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Some remarks on the homology of nilpotent groups

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Abstract. In this article we study the homology of nilpotent groups. In particular, a certain vanishing result for the homology and cohomology of nilpotent groups is proved.

Introduction

The vanishing problem for (co)homology of nilpotent groups asks: For a nilpotent group G and an RG-module M, R a commutative ring with 1, when the vanishing of the zero (co)homology of G with coefficients in M will result in the vanishing of all (co)homologies of G with coefficients in M?

Such vanishing results have many interesting applications [5], [6]. Simple examples show that certain finiteness conditions on M are needed [2], [5]. The most general results in this direction are due to Robinson [5, Theorem A and Theorem B]. He proved that if M is a Noetherian G-module (resp. an Artinian G-module), then we have the vanishing result for the homology (resp. cohomology) functors.

In this article we study a vanishing result which satisfy different type of finiteness conditions. As our main result we show that if H is a normal subgroup of finite index of a nilpotent group G such that G/H is l-torsion and if R is a principal ideal domain with $1/l \in R$ and M an RG-module, then $M_G = 0$ implies that $H_n(G, M_H) = 0$ for all $n \geq 0$. Similarly $M^G = 0$ implies that $H^n(G, M^H) = 0$ for all $n \geq 0$.

We generalize these results by removing the conditions $M_G = 0$ and $M^G = 0$. In fact, we show that for any $n \geq 0$ the natural maps of pairs

$$(\text{inc}, \text{cor}) : (H, M_G) \to (G, M_H), \quad (\text{inc}, \text{res}) : (H, M^G) \to (G, M^H)$$

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induce the isomorphisms

$$H_n(H, M_G) \simeq H_n(G, M_H), \qquad H^n(G, M^H) \simeq H^n(H, M^G),$$

respectively. More generally, we show that for any integers $n, r \geq 0$ we have the isomorphisms

$$H_n(H, H_r(G, M)) \simeq H_n(G, H_r(H, M)),$$
$$H^n(G, H^r(H, M)) \simeq H^n(H, H^r(G, M)).$$

1 A vanishing result

Theorem 1.1. *Let G be a nilpotent group and H a normal subgroup of G such that G/H is finite and l-torsion. Let R be a principal ideal domain with $l \in R^\times$ and M an RG-module.*

(i) *If $M_G = 0$, then for any $n \geq 0$, $H_n(G, M_H) = 0$.*

(ii) *If $M^G = 0$, then for any $n \geq 0$, $H^n(G, M^H) = 0$.*

In particular, if H acts trivially on M and $M_G = 0$ (resp. $M^G = 0$), then for any $n \geq 0$, $H_n(G, M) = 0$ (resp. $H^n(G, M) = 0$).

Proof. We prove the theorem in few steps:

Step 1. If G is finite and l-torsion, then the coinvariant and invariant functors

$$(-)_G,\ (-)^G : \mathcal{M}od_{RG} \to \mathcal{M}od_R$$

are exact: Set $F := ()_G$ and $F' := ()^G$. First let $l = |G|$. Since F is right exact and F' is left exact, to prove the claims it is sufficient to prove that the natural map $\alpha_G : M^G \to M_G$ is an isomorphism. If

$$\overline{N} : M_G \to M^G, \quad \overline{m} \mapsto Nm,$$

where $N := \sum_{g \in G} g \in RG$, then clearly $\overline{N} \circ \alpha_G$ and $\alpha_G \circ \overline{N}$ coincide with multiplication by $|G|$. Thus α_G is an isomorphism. The proof of the general case is by induction on the size of G and we may assume that $G \neq 1$. Since G is nilpotent, $Z(G) \neq 1$. Let H be a nontrivial cyclic subgroup of $Z(G)$. The map α_G coincides with the following composition of maps

$$M^G \xrightarrow{\simeq} (M^H)^{G/H} \xrightarrow{\alpha_H} (M_H)^{G/H} \xrightarrow{\alpha_{G/H}} (M_H)_{G/H} \xrightarrow{\simeq} M_G.$$

Now the claims follow from the above argument and the induction process.

Step 2. If G is l-torsion, then $H_n(G, M) = H^n(G, M) = 0$: This is a known fact. But here we give a direct proof of it. First let G be finite and $l = |G|$. If $P_\bullet \to M$ and $M \to I^\bullet$ are projective and injective resolutions of the RG-module M, respectively, then the claim follows from Step 1 and the following isomorphisms

$$H_n(G, M) \simeq H_n((P_\bullet)_G), \quad H^n(G, M) \simeq H_n((I^\bullet)^G)$$

(see [1, Chap. III.6, 1.4 and III.6, Exercise 1]). In general, since G is nilpotent and torsion, any finitely generated subgroup of G is finite. Hence we can write G as direct limit of its finite subgroups, e.g. $G = \varinjlim G_i$, where G_i's are finite. For the homology functor we have

$$H_n(G, M) \simeq \varinjlim H_n(G_i, M) = 0$$

(see [1, Chap. V.5, Exercise 3]). For the cohomology functor we have the spectral sequence

$$E_2^{p,q} = \varprojlim{}^p \, H^q(G_i, M) \Rightarrow H^{p+q}(G, M),$$

where $\varprojlim{}^p$ is the p-th derived functor of $\varprojlim$ [6, p. 297]. Now the claim follows from the finite case.

Step 3. If G is finite and l-torsion, then for any R-module N with the trivial action of G and any $n \geq 0$, we have the isomorphisms

$$\mathrm{Tor}_n^R(N, M)_G \simeq \mathrm{Tor}_n^R(N, M_G), \quad \mathrm{Ext}_R^n(N, M^G) \simeq \mathrm{Ext}_R^n(N, M)^G :$$

We start with the functor Tor. Since

$$\mathrm{Tor}_0^R(N, M)_G \simeq (N \otimes_R M) \otimes_G \mathbb{Z} \simeq N \otimes_R M_G \simeq \mathrm{Tor}_0^R(N, M_G),$$

the claim is true for $n = 0$. Let

$$0 \to N_{n-1} \to F \to N \to 0$$

be a short exact sequence of R-modules such that F is free. If $n \geq 2$, from the long exact sequence, we get the isomorphism $\mathrm{Tor}_n^R(N, M) \simeq \mathrm{Tor}_{n-1}^R(N_{n-1}, M)$. If we continue this process, we will find an R-module N_1 such that

$$\mathrm{Tor}_n^R(N, M) \simeq \mathrm{Tor}_1^R(N_1, M).$$

So it is sufficient to proof the claim for $n = 1$. From the exact sequence

$$0 \to N_1 \to F \to N \to 0$$

and Step 1 we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Tor}_1^R(N, M)_G & \longrightarrow & (N_1 \otimes_R M)_G & \longrightarrow & (F \otimes_R M)_G & \longrightarrow & (N \otimes_R M)_G & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Tor}_1^R(N, M_G) & \longrightarrow & N_1 \otimes_R M_G & \longrightarrow & F \otimes_R M_G & \longrightarrow & N \otimes_R M_G & \to 0.
\end{array}$$

Note that the three vertical maps on the right are isomorphisms. Now the claim follows from an easy diagram chase.

The proof of the claim for the functor Ext is similar. In fact here we should use an exact sequence $0 \to N \to I \to N^1 \to 0$, where I is an injective R-module.

Step 4. $H_n(H, M)_{G/H} \simeq H_n(G, M)$ and $H^n(G, M) \simeq H^n(H, M)^{G/H}$: We prove the first isomorphism. The second one can be proved in a similar way. From the extension $H \rightarrowtail G \twoheadrightarrow G/H$ we obtain the Lyndon-Hochschild-Serre homology spectral sequence

$$E^2_{p,q} = H_p(G/H, H_q(H, M)) \Rightarrow H_{p+q}(G, M).$$

By Step 2, we have $E^2_{p,q} = 0$. Now by an easy analysis of the spectral sequence we obtain the isomorphism $H_0(G/H, H_n(H, M)) \simeq H_n(G, M)$.

Step 5. The proof of the theorem: We prove (i). The proof of (ii) is similar. By Step 4, we have the isomorphism

$$H_n(G, M_H) \simeq H_n(H, M_H)_{G/H}.$$

Since the action of H on M_H is trivial, by the Universal Coefficient Theorem we have the exact sequence

$$0 \to H_n(H, R) \otimes_R M_H \to H_n(H, M_H) \to \mathrm{Tor}_1^R(H_{n-1}(H, R), M_H) \to 0.$$

By applying the functor $(.)_{G/H}$ we obtain the exact sequence

$$0 \to \big(H_n(H, R) \otimes_R M_H\big)_{G/H} \to H_n(H, M_H)_{G/H} \to \mathrm{Tor}_1^R(H_{n-1}(H, R), M_H)_{G/H} \to 0.$$

Now the claim follows from Step 3 and the assumption $M_G = 0$. □

2 The corestriction map for the homology of nilpotent groups

We say that a group G acts nilpotently on a G-module M, if M has a finite filtration of G-submodules

$$0 = M_0 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k = M,$$

such that the action of G on each quotient M_i/M_{i-1} is trivial.

The following theorem will be needed in the next section.

Theorem 2.1. *Let G be a nilpotent group and H a normal subgroup of G such that G/H is l-torsion. Let R be a commutative ring such that $l \in R^\times$. If M is an R-module with a nilpotent action of G, then for any $n \geq 0$ the natural maps*

$$\mathrm{cor}_H^G : H_n(H, M) \to H_n(G, M), \quad \mathrm{res}_H^G : H^n(G, M) \to H^n(H, M)$$

are isomorphisms. In particular, G/H acts trivially on $H_n(H, M)$ and $H^n(H, M)$.

Proof. The proof is by induction on the nilpotent class c of G. We prove the claim for cor_H^G. The claim for res_H^G can be proved in a similar way. Consider the lower central series of G:

$$1 = \gamma_{c+1}(G) \subset \gamma_c(G) \subset \cdots \subset \gamma_2(G) \subset \gamma_1(G) = G.$$

Let $c = 1$. Then $G' = \gamma_2(G) = 1$. So G is abelian. Let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M,$$

be a filtration of M such that G acts trivially on each quotient M_i/M_{i-1}. We prove this case by induction on k. If $k = 1$, then the action of G on $M = M_1$ is trivial. This implies that the action of G/H on $H_n(G, M)$ is trivial and therefore

$$H_n(H, M) = H_n(H, M)_{G/H} \simeq H_n(G, M)$$

(see Step 4, in the proof of Theorem 1.1). Now let $k > 1$ and set $M'_1 := M/M_1$. From the short exact sequence of G-modules $0 \to M_1 \to M \to M'_1 \to 0$, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H_{n+1}(H, M'_1) & \to & H_n(H, M_1) & \to & H_n(H, M) & \to & H_n(H, M'_1) & \to & H_{n-1}(H, M_1) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle \mathrm{cor}_H^G} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_2} \\
H_{n+1}(G, M'_1) & \to & H_n(G, M_1) & \to & H_n(G, M) & \to & H_n(G, M'_1) & \to & H_{n-1}(G, M_1),
\end{array}$$

where f_1, f_2, g_1 and g_2 are the natural corestriction maps. Since G acts trivially on M_1 and M'_1 has a filtration of length $k - 1$, by induction f_1, f_2, g_1 and g_2 are isomorphisms. Now an easy diagram chase shows that cor_H^G is an isomorphism. This proves the theorem for $c = 1$.

Now assume that the claim is true for nilpotent groups of class d, $1 \leq d \leq c - 1$. From the commutative diagram of extensions,

$$\begin{array}{ccccc}
\gamma_c(G) \cap H & \rightarrowtail & H & \twoheadrightarrow & H/(\gamma_c(G) \cap H) \\
\downarrow & & \downarrow & & \downarrow \\
\gamma_c(G) & \rightarrowtail & G & \twoheadrightarrow & G/\gamma_c(G),
\end{array}$$

we have the following morphism of Lyndon-Hochschild-Serre spectral sequences

$$\begin{array}{ccc}
E'^2_{p,q} = H_p(H/(\gamma_c(G) \cap H), H_q(\gamma_c(G) \cap H, M)) & \Longrightarrow & H_{p+q}(H, M) \\
\downarrow & & \downarrow \\
E^2_{p,q} = H_p(G/\gamma_c(G), H_q(\gamma_c(G), M)) & \Longrightarrow & H_{p+q}(G, M).
\end{array}$$

First note that $\gamma_c(G)/(\gamma_c(G) \cap H) \simeq \gamma_c(G)H/H \subseteq G/H$. So the groups $\gamma_c(G)/(\gamma_c(G) \cap H)$ and $(G/\gamma_c(G))/(H/(\gamma_c(G) \cap H))$ are l-torsion.

Since $\gamma_c(G)$ is abelian, by the first step of the induction, we have

$$H_q(\gamma_c(G) \cap H, M) \simeq H_q(\gamma_c(G), M).$$

Observe that $G/\gamma_c(G)$ is of nilpotent class $c - 1$. Since $\gamma_c(G) \subseteq Z(G)$, the conjugate action of $G/\gamma_c(G)$ on $\gamma_c(G)$ is trivial.

We show that the natural action of $G/\gamma_c(G)$ on $H_q(\gamma_c(G), M)$ is nilpotent. This can be done by induction on the length of the filtration of M. Assume that M has a filtration of length k as above. If $k = 1$, then $M = M_1$. Thus G acts trivially on M and so the action of $G/\gamma_c(G)$ on $H_q(\gamma_c(G), M)$ is trivial. Now let $k \geq 2$. From the short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

we obtain the long exact sequence

$$\cdots \to H_q(\gamma_c(G), M_1) \to H_q(\gamma_c(G), M) \to H_q(\gamma_c(G), M/M_1) \to \cdots .$$

By induction, the actions of $G/\gamma_c(G)$ on the modules $H_q(\gamma_c(G), M_1)$ and $H_q(\gamma_c(G), M/M_1)$ are nilpotent. It follows from the above exact sequence that $G/\gamma_c(G)$ acts nilpotently on $H_q(\gamma_c(G), M)$.

Since $H/(\gamma_c(G)\cap H) \hookrightarrow G/\gamma_c(G)$ and $H_q(\gamma_c(G), M) \simeq H_q(\gamma_c(G)\cap H, M)$, by induction on the nilpotent class of $G/\gamma_c(G)$, we have

$$H_p(H/(\gamma_c(G) \cap H), H_q(\gamma_c(G) \cap H, M)) \simeq H_p(G/\gamma_c(G), H_q(\gamma_c(G), M)).$$

Therefore $E'^2_{p,q} \simeq E^2_{p,q}$. Now by convergence of the spectral sequences, for any $n \geq 0$, we obtain the isomorphism

$$H_n(H, M) \simeq H_n(G, M).$$

Finally we know that $H_n(H, M)_{G/H} \simeq H_n(G, M)$ (see Step 4, in the proof of Theorem 1.1) Thus the map $H_n(H, M) \to H_n(H, M)_{G/H}$ is an isomorphism. This shows that G/H acts trivially on $H_n(H, M)$. □

Example 2.2. Easy examples show that in Theorem 2.1 the condition $l \in R^\times$ can not be removed. For example, if H is a proper finite subgroups of an abelian group G, then $H_1(H, \mathbb{Z}) \simeq H \subset G \simeq H_1(G, \mathbb{Z})$, which clearly is not an isomorphism.

The first part of Theorem 2.1 can be extended to all subgroups of finite index, as follows.

Corollary 2.3. *Let G be a nilpotent group and H a subgroup of finite index. Let R be a commutative ring such that $[G : H]! \in R^\times$. If M is an R-module with a nilpotent action of G, then, for any $n \geq 0$, the natural maps*

$$\mathrm{cor}_H^G : H_n(H, M) \to H_n(G, M), \quad \mathrm{res}_H^G : H^n(G, M) \to H^n(H, M)$$

are isomorphisms.

Proof. It is well-known that H has a subgroup L such that L is normal in G and

$$[G : L] \leq [G : H]!.$$

By Theorem 2.1, $\mathrm{cor}_L^G : H_n(L, M) \to H_n(G, M)$ and $\mathrm{cor}_L^H : H_n(L, M) \to H_n(H, M)$ are isomorphisms. Therefore

$$\mathrm{cor}_H^G : H_n(H, M) \to H_n(G, M)$$

is an isomorphism. The cohomology case can be treated similarly. □

3 Homology of nilpotent groups with corestriction coefficients

The following theorem generalizes Theorem 1.1.

Theorem 3.1. *Let G be a nilpotent group and H a normal subgroup of G such that G/H is finite and l-torsion. Let R be a principal ideal domain with $l \in R^\times$ and let M be an RG-module. Then for any $n \geq 0$,*

$$H_n(H, M_G) \simeq H_n(G, M_H), \qquad H^n(G, M^H) \simeq H^n(H, M^G),$$

which are induced by the pairs

$$(\mathrm{inc}, \mathrm{cor}) : (H, M_G) \to (G, M_H) \qquad \textit{and} \qquad (\mathrm{inc}, \mathrm{res}) : (H, M^G) \to (G, M^H),$$

respectively. More generally, for any integers $n, r \geq 0$, the above maps of pairs induce the isomorphisms

$$H_n(H, H_r(G, M)) \simeq H_n(G, H_r(H, M)),$$
$$H^n(G, H^r(H, M)) \simeq H^n(H, H^r(G, M)).$$

Proof. The Universal Coefficient Theorem and Step 1 of the proof of Theorem 1.1 gives us the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & (H_n(H,R) \otimes_R M_H)_{G/H} & \to & H_n(H, M_H)_{G/H} & \to & \mathrm{Tor}_1^R(H_{n-1}(H,R), M_H)_{G/H} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H_n(H,R) \otimes_R M_G & \to & H_n(H, M_G) & \to & \mathrm{Tor}_1^R(H_{n-1}(H,R), M_G) & \to 0.
\end{array}$$

By Theorem 2.1 the action of G/H on $H_n(H, R)$ is trivial. Thus by Step 3 of the proof of Theorem 1.1, the left and the right column maps of this diagram are isomorphisms. Hence $H_n(H, M_H)_{G/H} \simeq H_n(H, M_G)$ and therefore

$$H_n(G, M_H) \simeq H_n(H, M_H)_{G/H} \simeq H_n(H, M_G).$$

The other isomorphism can be proved in a similar way. In fact, one should prove that $H^n(H, M^G) \simeq H^n(H, M^H)^{G/H}$ first, and then use this result to prove the isomorphism $H^n(H, M^G) \simeq H^n(G, M^H)$.

The proof of the general case is similar. Just we should replace M_H and M_G, with $H_r(H, M)$ and $H_r(G, M)$, respectively. □

Corollary 3.2. *Let G be a nilpotent group, H be a subgroup of G such that G/H is l-torsion, $R = \mathbb{Z}[1/l]$, and M be an RG-module. Then for any $n \geq 0$, $H_n(G, M_H) \simeq H_n(H, M_G)$. In particular, if the action of H on M is trivial, then*

$$H_n(G, M) \simeq H_n(H, M_G).$$

Proof. First notice that the group G/H can be written as direct limit of its finite subgroups, e.g. $G/H = \varinjlim G_i/H$. (Note that finitely generated torsion subgroups of nilpotent groups are finite.) Hence by Theorem 3.1,

$$H_n(G, M_H) \simeq \varinjlim H_n(G_i, M_H) \simeq \varinjlim H_n(H, M_{G_i})$$
$$\simeq H_n(H, \varinjlim M_{G_i}) \simeq H_n(H, M_G).$$

(see [1, Exercise 3, Chapter V.5]). □

Example 3.3. As an application of Theorem 3.1, we study the homology of special linear groups. Let R be a commutative ring. The conjugate action of $R^\times$ on $\mathrm{SL}_n(R)$, given by

$$a.A := \mathrm{diag}(a, I_{n-1}).A.\,\mathrm{diag}(a^{-1}, I_{n-1}),$$

induces a natural action of $R^\times$ on $H_q(\mathrm{SL}_n(R), \mathbb{Z})$. Since

$$\begin{aligned} a^n.A &= \mathrm{diag}(a^n, I_{n-1}).A.\,\mathrm{diag}(a^{-n}, I_{n-1}) \\ &= \mathrm{diag}(a^{n-1}, a^{-1}I_{n-1}).aI_n.A.a^{-1}I_n.\,\mathrm{diag}(a^{-(n-1)}, aI_{n-1}) \\ &= \mathrm{diag}(a^{n-1}, a^{-1}I_{n-1}).A.\,\mathrm{diag}(a^{-(n-1)}, aI_{n-1}), \end{aligned}$$

the action of $R^{\times n}$ on $H_q(\mathrm{SL}_n(R), \mathbb{Z})$ is trivial [1, Chap. II, Proposition 6.2]. Since $R^\times/R^{\times n}$ is a n-torsion group, by Corollary 3.2

$$H_p(R^\times, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H_p(R^\times, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^\times}).$$

We say that a commutative ring R is a *ring with many units* if for any $n \geq 2$ and for any finite number of surjective linear forms $f_i : R^n \to R$, there exists a $v \in R^n$ such that, for all i, $f_i(v) \in R^\times$. Important examples of rings with many units are semi-local rings with infinite residue fields. For more about these rings please see [3, Section 1] and [4, Section 2]. Now one can show that if $q \leq n$ are nonnegative integers, then

$$H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^\times} \simeq H_q(\mathrm{SL}(R), \mathbb{Z}[1/n]),$$

provided that R is a ring with many units [4, Section 3]. Combining these results, we obtain the isomorphism

$$H_p(R^\times, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H_p(R^\times, H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])).$$

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