}}{(\lambda z-\mu)(1-z)},\tag{8}$$

$$\boldsymbol{G}_2(z)=\frac{\gamma\bar{p}(\boldsymbol{G}_0(1)-\pi_{0,0})}{\alpha+\lambda(1-z)},\tag{9}$$

where

$$K_1(z)=\int_0^z(1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)-1}x^{\frac{q\eta}{\xi}}[\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}e^{-\frac{\lambda}{\xi}x}dx,$$
$$K_2(z)=\int_0^z(1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)}x^{\frac{q\eta}{\xi}-1}[\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}e^{-\frac{\lambda}{\xi}x}dx,$$
$$K_3(z)=\int_0^z(1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)}x^{\frac{q\eta}{\xi}}[\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}e^{-\frac{\lambda}{\xi}x}\frac{1}{\alpha+\lambda(1-x)}dx,$$
$$\beta=\lambda+\eta+\phi+\gamma,$$
$$A_1=\mu\pi_{1,1}+(\phi+\bar{q}\eta)\pi_{0,0}+\bar{q}\eta\pi_{1,0},\quad A_2=q\eta\pi_{0,0}\quad\text{and}\quad A_3=\gamma\lambda\bar{p}\pi_{0,0}.\tag{10}$$

Proof. Multiplying the power of z^n in (6), adding over all possible values of n, and using (5), we obtain

$$[\alpha+\lambda(1-z)]\mathbf{G}_2(z)=\gamma\bar{p}[\mathbf{G}_0(z)-\pi_{0,0}],\tag{11}$$

so, we have

$$\mathbf{G}_2(z)=\frac{\gamma\bar{p}(\mathbf{G}_0(1)-\pi_{0,0})}{\alpha+\lambda(1-z)}.\tag{12}$$

In the same way, using (1) and (2), we get

$$\mathbf{G}_0(z)(g(z)+\frac{\alpha\gamma\bar{p}z}{\alpha+\lambda(1-z)})+\xi z(1-z)\mathbf{G}_0'(z)\tag{13}$$
$$=A_2(1-z)-z(\frac{\gamma\lambda\bar{p}\pi_{0,0}(1-z)}{\alpha+\lambda(1-z)}+A_1+\gamma p\mathbf{G}_0(1)),$$

where

$$g(z) = \lambda z^2 - \beta z + q\eta. \tag{14}$$

Similarly, multiplying (4) and (5) with the power of z^n and adding over n, we have

$$(\lambda z - \mu)(1-z)\mathbf{G}_1(z) = (\phi z + \bar{q}\eta)\mathbf{G}_0(z) - A_1 z - \bar{q}\eta(1-z)\pi_{0,0}. \tag{15}$$

For $\xi \neq 0$, $z \neq 0$ and $|z| < 1$, the formula (13) becomes

$$\begin{aligned}\frac{d}{dz}\mathbf{G}_0(z) + \frac{1}{\xi}\mathbf{G}_0(z)\left[\frac{g(z)}{z(1-z)} + \frac{\alpha\gamma\bar{p}}{(1-z)(\alpha+\lambda(1-z))}\right]& \\ = \frac{1}{\xi}\left[\frac{A_2}{z} - \frac{A_1 + \gamma p\mathbf{G}_0(1)}{(1-z)} - \frac{\gamma\lambda\bar{p}\pi_{0,0}}{\alpha+\lambda(1-z)}\right]&.\end{aligned} \tag{16}$$

Solving the linear differential equation (16), we obtain an integrating factor as

$$e^{\int_0^z \frac{g(x)}{\xi x(1-x)} + \frac{\alpha\gamma\bar{p}}{\xi(1-x)(\alpha+\lambda(1-x))}dx} = (1-z)^{\frac{1}{\xi}(\beta-q\eta-\lambda)} z^{\frac{q\eta}{\xi}} \left[\frac{(\alpha+\lambda)(1-z)}{\lambda(1-z)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}z}. \tag{17}$$

Multiplying both sides of (16) by the integrating factor, we have

$$\begin{aligned}&\frac{d}{dz}[(1-z)^{\frac{\beta-q\eta-\lambda}{\xi}} z^{\frac{q\eta}{\xi}} \left[\frac{(\alpha+\lambda)(1-z)}{\lambda(1-z)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}z}\mathbf{G}_0(z)] \\ &= \frac{1}{\xi}\left[\frac{A_2}{z} - \frac{A_1+\gamma p\mathbf{G}_0(1)}{1-z} - \frac{\gamma\lambda\bar{p}\pi_{0,0}}{\alpha+\lambda(1-z)}\right](1-z)^{\frac{1}{\xi}(\beta-q\eta-\lambda)} z^{\frac{q\eta}{\xi}} \left[\frac{(\alpha+\lambda)(1-z)}{\lambda(1-z)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}z}.\end{aligned} \tag{18}$$

Integrating both sides of (18) from 0 to z, we obtain

$$\mathbf{G}_0(z) = \frac{A_2K_2(z) - (\gamma p\mathbf{G}_0(1) + A_1)K_1(z) - A_3K_3(z)}{\xi(1-z)^{\frac{1}{\xi}(\beta-q\eta-\lambda)} z^{\frac{q\eta}{\xi}} \left[\frac{(\alpha+\lambda)(1-z)}{\lambda(1-z)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}z}}. \tag{19}$$

Replacing z by 1 in (13), we have

$$A_1 = (\phi + \bar{q}\eta)\mathbf{G}_0(1). \tag{20}$$

Now using (20), formula (15) can be rewritten as

$$\mathbf{G}_1(z) = \frac{(\phi z + \bar{q}\eta)\mathbf{G}_0(z) - (\phi + \bar{q}\eta)\mathbf{G}_0(1)z - \bar{q}\eta(1-z)\pi_{0,0}}{(\lambda z - \mu)(1-z)}. \tag{21}$$

□

Remark 3.2. Setting $\gamma = 0$ in (13), we get

$$\mathbf{G}_0(z)g(z) + \xi z(z-1)\mathbf{G}_0'(z) = zA_1 - A_2(1-z)$$

which matches with Goswami (2014) results.

Corollary 3.3. *The steady-state probabilities that the server is in operational vacation, busy or under repair, are given respectively by*

$$\begin{aligned}\boldsymbol{G}_0(1) &= \rho\alpha(\mu-\lambda)[\lambda\gamma\bar{p}K_3(1) - q\eta K_2(1)] \times \Pi^{-1},\\ \boldsymbol{G}_1(1) &= \frac{\boldsymbol{G}_0'(1)(\phi+\bar{q}\eta) - \bar{q}\eta(\boldsymbol{G}_0(1)-\pi_{0,0})}{\mu-\lambda},\\ \boldsymbol{G}_2(1) &= \frac{\gamma\bar{p}(\boldsymbol{G}_0(1)-\pi_{0,0})}{\alpha},\end{aligned}$$

where

$$\begin{aligned}\rho &= \eta\bar{q}+\gamma p+\phi+\xi,\\ \Pi &= [(\alpha\lambda+\lambda\gamma\bar{p}-\alpha\eta q)(\phi+\bar{q}\eta) - \rho(\alpha\eta\bar{q}-(\gamma\bar{p}+\alpha)(\mu-\lambda))][\lambda\gamma\bar{p}K_3(1)-q\eta K_2(1)]\\ &\quad -(\gamma p+\phi+\bar{q}\eta)K_1(1)[(\alpha\eta q-\lambda\gamma\bar{p})(\phi+\bar{q}\eta)+\rho(\alpha\eta\bar{q}-\gamma\bar{p}(\mu-\lambda))],\\ K_1(1) &= \int_0^1 (1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)-1} x^{\frac{q\eta}{\xi}} [\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}x}dx,\\ K_2(1) &= \int_0^1 (1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)} x^{\frac{q\eta}{\xi}-1} \left[\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}\right]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}x}dx, \qquad (22)\\ K_3(1) &= \int_0^1 (1-x)^{\frac{1}{\xi}(\beta-q\eta-\lambda)} x^{\frac{q\eta}{\xi}} [\frac{(\alpha+\lambda)(1-x)}{\lambda(1-x)+\alpha}]^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} e^{-\frac{\lambda}{\xi}x} \frac{1}{\alpha+\lambda(1-x)}dx,\\ \pi_{0,0} &= \frac{(\gamma p+\phi+\bar{q}\eta)K_1(1)}{q\eta K_2(1)-\lambda\gamma\bar{p}K_3(1)}\boldsymbol{G}_0(1).\end{aligned}$$

Proof. Putting $z=1$ in (12), we have

$$\mathbf{G}_2(1) = \frac{\gamma\bar{p}(\mathbf{G}_0(1)-\pi_{0,0})}{\alpha}. \qquad (23)$$

Using l'Hopital's rule in (8), we have

$$\mathbf{G}_1(1) = \frac{\mathbf{G}_0'(1)(\phi+\bar{q}\eta) - \bar{q}\eta(\mathbf{G}_0(1)-\pi_{0,0})}{\mu-\lambda}. \qquad (24)$$

Since $\mathbf{G}_1(1) = 1-\mathbf{G}_0(1)-\mathbf{G}_2(1)$, from (23) and (24), we get the followings

$$\mathbf{G}_0'(1) = \frac{1}{\phi+\bar{q}\eta}\left((\mu-\lambda)[1-\mathbf{G}_0(1)] + \frac{1}{\alpha}(\bar{q}\eta\alpha-\gamma\bar{p}(\mu-\lambda))[\mathbf{G}_0(1)-\pi_{0,0}]\right). \qquad (25)$$

Differentiating (13) at $z=1$, we obtain

$$\mathbf{G}_0'(1) = \frac{\mathbf{G}_0(1)(\alpha\lambda+\lambda\gamma\bar{p}-\alpha\eta q)+\pi_{0,0}(\alpha q\eta-\lambda\gamma\bar{p})}{[\alpha(\eta\bar{q}+\gamma p+\phi+\xi)]}. \qquad (26)$$

From (25) and (26) we get

$$\begin{aligned}\rho\alpha(\mu-\lambda) =& \{[\alpha\lambda+\lambda\gamma\bar{p}-\alpha\eta q](\phi+\bar{q}\eta)-\rho[\alpha\eta\bar{q}-(\gamma\bar{p}+\alpha)(\mu-\lambda)]\}\mathbf{G}_0(1) \\ &+\pi_{0,0}\{[\alpha\eta q-\lambda\gamma\bar{p}](\phi+\bar{q}\eta)+\rho[\alpha\eta\bar{q}-\gamma\bar{p}(\mu-\lambda)]\}.\end{aligned} \tag{27}$$

When z tends to 1 in (19) and, using (10) and (20), we get

$$\begin{aligned}\mathbf{G}_0(1) =& \frac{-(\gamma p+\phi+\bar{q}\eta)\mathbf{G}_0(1)K_1(1)+q\eta\pi_{0,0}K_2(1)-\lambda\gamma\bar{p}\pi_{0,0}K_3(1)}{\xi e^{-\frac{\lambda}{\xi}}\left(\frac{\alpha+\lambda}{\alpha}\right)^{\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}} \\ &\times \lim_{z\longrightarrow 1}(1-z)^{-\frac{1}{\xi}(\beta-\lambda-q\eta+)-\frac{2\lambda\gamma\bar{p}}{\alpha\xi}}.\end{aligned} \tag{28}$$

As $0 \leqslant \mathbf{G}_0(1) = \sum_0^\infty \pi_{n,0} \leqslant 1$ and $\lim_{z\longrightarrow 1}(1-z)^{-\frac{1}{\xi}(\beta-\lambda-q\eta)-\frac{2\lambda\gamma\bar{p}}{\alpha\xi}} \to \infty$, so we must have

$$-(\gamma p+\phi+\bar{q}\eta)\mathbf{G}_0(1)K_1(1)+q\eta\pi_{0,0}K_2(1)-\lambda\gamma\bar{p}\pi_{0,0}K_3(1)=0. \tag{29}$$

Using (27) and (29), we obtain

$$\pi_{0,0} = \frac{(\gamma p+\phi+\bar{q}\eta)K_1(1)}{q\eta K_2(1)-\lambda\gamma\bar{p}K_3(1)}\mathbf{G}_0(1). \tag{30}$$

So

$$\mathbf{G}_0(1) = \rho\alpha(\mu-\lambda)[\lambda\gamma\bar{p}K_3(1)-q\eta K_2(1)]\times\Pi^{-1},$$

where

$$\begin{aligned}\Pi =& [(\alpha\lambda+\lambda\gamma\bar{p}-\alpha\eta q)(\phi+\bar{q}\eta)-\rho(\alpha\eta\bar{q}-(\gamma\bar{p}+\alpha)(\mu-\lambda))]\times[\lambda\gamma\bar{p}K_3(1)-q\eta K_2(1)] \\ &-(\gamma p+\phi+\bar{q}\eta)K_1(1)\times[(\alpha\eta q-\lambda\gamma\bar{p})(\phi+\bar{q}\eta)+\rho(\alpha\eta\bar{q}-\gamma\bar{p}(\mu-\lambda))].\end{aligned}$$

□

4 Performance measures

In this section, we will compute some important performance measures of the proposed queuing system. Let L_i, $i = 0, 1, 2$ be the number of customers in the server's states respectively in operational vacation period, busy period and repair period . Let T be the total sojourn time of a customer in the system measured from the moment of arrival until departure, where the latter occurs at the end of service or at his abandonment.

Theorem 4.1. *The mathematical expectation of T is given by*

$$E(T) = \frac{1}{\lambda}(E(L_0)+E(L_1)+E(L_2)) \tag{31}$$

where

$$E(L_0) = \frac{\rho(\mu-\lambda)[\lambda\gamma\bar{p}K_3(1) - q\eta K_2(1)] \times \Pi^{-1}}{(\eta\bar{q}+\gamma p+\phi+\xi)} \left[(\alpha\lambda + \lambda\gamma\bar{p} - \alpha\eta q) + \frac{(\alpha q\eta - \lambda\gamma\bar{p})(\gamma p+\phi+\bar{q}\eta)}{q\eta k_2(1) - \gamma\lambda\bar{p}k_3(1)}\right],$$

$$E(L_1) = \frac{\phi+\bar{q}\eta}{\mu-\lambda} \frac{\boldsymbol{G}_0(1)(\lambda\alpha^2+\gamma\lambda\bar{p}) + E(L_0)[\alpha^2(\lambda-\eta-\gamma p-\phi-\xi)+\alpha\gamma\lambda\bar{p}] + \gamma\lambda\bar{p}\pi_{0,0}(\alpha+\lambda)}{\alpha^2(\gamma p+\phi+\eta\bar{q}+2\xi)}$$
$$+ \frac{1}{(\phi+\bar{q}\eta)(\mu-\lambda)} \times [(\phi\mu+\lambda\bar{q}\eta)(1-\boldsymbol{G}_0(1)) + \phi\bar{q}\eta(\boldsymbol{G}_0(1)-\pi_{0,0})],$$

$$E(L_2) = \frac{\gamma\bar{p}}{\alpha}\left(E(L_0) + \frac{\lambda}{\alpha}\boldsymbol{G}_0(1)[1 + \frac{(\gamma p+\phi+\bar{q}\eta)K_1(1)}{q\eta K_2(1) - \lambda\gamma\bar{p}K_3(1)}]\right).$$

Proof. Using the expression of $\pi_{0,0}$ and $\mathbf{G}_0(1)$ of Corollary 3.3 and in (26), we get the expected number of customers in the server's operational vacation $E(L_0)$.

$$E(L_0) = \frac{\rho(\mu-\lambda)[\lambda\gamma\bar{p}K_3(1) - q\eta K_2(1)] \times \Pi^{-1}}{(\eta\bar{q}+\gamma p+\phi+\xi)} \times \left[(\alpha\lambda + \lambda\gamma\bar{p} - \alpha\eta q) + \frac{(\alpha q\eta - \lambda\gamma\bar{p})(\gamma p+\phi+\bar{q}\eta)}{q\eta k_2(1) - \gamma\lambda\bar{p}k_3(1)}\right].$$

Next, we can determine the expected number of customers in busy period $E(L_1)$ from (21) by using l'Hopital's rule as follows

$$E(L_1) = \mathbf{G}_1'(1) = \frac{\phi+\bar{q}\eta}{\mu-\lambda}\frac{\mathbf{G}_0''(1)}{2} + \frac{1}{(\phi+\bar{q}\eta)(\mu-\lambda)} \times \mathbf{H}, \tag{32}$$

where

$$\mathbf{H} = [(\phi\mu+\lambda\bar{q}\eta)(1-\mathbf{G}_0(1)) + \phi\bar{q}\eta(\mathbf{G}_0(1)-\pi_{0,0})].$$

Differentiating (13) twice at $z = 1$, we obtain

$$\mathbf{G}_0''(1)\left[g(1)+\gamma\bar{p}-2\xi\right] + 2\mathbf{G}_0'(1)[g'(1) + \frac{\gamma\bar{p}(\alpha+\lambda)}{\alpha} - \xi] \tag{33}$$

$$= -\mathbf{G}_0(1)\left[g''(1) + \frac{2\gamma\bar{p}\lambda}{\alpha^2}\right] + 2\gamma\bar{p}\lambda\pi_{0,0}\frac{\alpha+\lambda}{\alpha^2}. \tag{34}$$

It is easy to see that

$$g(1) = -(\gamma+\phi+\eta\bar{q}), \qquad g'(1) = \lambda-\gamma-\eta-\phi, \qquad g''(1) = 2\lambda.$$

Thus, from (33), we get

$$\frac{\mathbf{G}_0''(1)}{2} = \frac{\mathbf{G}_0(1)(\lambda\alpha^2+\gamma\lambda\bar{p}) + E(L_0)[\alpha^2(\lambda-\eta-\gamma p-\phi-\xi)+\alpha\gamma\lambda\bar{p}] + \gamma\lambda\bar{p}\pi_{0,0}(\alpha+\lambda)}{\alpha^2(\gamma p+\phi+\eta\bar{q}+2\xi)}. \tag{35}$$

Using (25) and (32) in (34), we get $E(L_1)$ by

$$E(L_1) = \frac{\phi+\bar{q}\eta}{\mu-\lambda}\frac{\mathbf{G}_0(1)(\lambda\alpha^2+\gamma\lambda\bar{p})+E(L_0)[\alpha^2(\lambda-\eta-\gamma p-\phi-\xi)+\alpha\gamma\lambda\bar{p}]+\gamma\lambda\bar{p}\pi_{0,0}(\alpha+\lambda)}{\alpha^2(\gamma p+\phi+\eta\bar{q}+2\xi)} + \frac{1}{(\phi+\bar{q}\eta)(\mu-\lambda)}\left[(\phi\mu+\lambda\bar{q}\eta)(1-\mathbf{G}_0(1))+\phi\bar{q}\eta(\mathbf{G}_0(1)-\pi_{0,0})\right].$$

Differentiating (11) at $z=1$, we have

$$E(L_2) = \mathbf{G}_2'(1) = \frac{1}{\alpha}\left[\gamma\bar{p}E(L_0)+\lambda\mathbf{G}_2(1)\right]. \tag{36}$$

From (12) and (30), we get

$$E(L_2) = \frac{\gamma\bar{p}}{\alpha}\left(E(L_0)+\frac{\lambda}{\alpha}\mathbf{G}_0(1)[1+\frac{(\gamma p+\phi+\bar{q}\eta)K_1(1)}{q\eta K_2(1)-\lambda\gamma\bar{p}K_3(1)}]\right). \tag{37}$$

The expected number of customers in the system is obtained by

$$E(L) = E(L_0)+E(L_1)+E(L_2),$$

then, according to the Little's rule, we get the result. □

Corollary 4.2. *The variance of the number of customers in each state is given by*

$$V(L_i) = \boldsymbol{G}_i''(1)+E(L_i)-E^2(L_i), \qquad i=0,1,2.$$

Proof. To obtain the variance of the number of customers in each state, we can calculate only $\mathbf{G}_j''(1)$ for $j=0,1,2$. From (34), we get

$$\mathbf{G}_0''(1) = 2\frac{\mathbf{G}_0(1)(\lambda\alpha^2+\gamma\lambda\bar{p})+E(L_0)[\alpha^2(\lambda-\eta-\gamma p-\phi-\xi)+\alpha\gamma\lambda\bar{p}]+\gamma\lambda\bar{p}\pi_{0,0}(\alpha+\lambda)}{\alpha^2(\gamma p+\phi+\eta\bar{q}+2\xi)}. \tag{38}$$

Differentiating (11) twice at $z=1$, we obtain

$$\mathbf{G}_2''(1) = \frac{\gamma\bar{p}\mathbf{G}_0''(1)+2\lambda\mathbf{G}_2'(1)}{\alpha}. \tag{39}$$

We differentiate (15) three times at $z=1$

$$\mathbf{G}_1''(1) = \frac{\phi\mathbf{G}_0''(1)+2\lambda\mathbf{G}_1'(1)}{\mu-\lambda}+\frac{\mathbf{G}_0^{(3)}(1)(\phi+\bar{q}\eta)}{3(\mu-\lambda)}. \tag{40}$$

Finally, to calculate $\mathbf{G}_0^{(3)}(1)$, we differentiate (13) three times and apply to $z=1$, we get

$$\mathbf{G}_0^{(3)}(1) = \frac{2\mathbf{G}''(1)[\alpha^3(\lambda-\eta-\phi-\gamma p-3\xi)+\gamma\bar{p}\lambda\alpha^2]+2\lambda\alpha E(L_0)[\alpha^2+\gamma\bar{p}(\alpha+\lambda)]}{\alpha^3(\phi+6\xi+\gamma p+\eta\bar{q})} - \frac{2\gamma\lambda^2\bar{p}\pi_{0,0}(\alpha+2+3\lambda)}{\alpha^3(\phi+6\xi+\gamma p+\eta\bar{q})}.$$

□

4.1 Sojourn time

Let T_{ser} be the total sojourn time of a customer who finishes his service before leaving the system. Denote by $T_{n,j}$ the conditional sojourn time of a customer who does not abandon the system, given that, the state on arrival is (n, j). Then, we have

$$E(T_{n,1}) = \frac{n+1}{\mu} \qquad n = 1, 2, \ldots. \tag{41}$$

Let $p_{(n,0)(n,1)}$, $p_{(n,0)(n+1,0)}$, $p_{(n+1,0)(n,1)}$, $p_{(n,0)(n-1,0)}$, $p_{(n+1,0)(n+1,2)}$ the successive probabilities of transition from the state $(n, 0)$ to the state $(n, 1)$, from the state $(n, 0)$ to the state $(n + 1, 0)$, from the state $(n + 1, 0)$ to the state $(n, 1)$, from the state $(n, 0)$ to the state $(n - 1, 0)$ and from the state $(n + 1, 0)$ to the state $(n + 1, 2)$.

Using the approach of Altman and Yechiali (2006), we determine $E(T_{n,0})$. For $n \geq 1$, we get

$$\begin{aligned} E(T_{n,0}) = {} & p_{(n+1,0)(n+1,1)}[\frac{1}{\alpha_{n+1}} + E(T_{n,1})] \\ & + p_{(n+1,0)(n+2,0)}[\frac{1}{\alpha_{n+1}} + E(T_{n,0})] \\ & + p_{(n+1,0)(n,0)}[\frac{1}{\alpha_{n+1}} + E(T_{n-1,0})] \\ & + p_{(n+1,0)(n,1)}[\frac{1}{\alpha_{n+1}} + E(T_{n-1,1})] \\ & + p_{(n+1,0)(n+1,2)}[\frac{1}{\alpha_{n+1}} + E(T_{n,2}) + E(T_{n,0})] \end{aligned} \tag{42}$$

where

$$\alpha_n = \lambda + \phi + \eta + \bar{p}\gamma + n\xi \qquad \text{for any } n.$$

The mean sojourn time of the marked customer during the transition from state $(n+1, 0)$ to any other state is

$$\frac{1}{\alpha_{n+1}}.$$

The second term of the right hand side of equation (42) follows because a new arrival does not modify the sojourn time of a customer present in the system. In the next term, as the marked customer is not impatient, n customers can leave the system and the fourth term follows due to the vacation interruption while the fifth term follows due to repair period. Given that

$$p_{(n+1,0)(n+1,1)} = \frac{\phi}{\alpha_{n+1}},$$
$$p_{(n+1,0)(n+2,0)} = \frac{\lambda}{\alpha_{n+1}},$$
$$p_{(n+1,0)(n,0)} = \frac{\bar{q}\eta}{\alpha_{n+1}},$$
$$p_{(n+1,0)(n,1)} = \frac{n\xi + q\eta}{\alpha_{n+1}},$$
$$p_{(n+1,0)(n+1,2)} = \frac{\bar{p}\gamma}{\alpha_{n+1}},$$

then, $E(T_{n,0})$ becomes

$$\begin{aligned} E(T_{n,0}) = & \frac{\phi}{\alpha_{n+1}}[\frac{1}{\alpha_{n+1}} + E(T_{n,1})] \\ & + \frac{\lambda}{\alpha_{n+1}}[\frac{1}{\alpha_{n+1}} + E(T_{n,0})] \\ & + \frac{n\xi + q\eta}{\alpha_{n+1}}[\frac{1}{\alpha_{n+1}} + E(T_{n-1,0})] \\ & + \frac{\bar{q}\eta}{\alpha_{n+1}}[\frac{1}{\alpha_{n+1}} + E(T_{n-1,1})] \\ & + \frac{\bar{p}\gamma}{\alpha_{n+1}}[\frac{1}{\alpha_{n+1}} + E(T_{n,2}) + E(T_{n,0})]. \end{aligned} \tag{43}$$

On the other hand, for $n \geq 1$, we have

$$E(T_{n,2}) = p_{(n,2)(n,0)}[\frac{1}{\beta_n} + E(T_{n,0}]) + p_{(n,2)(n+1,2)}[\frac{1}{\beta_n} + E(T_{n,2}]). \tag{44}$$

The mean sojourn time of the marked customer during the transition from the state $(n+1, 2)$ to any other state is

$$\frac{1}{\beta_{n+1}} = \frac{1}{\alpha + \lambda}.$$

The first term of the right hand side of equation (44) follows because the repair period is finished and the second term follows because a new arrival does not vary the sojourn time of the present customer in the system. The transition probabilities in (44) are given then by

$$p_{(n,2)(n,0)} = \frac{\alpha}{\lambda + \alpha} \qquad \text{and} \qquad p_{(n,2)(n+1,2)} = \frac{\lambda}{\lambda + \alpha}. \tag{45}$$

From (44), we get

$$E(T_{n,2}) = \frac{1}{\alpha} + E(T_{n,0}). \tag{46}$$

As

$$E(T_{0,0}) = \frac{\phi}{\alpha_1}(\frac{1}{\alpha_1} + \frac{1}{\mu}) + \frac{\lambda}{\alpha_1}(\frac{1}{\alpha_1} + E(T_{0,0})) + \frac{\eta}{\alpha_1}\frac{1}{\alpha_1}, \tag{47}$$

we have then from (47),

$$E(T_{0,0}) = \frac{1}{\phi + \eta + \xi + \gamma\bar{p}} \left[\frac{\phi + \lambda + \eta}{\alpha_1} + \frac{\phi}{\mu}\right]. \tag{48}$$

Recursively iterating (43) using (46) and (48), we have

$$\begin{aligned} E(T_{n,0}) = & \frac{1}{\phi + \eta + (n+1)\xi - \gamma\bar{p}} \quad [\frac{\alpha_n}{\alpha_{n+1}} + \frac{(n+1)\phi + n\eta\bar{q}}{\mu} + \frac{\gamma\bar{p}}{\alpha}] \\ & + \prod_{j=1}^{n} \frac{j\xi + q\eta}{\phi + \eta + (j+1)\xi - \gamma\bar{p}} \times \frac{1}{\phi + \eta + \xi + \gamma\bar{p}} \left[\frac{\phi + \lambda + \eta}{\alpha_1} + \frac{\phi}{\mu}\right] \\ & + \sum_{i=1}^{n+1} [\frac{\alpha_{i-1}}{\alpha_i} + \frac{i\phi + (i-1)\eta\bar{q}}{\mu} + \frac{\gamma\bar{p}}{\alpha}] \times \prod_{j=i}^{n} \frac{j\xi + q\eta}{\phi + \eta + (j+1)\xi - \gamma\bar{p}}. \end{aligned}$$

Other performance measures are studied such as $\mathbb{P}_{ser}$ the proportion of served customers and $\mathbb{Q}$ the abandonment rate due to impatience and the strong disaster. Given that the expected number of served customers per unit of time is

$$\mu\mathbf{G}_1(1) + \eta[\mathbf{G}_0(1) - \pi_{0,0} + \mathbf{G}_2(1)],$$

so, $\mathbb{P}_{ser}$ the proportion of served customers is as follows

$$\mathbb{P}_{ser} = \frac{1}{\lambda}[\mu\mathbf{G}_1(1) + \eta(\mathbf{G}_0(1) - \pi_{0,0} + \mathbf{G}_2(1))],$$

and $\mathbb{Q}$ the abandonment rate of a customer due to impatience and the strong disaster is given by

$$\mathbb{Q} = \lambda(1 - \mathbb{P}_{ser}) = \lambda - [\mu\mathbf{G}_1(1) + \eta(\mathbf{G}_0(1) - \pi_{0,0} + \mathbf{G}_2(1))].$$

5 Numerical Results

In this section, based on the theoretical results obtained previously, we present some numerical examples to study the impact of the model's different parameters on the mean and variance of the number of customers in the system for the different periods.

Numerical results of the most performance measures such as:

- The mean number of customers when the server is in operational vacation period $E(L_0)$, when the server is in busy period $E(L_1)$, when the server is under repair $E(L_2)$.
- The expected number of customers in the system $E(L)$.

- The variance of the number of customers in the server's operational vacation $V(L_0)$, in the busy period $V(L_1)$ and in the repair period $V(L_2)$.

are presented by varying p and α in (Table 1), and γ, ϕ and q in (Table 2).

The parameters in Table 1 are chosen such as $\lambda = 2$, $\mu = 4$, $\phi = 1.6$, $\eta = 1.5$, $\gamma = 0.5$, $\xi = 0.8$ and $q = 0.75$. The parameters used in Table 2 are the same as those in Table 1 except for $\alpha = 2.5$ and $p = 0.9$.

Table 1: The effect of α and p on the studied performance measures

α	p	$E(L_0)$	$E(L_1)$	$E(L_2)$	$E(L)$	$V(L_0)$	$V(L_1)$	$V(L_2)$
2.2		0.36	0.9359	0.1028	1.3987	0.7586	2.2331	0.3631
2.5	0.3	0.3599	0.9105	0.0857	1.3561	0.7295	2.196747	0.2854
3		0.3598	0.8809	0.0666	1.3073	0.697	2.1512	0.2053
2.2		0.3552	0.8703	0.0712	1.2967	0.6948	2.1199	0.2486
2.5	0.5	0.3551	0.8518	0.0594	1.2664	0.6724	2.0919	0.1953
3		0.3552	0.8305	0.0462	1.2319	0.6474	2.0572	0.1405
2.2		0.3511	0.8059	0.0413	1.1982	0.6252	2.0014	0.1416
2.5	0.7	0.3511	0.7947	0.0345	1.1802	0.6107	1.9835	0.1114
3		0.3512	0.7819	0.0268	1.1599	0.5946	1.9616	0.0802
2.2		0.3477	0.7434	0.0132	1.1043	0.5493	1.8792	0.0443
2.5	0.9	0.3478	0.7396	0.0110	1.0984	0.5442	1.8729	0.0349
3		0.3478	0.7354	0.0086	1.0918	0.5384	1.8653	0.0253

From Table 1, we observe that the effect of increasing the repair time's rate α and the probability of abandoning the system p shows a decrease in the expected number of customers in the different periods. It can also be seen that the variance of the number of customers in each server's state of the system decreases due to the same α and p changes. According to Table 2, for a given ϕ value, the performance measures $E(L_0)$, $E(L_1)$, $E(L)$

Table 2: The effect of γ, q and ϕ on the studied performance measures

γ	q	ϕ	$E(L_0)$	$E(L_1)$	$E(L_2)$	$E(L)$	$V(L_0)$	$V(L_1)$	$V(L_2)$
0.8	0.2		0.3105	0.7388	0.0159	1.0652	0.4774	1.821	0.0496
1.2	0.5	1.6	0.304	0.6876	0.0239	1.0154	0.4810	1.7386	0.0744
1.4	0.8		0.3027	0.6537	0.0282	0.9847	0.4968	1.6953	0.0885
0.8	0.2		0.2926	0.7604	0.0151	1.0681	0.4476	1.8575	0.0467
1.2	0.5	1.8	0.2865	0.7134	0.0226	1.0225	0.4515	1.7841	0.0701
1.4	0.8		0.2852	0.6827	0.0267	0.9945	0.4658	1.7470	0.0833
0.8	0.2		0.2845	0.7702	0.0147	1.0694	0.4339	1.8733	0.0453
1.2	0.5	1.9	0.2786	0.7251	0.022	1.0256	0.438	1.8039	0.0681
1.4	0.8		0.2772	0.6958	0.0259	0.9989	0.4516	1.7696	0.0809

and $V(L_1)$ decrease while the performances $E(L_2)$, $V(L_0)$ and $V(L_2)$ increase as γ and

q increase. However, for fixed γ and q parameters, all performances decrease when ϕ increases.

Now, we present in Figure 1, the effect of η the rate of the service time in the server's operational vacation and α the rate of the repair time, on the values of $\pi_{0,0}$ by taking $\mu = 1.9$, $\lambda = 1.7$, $\phi = 0.9$, $q = 0.5$, $p = 0.5$, $\xi = 0.4$ and $\gamma = 0.2$.

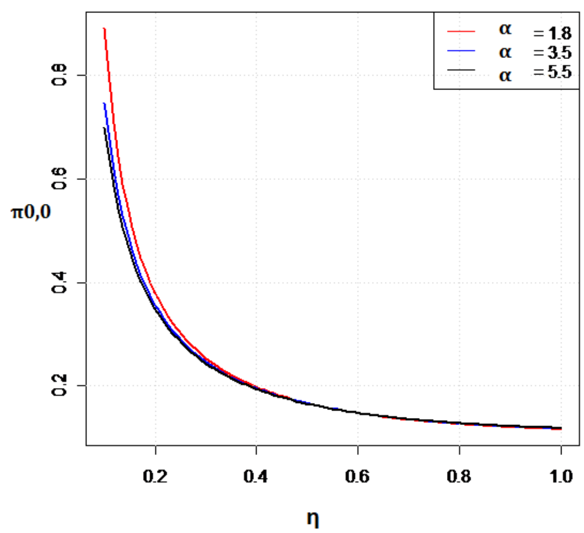

Figure 1: The effect of η and α on $\pi_{0,0}$

From Figure 1, we observe that $\pi_{0,0}$ decreases when η increases, also, when η and α increase, the $\pi_{0,0}$ decreases rapidly and tends to zero.

The Figures 2 and 3 represent the effect of α and γ respectively on $\pi_{0,0}$ and $E(L)$, where the parameters are taken to be $\mu = 3$, $\lambda = 2$, $\phi = 0.8$, $\eta = 1.5$, $\xi = 0.5$, $p = 0.4$ and $q = 0.5$.

According to Figures 2 and 3, we can say that there is an inverse relationship between $\pi_{0,0}$ and $E(L)$ as the values of α increase.

In Figure 4, we show the impact of the probability p on the expected number of customers in the system $E(L)$ for different values of α when $\mu = 5$, $\lambda = 1.9$, $\phi = 1.5$, $\eta = 1.2$, $\xi = 1.3$, $\gamma = 2$ and $q = 0.5$. Figure 5 presents the impact of ξ on the expected number of customers in the system $E(L)$ for different values of γ when the parameters are taken such as $\mu = 6$, $\lambda = 2.5$, $\phi = 1.5$, $\alpha = 2.7$, $\eta = 1.8$, $p = 0.7$, $q = 0.5$.

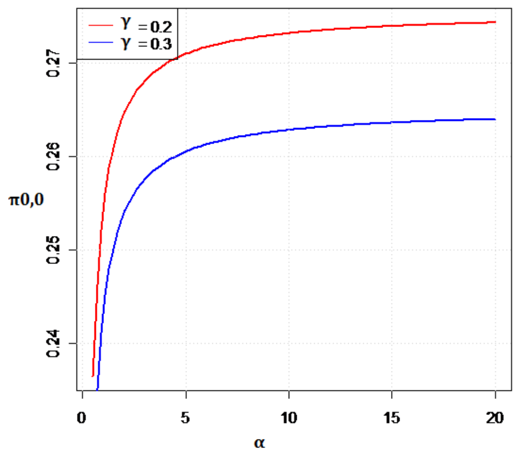

Figure 2: The effect of γ and α on $\pi_{0,0}$

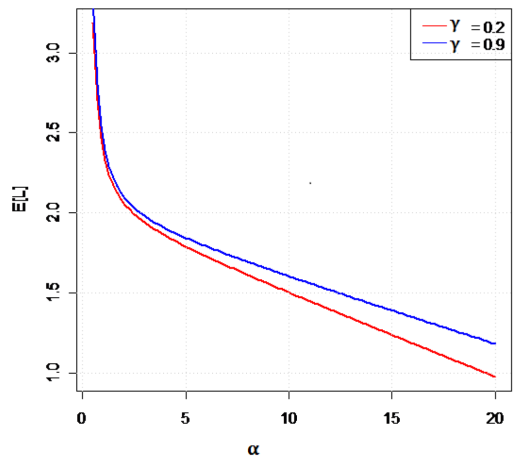

Figure 3: The effect of γ and α on $E(L)$

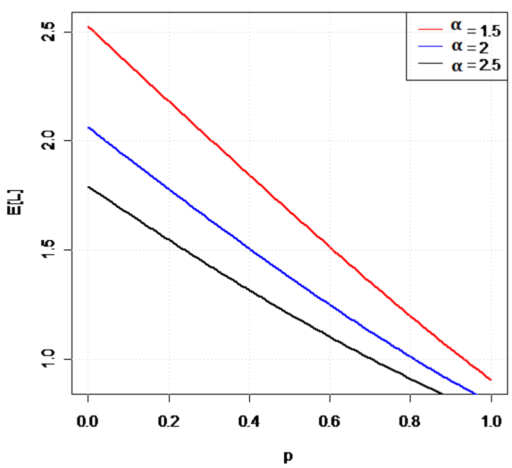

Figure 4: The effect of p and α on $E(L)$

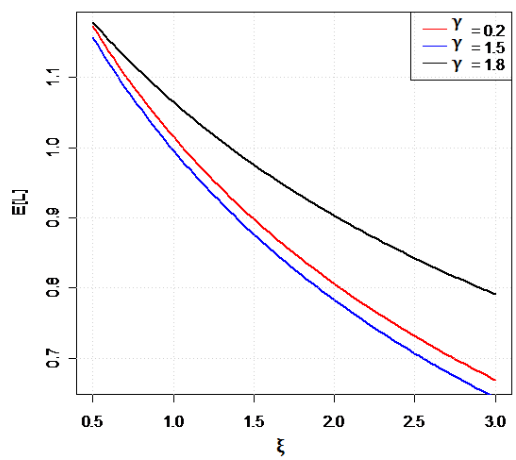

Figure 5: The effect of ξ and γ on $E(L)$

Figure 4 indicates that $E(L)$ decreases as p increases, for different values of α. On the other hand, for a given p, it is clear that the increasing of α leads to the decreasing of $E(L)$ rapidly.

We observe from Figure 5 that as ξ increases, $E(L)$ decreases rapidly. Figure 5 confirms that the disaster lowers the expected number of customers, with more losses of the customers when the strong Bernoulli disaster occurs.

In Figure 6, we measure the impact of the vacation rate ϕ on the variance of the number of customers in the server's operational vacation $V(L_0)$ for different values of γ by taking $\mu = 6$, $\lambda = 3$,$\eta = 1.8$,$\xi = 0.8$, $\alpha = 3.5$,$q = 0.5$ and $p = 0.7$.

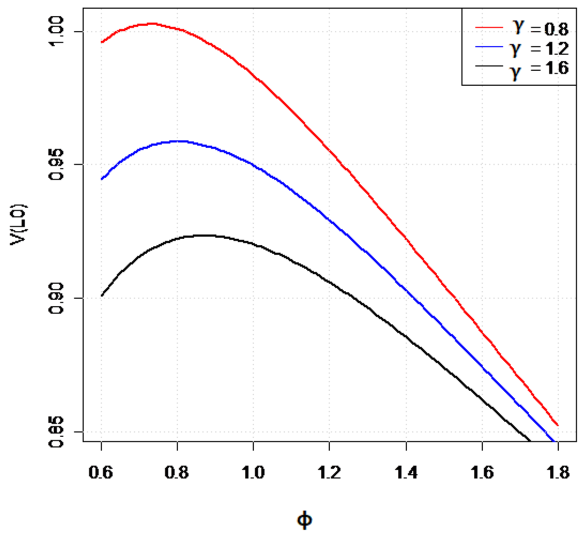

Figure 6: The effect of ϕ and γ on $V(L_0)$

The Figure 6 indicates that $V(L_0)$ decreases rapidly from a certain value of ϕ when γ is equal to 0.8, 1.2 and 1.6. The variance of the number of customers in the server's operational vacation takes his maximum value when ϕ is around 0.8 for a given γ.

Now we demonstrate the influence of ϕ on the variance of the number of customers in repair period $V(L_2)$ but for different values of q in Figure 7 by taking $\mu = 5$, $\lambda = 3$, $\eta = 1.8$, $\xi = 0.8$, $\alpha = 3.5$, $p = 0.7$ and $\gamma = 0.8$.

It can be observed from Figure 7 that the variance $V(L_2)$ decreases with the increasing values of ϕ for the q values 0.3, 0.7 and 1.

Finally, in Figure 8, we present the effect of ϕ and q on the variance of the number of customers in busy period $V(L_1)$, in the case of $\mu = 4$, $\lambda = 2$,$\eta = 1.2$, $\xi = 0.8$, $\alpha = 3$, $\gamma = 0.8$ and $p = 0.5$.

The Figure 8 indicates that $V(L_1)$ increases as ϕ increases when q is equal to 0.3, 0.7, and 1.

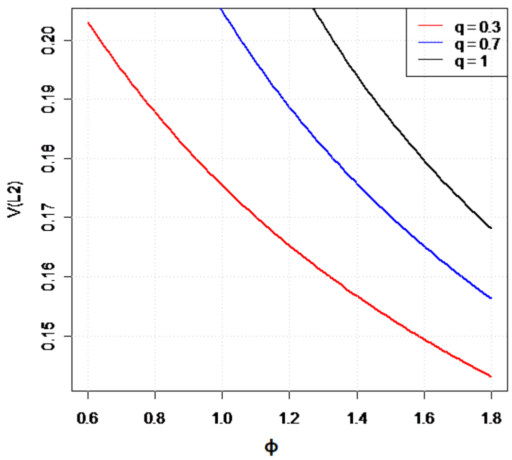

Figure 7: The effect of ϕ and q on $V(L_2)$

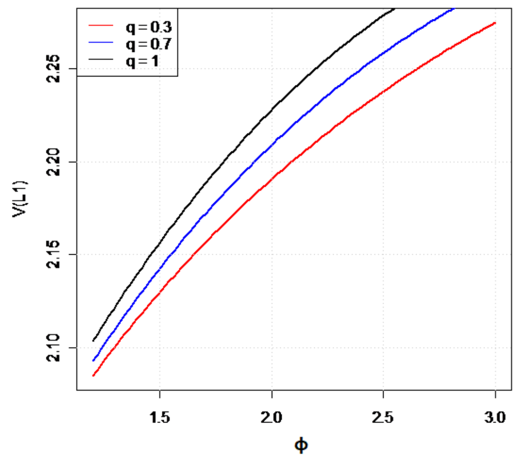

Figure 8: The effect of ϕ and q on $V(L_1)$

To conclude on the numerical study, we can say that all the numerical results confirm theoretical results of the performance measures of our system in each server's state for all the possible values of the parameters. Also, the illustrated variances of the number of customers in the server's states are all quite small, therefore the practical model data is close to its mean in all periods, which reflects the stochastic behavior of the proposed model.

6 Conclusion

The performance measures such as the number of customers in the system, in the operational vacation period, in a busy period, in a repair period, together with the variance in each server's state were derived. The steady-state sojourn time distribution of an arbitrary customer was also obtained. Finally, some numerical examples to demonstrate the impact of input parameters on the average queue length and the variance in each server's state were presented.

The future extension of this research work is that the proposed queuing model can be extended to a queuing network model with two or more identified servers whose performances will be generated by the developed performance measures of our model and then, an application on a specific network, for instance, the Network on Chip will be implemented.

References

[1] Altman,E., & Yechiali,U.: Analysis of customers' impatience in queues with server vacations. *queuing systems* 52 (4) (2006) 261–279.

[2] Bouchentouf,A.A., Guendouzi, A., & Kandouci, A.: Performance and economic analysis of Markovian Bernoulli feedback queuing system with, vacations ,waiting sever and impatient customers. *Acta Universitatis Sapientiae, Mathematica* 10 (2) (2018) 218–241.

[3] Chakravarthy,S.R.: A disaster queue with Markovian arrivals and impatient customers. *Applied Mathematics and Computation* 214 (1) (2009) 48–59.

[4] Dimou,S.,& Economou,A.: The single server queue with catastrophes and geometric reneging. *Methodology and Cumputing in Applied Probability* 15 (3) (2013) 595–621.

[5] Goswami, V.: Analysis of impatient customers in queues with Bernoulli schedule working vacations and vacation interruption. *Journal of Stochastics* (2014) .

[6] Li,J., & Liu,L.: Performance analysis of a complex queuing system with vacations in random environment. *Advances in Mechanical Engineering* 9 (8) (2017) 1–9.

[7] Li, J., & Tian, N.: The M/M/1 queue with working vacations and vacation interruptions. *Journal of Systems Science and Systems Engineering* 16 (1) (2007) 121–127.

[8] Yue,D.,Yue, W., & Xu,G.: Analysis of customers' impatience in an M/M/1 queue with working vacations. *Journal of Industrial & Management Optimization* 8 (4) (2012) 895–908.

[9] Yue,D., Yue, W., & Zhao, G.: Analysis of an M/M/1 queue with vacations and impatience timers which depend on server's states. *Journal of Industrial & Management Optimization* 12 (2) (2016) 653–666.

[10] Ye,J., Liu,L., & Jiang, T.: Analysis of a single-server queue with disasters and repairs under Bernoulli vacation schedule. *Journal of Systems Science and Information* 4 (6) (2016) 547–559.

[11] Yechiali ,U.: Queues with system disasters and impatient customers when system is down. *queuing Systems* 56 (3-4) (2017) 195–202.

[12] Yue,D., Yue, W.,& Xu,G.: Analysis of a queuing System with Impatient Customers and Working Vacations. *In Proceedings of the 6th international conference on queuing theory and network applications* (2011) 208–2012.

Received: April 28, 2021
Accepted for publication: March 31, 2022
Communicated by: Pasha Zusmanovich