

Cantor series expansions of rational numbers

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Abstract. This survey is devoted to necessary and sufficient conditions for a rational number to be representable by a Cantor series. Necessary and sufficient conditions are formulated for the case of an arbitrary sequence (q_k) .

1 Introduction

Let $Q \equiv (q_k)$ be a fixed sequence of positive integers, $q_k > 1$, Θ_k be a sequence of the sets $\Theta_k \equiv \{0, 1, \dots, q_k - 1\}$, and $\varepsilon_k \in \Theta_k$.

The Cantor series expansion

$$\frac{\varepsilon_1}{q_1} + \frac{\varepsilon_2}{q_1 q_2} + \dots + \frac{\varepsilon_k}{q_1 q_2 \dots q_k} + \dots \tag{1}$$

of $x \in [0, 1]$, first studied by G. Cantor in [2], is a natural generalization of the b-ary expansion

$$\frac{\alpha_1}{b} + \frac{\alpha_2}{b^2} + \dots + \frac{\alpha_n}{b^n} + \dots$$

of numbers from the closed interval $[0, 1]$. Here b is a fixed positive integer, $b > 1$, and $\alpha_n \in \{0, 1, \dots, b - 1\}$.

By $x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^Q$ denote a number $x \in [0, 1]$ represented by series (1). This notation is called *the representation of x by Cantor series* (1).

We note that certain numbers from $[0, 1]$ have two different representations by Cantor series (1), i.e.,

$$\Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_{m-1} \varepsilon_m 000\dots}^Q = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_{m-1} [\varepsilon_{m-1} - 1] [q_{m+1} - 1] [q_{m+2} - 1] \dots}^Q = \sum_{i=1}^m \frac{\varepsilon_i}{q_1 q_2 \dots q_i}.$$

Such numbers are called *Q-rational*. The other numbers in $[0, 1]$ are called *Q-irrational*.

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Cantor series expansions have been intensively studied from different points of view during the last century. The metric, probability, and fractal theories of number representations by positive Cantor series were studied by a number of researchers. Also, functions and fractal sets defined in terms of Cantor series expansions were investigated. These problems were considered by the following researchers: P. Erdős, J. Galambos, G. Iommi, P. Kirschenhofer, T. Komatsu, V. Laohakosol, B. Li, M. Paštéka, S. Prugsapitak, J. Rattanamoong, A. Rényi, B. Skorulski, R. F. Tichy, P. Turán, Yi Wang, M. S. Waterman, H. Wegmann, Liu Wen, Zhixiong Wen, Lifeng Xi, and other mathematicians.

Such investigations can be divided into two groups. The first is the investigation of the fractional parts of real numbers represented by Cantor series (1), and the other is the investigation of representations of non-negative integers represented by positive Cantor series of the form

$$n = \sum_{k=1}^{\infty} \varepsilon_k q_1 q_2 \cdots q_k,$$

where $\varepsilon_k \in \Theta_k$.

We give a brief description of these investigations.

A number of researches are devoted to studying various types of the normality of numbers represented by the Cantor series. In these papers, the notions of \mathbb{Q} -distribution normality, \mathbb{Q} -normality, and \mathbb{Q} -ratio normality, are studied. For example, in the papers [1], [27], [28], the notion of \mathbb{Q} -distribution normality is investigated. Indeed, one can note the following investigations: relations between various types of normality (e.g., see [1], [26]); the average value of the function of the sum of digits in the Cantor series representation of a number (see [20] and references in the last-mentioned article); behaviour of the frequency of the most frequently used digit among the first digits in the representation of a number (e.g., see [4]); necessary, sufficient, necessary and sufficient conditions for a number to be a number having the property of certain type normality (see [27], [26], [28]); the completeness of the Lebesgue measure, the density, topological properties, the Hausdorff measure of a set whose elements are numbers having the property of the normality of a certain type (e.g., see [27], [28]); the rationality and irrationality of a number which has the property of the normality of a certain type (see [26]), etc. Note that, in the papers [4], [5], [33], [34], [36], [49], P. Erdős, A. Rényi, and P. Turán introduced and studied the problem on normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions. Some investigations of Cantor series expansions were published by J. Galambos in [8], [9].

In some papers, certain generalizations of real numbers representations by the Cantor series are studied. For example, properties of digits (sequences of digits) of the polyadic number α as functions (sequences of functions) of α are studied in [32]; in [23], the notion of a complex Cantor series is introduced, and the \mathbb{Q} -algebraic and \mathbb{Q} -linearly independence of numbers represented by Cantor series are investigated; matrix expansions are studied in [51]; the papers [38], [44] are devoted to certain generalizations of alternating Cantor series.

In certain papers, fractal properties of representations of real numbers by positive

Cantor series and fractal properties of certain type sets whose elements are represented by a positive Cantor series, are studied (e.g., see [19], [50], [29], [26]). For example, in [19], the Hausdorff-Besicovitch dimensions of sets whose elements are defined in terms of the frequencies of digits, are investigated. The paper [52] is devoted to studying the conditions under which the family of all possible rank cylinders $\Delta_{c_1 c_2 \dots c_n}^Q$ is faithful for the Hausdorff-Besicovitch dimension calculation. Sets whose elements have a restriction on using digits in their own representations are studied in [29]. In the last-mentioned article, the formula for a calculation of the Hausdorff dimension of the following set is proved, and conditions for the equality of the Hausdorff, packing, and box dimensions of this set, are discovered:

$$R_I(Q) = \left\{ x : x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k}, \varepsilon_k \in I_k \subseteq \Theta_k \right\}.$$

Here the condition

$$\lim_{k \rightarrow \infty} \frac{\log q_k}{\log q_1 q_2 \dots q_k} = 0$$

holds.

Also, we can note several investigations of functions. The arguments or values of these functions are defined by positive [2] or alternating [39] Cantor series. In [53], properties of the following function were investigated:

$$u = f(x) = \sum_{k=1}^{\infty} \frac{u_k}{k(k+1)},$$

where $u_1 = 1$ and for $k = 1, 2, \dots$

$$u_{k+1} = \begin{cases} -\frac{u_k}{k}, & \text{if } \varepsilon_{k+1} = 0 \text{ but } \varepsilon_k \neq 0, \\ & \text{or if } \varepsilon_{k+1} = q_{k+1} - 1 \text{ but } \varepsilon_k \neq q_k - 1; \\ u_k, & \text{otherwise.} \end{cases}$$

Here x is represented by series (1). This function is well-defined and continuous. Also, $u = f(x)$ is nowhere differentiable when $q_k \geq 3$ for all $k = 1, 2, \dots$, and the condition

$$\lim_{k \rightarrow \infty} \frac{q_1 q_2 \dots q_k}{k!} = \infty$$

holds. The last-mentioned function is a function with a complicated local structure. Certain examples of functions with a complicated local structure are described in [42], [37], [40]. In the paper [29], the following function are studied:

$$\psi_{P,Q}(x) = \sum_{k=1}^{\infty} \frac{\min(E_k, q_k - 1)}{q_1 q_2 \dots q_k},$$

where

$$x = E_0 + \sum_{k=1}^{\infty} \frac{E_k}{p_1 p_2 \dots p_k}, \quad \varepsilon_0 + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k}.$$

Here $E_0, \varepsilon_0 \in \mathbb{Z}$, $Q \equiv (q_k)$ and $P \equiv (p_k)$ are sequences of positive integers, that greater than 1. Also, $E_k \neq p_k - 1$ and $\varepsilon_k \neq q_k - 1$ infinitely often, $E_k \in \{0, 1, \dots, p_k - 1\}$ and $\varepsilon_k \in \Theta_k$.

In the present article, the main attention is given to necessary and sufficient conditions for x (represented by Cantor series with an arbitrary basic sequence (q_k)) to be rational.

Remark 1.1. In the present article, we use the following notations: \mathbb{N} , \mathbb{Z}_0 , \mathbb{Z} , \mathbb{Q} , and \mathbb{I} . Here by \mathbb{N} denote the set of all positive integers and by \mathbb{Z}_0 denotes the set $\mathbb{N} \cup \{0\}$, \mathbb{Z} is the set of all integers, and \mathbb{Q} is the set of all rational numbers, and \mathbb{I} is the set of all irrational numbers.

2 Description of research of the main problem

The problem of expansions of rational/irrational numbers in terms of generalizations of the b-ary numeral system is difficult. A version of this problem for Cantor series (1) was introduced in the paper [2] in 1869 and has been studied by a number of researchers. For example, G. Cantor, P. A. Diananda, A. Oppenheim, P. Erdős, J. Hančl, E. G. Straus, P. Rucki, R. Tijdeman, P. Kuhapatanakul, V. Laohakosol, D. Marques, Pingzhi Yuan and other scientists studied this problem.

In the monograph [8], Prof. János Galambos called the problem on representations of rational numbers by Cantor series (1) as *the fourth open problem*, and wrote the following:

“Problem Four. Give a criterion of rationality for numbers given by a Cantor series. What one should seek here is a directly applicable criterion. A general sufficient condition for rationality would also be of interest, in which the quoted theorems of Diananda and Oppenheim (including the abstract criterion by condensations) can be guides or useful tools.

If in a Cantor series, negative and positive terms are permitted, somewhat less is known about the rationality or irrationality of the resulting sum. G. Lord (personal communication) tells me that the condensation method can be extended to this case as well, but still, the results are less complete than in the case of ordinary Cantor series.” ([8, p. 134]).

The paper [46] is devoted to the last-mentioned discussion and to expansions of rational numbers by sign-variable Cantor series. For fullness, one can note the following result of Diananda and Oppenheim noted by J. Galambos.

Theorem 2.1 ([3]). *A necessary and sufficient condition that x given by (1) shall be rational is this: coprime integers h, k , $0 \leq h \leq k$, an integer N and a condensation shall exist such that*

$$A_i = \frac{h}{k}(B_i - 1)$$

for all $i \geq N$.

Here

$$x = X = A_0 + \frac{A_1}{B_1} + \frac{A_2}{B_1 B_2} + \dots + \frac{A_n}{B_1 B_2 \dots B_n} + \dots,$$

where $A_0 = \varepsilon_0$ is the integer part of x ,

$$B_1 = q_1 q_2 \cdots q_{i_1}, B_2 = q_{i_1+1} q_{i_2+1} \cdots q_{i_1+i_2}, \dots,$$

and $B_i \geq 2, 0 \leq A_i \leq B_i - 1$,

$$\frac{\varepsilon_1}{q_1} + \frac{\varepsilon_2}{q_1 q_2} + \cdots + \frac{\varepsilon_{i_1}}{q_1 q_2 \cdots q_{i_1}} = \frac{A_1}{B_1}.$$

We begin with a brief description of investigations of rational numbers represented by the Cantor series.

Much research [48], [10], [13], [11], [14] has been devoted to necessary or/and sufficient conditions for a rational number to be representable by Cantor series (1) such that sequences (q_k) and (ε_k) are sequences of integers. In some papers (see [13], [48],[7], [11], [26]), the case of Cantor series for which sequences (q_k) and (ε_k) are sequences of integers and the condition $\mathbb{Z} \ni q_k > 1$ holds for all $k \in \mathbb{N}$, is investigated. However, the main problem of the present article is studied for the case of series (1) (e.g., see [2], [3], [24], [31]) and still for the case of Cantor series of a special type (e.g., see [15], [17], [16], [13]). For example, in the papers [3], [14], [6], [21], Ahmes series are considered. The last series is Cantor series (1) for which $\varepsilon_k = \text{const} = 1$ holds for all $k \in \mathbb{N}$.

In the papers [3], [10], [11], [13], [14], [15],[7], [31], [48], necessary and sufficient conditions for a rational (irrational) number to be representable by a Cantor series are studied, and sufficient conditions are investigated in the papers [7], [3], [13], [24], [31], [48]. Although much research has been devoted to the problem of representations of rational (irrational) numbers by Cantor series for which sequences (q_k) and (ε_k) are sequences of special types (see [2],[7], [10], [11], [13], [24], [31],[48]), little is known about necessary and sufficient conditions of the rationality (irrationality) for the case of an arbitrary sequence (q_k) (see [3], [13], [41], [43], [39], [45], [46], [48]).

Finally, several papers (see [12], [21], [24], [48]) were devoted to investigations of conditions of the rationality or irrationality of numbers represented by series of the form $\sum_{k=1}^{\infty} \frac{a_k}{b_k}$. Furthermore, in [24], a necessary and sufficient condition of the rationality of the sum $\sum_{k=1}^{\infty} \frac{a_k(-1)^{k+1}}{b_k}$ is proved for the case of certain properties which are satisfied by sequences (a_k) and (b_k) .

Let us consider our problem more in detail.

3 Cantor's investigations, finite expansions, and conditions for finite expansions of rational numbers

Let us begin with a consideration of the results presented in the first paper on this topic (i.e., [2]). In [2], G. Cantor proved a fact that an arbitrary number $x \in [0, 1)$ is a rational number if and only if (ε_k) is ultimately periodic under the condition when a sequence (q_k) is periodic. In addition, one can note the following theorem which necessity was given in [2] with the other formulation and with a more complicated proof for the case of positive Cantor series.

Theorem 3.1. *A rational number $\frac{p}{r}$ has a finite expansion by a positive or sign-variable Cantor series if and only if there exists a number n_0 such that*

$$q_1 q_2 \dots q_{n_0} \equiv 0 \pmod{r}.$$

The interest in the last theorem can be explained ([41], [43], [39], [46]) by the fact that there exist certain sequences (q_k) such that all rational numbers represented by Cantor series (positive or sign-variable) have finite expansions. For example, all rational numbers represented by the following representations have finite expansions.

$$x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{-(2k)} \equiv \sum_{k=1}^{\infty} \frac{(-1)^k \varepsilon_k}{2 \cdot 4 \cdot 8 \cdot \dots \cdot 2k}, \text{ where } \varepsilon_k \in \{0, 1, \dots, 2k-1\};$$

$$x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(k+1)!} \equiv \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2 \cdot 3 \cdot 4 \cdot \dots \cdot (k+1)}, \text{ where } \varepsilon_k \in \{0, 1, \dots, k\}.$$

It is easy to see that there exist sequences (q_k) and (ε_k) such that a finite expansion is a necessary or/and sufficient condition of the rationality of any number represented by a Cantor series. Several papers were devoted to such investigations. For example, see [24], [11]. Let us consider several related results.

In 2006, J. Sondow gave a geometric proof of the irrationality of the number e [47]. In [30], the following statement was proved by a generalization to Sondow's construction.

Theorem 3.2 ([30]). *Let $x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k}$. Suppose that each prime divides infinitely many of the q_k . Then $x \in \mathbb{I}$ if and only if both $0 < \varepsilon_k < q_k - 1$ hold infinitely often.*

For example, in [11], attention is given to conditions of finite expansions of rational numbers by positive and sign-variable Cantor expansions. That is, $\sum_{n=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k} \in \mathbb{Q}$ if and only if $\varepsilon_k = 0$ for every sufficiently large positive integer k under one of the following two systems of conditions:

- System 1 of conditions (the case of sign-variable series): suppose (q_k) is a sequence of positive integers greater than one, (ε_k) is a sequence of integers such that the condition

$$\liminf_{k \rightarrow \infty} \frac{|\varepsilon_k| + 1}{q_k} = 0$$

holds and for every sufficiently large positive integer k

$$|\varepsilon_{k+1}| \leq \frac{1}{2} \max(|\varepsilon_k|, 1) q_{k+1}.$$

- System 2 of conditions (the case of positive series): suppose (q_k) is a sequence of positive integers greater than one and $K \in (0, 1)$, (ε_k) is a sequence of non-negative integers such that the condition

$$\liminf_{n \rightarrow \infty} \frac{\varepsilon_k + 1}{q_k} = 0$$

holds and for every sufficiently large positive integer k

$$\varepsilon_{k+1} \leq K \max(\varepsilon_k, 1) q_{k+1}.$$

4 The shift operator and related investigations

We must note that the notion of the shift operator plays an important role in investigations of expansions of rational numbers defined by the Cantor series (positive, alternating, or sign-variable).

We begin with definitions. Let N_B be a fixed subset of positive integers,

$$\rho_k = \begin{cases} 1 & \text{if } k \in N_B \\ 2 & \text{if } k \notin N_B, \end{cases}$$

and $Q \equiv (q_k)$ be a fixed sequence of positive integers such that $q_k > 1$ for all $n \in \mathbb{N}$. Then we get the following representation of real numbers

$$x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)} \equiv \frac{(-1)^{\rho_1} \varepsilon_1}{q_1} + \frac{(-1)^{\rho_2} \varepsilon_2}{q_1 q_2} + \dots + \frac{(-1)^{\rho_k} \varepsilon_k}{q_1 q_2 \dots q_k} + \dots, \quad (2)$$

where $\varepsilon_k \in \{0, 1, \dots, q_k - 1\}$.

The last representation is called *the representation of a number x by a sign-variable Cantor series* or *the quasi-nega- Q -representation*. It is easy to see that we get a positive Cantor series whenever $N_B = \emptyset$.

Define *the shift operator σ of expansion (2)* by the rule

$$\sigma(x) = \sigma \left(\Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)} \right) = \sum_{k=2}^{\infty} \frac{(-1)^{\rho_k} \varepsilon_k}{q_2 q_3 \dots q_k} = q_1 \Delta_{0 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)}.$$

Clearly,

$$\begin{aligned} \sigma^n(x) &= \sigma^n \left(\Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)} \right) \\ &= \sum_{k=n+1}^{\infty} \frac{(-1)^{\rho_k} \varepsilon_k}{q_{n+1} q_{n+2} \dots q_k} = q_1 \dots q_n \Delta_{\underbrace{0 \dots 0}_n \varepsilon_{n+1} \varepsilon_{n+2} \dots}^{(\pm Q, N_B)}. \end{aligned} \quad (3)$$

The following theorem is the most general statement on the representation of rational numbers for any sequences (q_k) , (ε_k) , and an arbitrary set N_B .

Theorem 4.1 ([41], [43], [39], [45]). *A number x represented by series (2) is rational for the case of any $N_B \subseteq \mathbb{N}$ if and only if there exist numbers $n \in \mathbb{Z}_0$ and $m \in \mathbb{N}$ such that $\sigma^n(x) = \sigma^{n+m}(x)$.*

The last theorem can be formulated by the following way.

Theorem 4.2 ([41], [43], [45]). *A number $x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)}$ is rational if and only if there exist numbers $n \in \mathbb{Z}_0$ and $m \in \mathbb{N}$ such that*

$$\Delta_{\underbrace{0 \dots 0}_n \varepsilon_{n+1} \varepsilon_{n+2} \dots}^{(\pm Q, N_B)} = q_{n+1} \dots q_{n+m} \Delta_{\underbrace{0 \dots 0}_{n+m} \varepsilon_{n+m+1} \varepsilon_{n+m+2} \dots}^{(\pm Q, N_B)}.$$

Let us recall several auxiliary statements which are true for positive Cantor series but do not hold for the general case of sign-variable Cantor series (i.e., for certain sets N_B).

Lemma 4.3 ([41], [43]). *Let n_0 be a fixed positive integer. Then the condition $\sigma^n(x) = \text{const}$ holds for all $n \geq n_0$ if and only if $\frac{\varepsilon_n}{q_n-1} = \text{const}$ for all $n > n_0$.*

Lemma 4.4 ([41], [43]). *Suppose we have $q = \min_{n \in \mathbb{N}} q_n$ and fixed $\varepsilon \in \{0, 1, \dots, q - 1\}$. Then the condition $\sigma^n(x) = x = \frac{\varepsilon}{q-1}$ holds if and only if the condition $\frac{q_n-1}{q-1}\varepsilon = \varepsilon_n \in \mathbb{Z}_0$ holds for all $n \in \mathbb{N}$.*

Let us consider cases when the condition $\frac{\varepsilon_k}{q_k-1} = \text{const}$ (the last equality holds for all k greater than some fixed k_0) is a necessary and/or sufficient condition for a rational number to be representable by a positive Cantor series. For more information, see [3], [13], [48].

In [13], J. Hančl and R. Tijdeman formulated certain conditions of the irrationality of a number represented by Cantor series (1) when sequences (q_k) and (ε_k) are sequences of positive integers and $q_k > 1$ for all $k \in \mathbb{N}$. Applications of the shift operator to representations of rational numbers by such series are considered. This article is partially devoted to conditions under which the condition $\frac{\varepsilon_k}{q_k-1} = \text{const}$ is a necessary and sufficient condition of the rationality of numbers represented by such expansions. In particular, the following cases are considered:

$$\liminf_{k \rightarrow \infty} \left(\frac{\varepsilon_{k+1}}{q_{k+1}} - \frac{\varepsilon_k}{q_k} \right) = 0, \quad \varepsilon_k = o(q_{k-1}q_k), \quad \varepsilon_{k+1} - \varepsilon_k = o(q_{k-1}q_k).$$

Also, in [13], the authors noted that sum (1) is equal to a rational number if $\frac{\varepsilon_k}{q_k-1} = \text{const}$ holds for all k greater than some number n_0 . Let us recall some results.

Lemma 4.5 ([13]). *If $S = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k} = \frac{r}{p}$ holds for a certain $r \in \mathbb{Z}$ and $p \in \mathbb{N}$, then $pS_N \in \mathbb{Z}$ for all $N \in \mathbb{N}$.*

Here $S = \sigma^0(x)$ and $S_N = \sigma^{N-1}(x)$. That is,

$$S_N = \sum_{k=N}^{\infty} \frac{\varepsilon_k}{q_N \dots q_k}.$$

Proposition 4.6 ([13]). *If (S_k) is bounded from below and for every $\varepsilon > 0$ we have*

$$S_{k+1} - S_k < \varepsilon$$

for $k \geq k_0(\varepsilon)$, then $S = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{q_k-1} = \text{const}$ for $N > N_0$.

Corollary 4.7 ([13]). *If (ε_k) is a sequence of positive integers such that $\varepsilon_{k+1} - \varepsilon_k = o(k)$, then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{k!} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{k-1} = \text{const}$ for k greater than some k_1 .*

Theorem 4.8 ([13]). *Let (q_k) be a sequence of positive integers which is monotonic and satisfies $\varepsilon_k = o(q_k^2)$. Then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \dots q_k} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{q_k-1} = \text{const}$ for $k \geq k_0$.*

Theorem 4.9 ([13]). *Let (q_k) and (ε_k) be sequences of integers such that $q_k > 1$ for all $k \in \mathbb{N}$. If $\left(\frac{\varepsilon_k}{q_k}\right)$ is bounded from below, $\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{q_{k-1}q_k} = 0$, and for each $\varepsilon > 0$ there exists $k_0(\varepsilon)$ such that the condition $\frac{\varepsilon_{k+1}}{q_{k+1}} < \frac{\varepsilon_k}{q_k} + \varepsilon$ holds for $k > k_0(\varepsilon)$, then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{q_{k-1}} = \text{const}$ for $k \geq N_0$.*

Theorem 4.10 ([13]). *Let (q_k) be a monotonic sequence of positive integers satisfying $\lim_{k \rightarrow \infty} \frac{q_k}{\log k} = \infty$. Then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{q_{k-1}} = \text{const}$ for $k \geq k_0$.*

Theorem 4.11 ([13]). *Let (q_k) be an unbounded monotonic sequence of positive integers. Then $\sum_{k=1}^{\infty} \frac{k}{q_1 q_2 \cdots q_k} \in \mathbb{Q}$ if and only if $\frac{k}{q_{k-1}} = \text{const}$ for $k \geq k_0$.*

Results obtained in [13] were generalized and corrected by Robert Tijdeman and Pingzhi Yuan in paper [48]. In particular, results are generalized for the cases when $\varepsilon_k = k$ and $q_k \rightarrow \infty$, $q_k = k$ and $\varepsilon_{k+1} - \varepsilon_k = O(k)$. In the last-mentioned article, it is shown that, in order that the condition $\frac{\varepsilon_k}{q_{k-1}} = \text{const}$ for all $k \geq k_0$ is a necessary and sufficient condition of the rationality, one can neglect the condition $\varepsilon_k = o(q_k^2)$ in the system of conditions: $\varepsilon_k = o(q_k^2)$, $\varepsilon_k \geq 0$, $\varepsilon_{k+1} - \varepsilon_k < \varepsilon q_k$ for $k \geq k_1(\varepsilon)$. We note the following statements.

Theorem 4.12 ([48]). *Let (q_n) be a monotonic integer sequence with $q_n > 1$ for all n and (ε_n) be an integer sequence such that $\varepsilon_{n+1} - \varepsilon_n = o(q_{n+1})$. Then $\sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_1 q_2 \cdots q_n} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_n}{q_{n-1}} = \text{const}$ for all n greater than some n_0 .*

Theorem 4.13 ([48]). *Let (q_k) be a monotonic sequence of positive integers, $q_k > 1$. Let (ε_k) be a sequence of positive integers satisfying*

$$\limsup_{k \rightarrow \infty} \frac{\varepsilon_{k+1} - \varepsilon_k}{q_k} \leq 0.$$

Then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} \in \mathbb{Q}$ if and only if $\frac{\varepsilon_k}{q_{k-1}} = \text{const}$ for all k greater than some k_0 .

In addition, the following sufficient condition of the irrationality is proved.

Theorem 4.14 ([48]). *Let $q_k > 1$ be such that $\varepsilon_k = O(q_k)$ for all k and $\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{q_k} = \alpha \in \mathbb{I}$. Then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} \in \mathbb{I}$.*

The last statement with the condition $0 \leq \varepsilon_k < q_k$ without $\varepsilon_k = O(q_k)$ was proved in [31].

Finally, in [48], the following denotations are used in proofs:

$$S = \sum_{k=1}^{\infty} \frac{\varepsilon_k^*}{q_1^* q_2^* \cdots q_k^*}, \quad S_{n_k} = \sum_{j=1}^{n_k} \frac{\varepsilon_j^*}{q_1^* q_2^* \cdots q_j^*}, \quad R_{n_k} = \sum_{j=k+1}^{\infty} \frac{\varepsilon_j^*}{q_{k+1}^* q_{k+2}^* \cdots q_j^*}.$$

Here (n_k) is a subsequence of positive integers, $n_0 = 1$,

$$\varepsilon_k^* = \varepsilon_{n_k-1} + \varepsilon_{n_k-2} q_{n_k-1} + \cdots + \varepsilon_{n_k-1} q_{n_k-1} q_{n_k-2} \cdots q_{n_k-1+1},$$

and $q_k^* = q_{n_k-1}q_{n_k-2}\cdots q_{n_k-1}$, $k = 1, 2, 3, \dots$. For series (1), where (q_n) and (ε_n) are sequences of integers such that $q_n > 0$ for all $n \in \mathbb{N}$ and series (1) converges, the following statements are true.

Lemma 4.15 ([48]). *Using the notation above, if there exists a subsequence (n_k) of positive integers such that $R_{n_k} = R_{n_{k+1}}$ for $k = 1, 2, \dots$, then $S \in \mathbb{Q}$.*

Proposition 4.16 ([48]). *If (R_n) is bounded from below and there exists a subsequence (n_k) of positive integers with $R_{n_{k+1}} - R_{n_k} < \varepsilon$ for $k \geq k_0(\varepsilon)$, then $S \in \mathbb{Q}$ if and only if $R_{n_k} = R_{n_{k+1}}$ for all large k .*

In [31], A. Oppenheim studied sufficient conditions of the irrationality of numbers represented by Cantor series (1) and, also, alternating series (1) such that $|\varepsilon_i| < q_i - 1$ for $i = 1, 2, 3, \dots$, and $\varepsilon_m \varepsilon_n < 0$ for some $m > i$ and $n > i$ when i is any fixed integer. Also, in [31], the main results obtained by using some results from [2] and sums of the form

$$x_{i_k} = \frac{\varepsilon_{i_k}}{q_{i_k}} + \frac{\varepsilon_{i_{k+1}}}{q_{i_k}q_{i_{k+1}}} + \frac{\varepsilon_{i_{k+2}}}{q_{i_k}q_{i_{k+1}}q_{i_{k+2}}} + \dots,$$

where (i_k) is some subsequence of positive integers, and by investigation of the limit of $c_{i_k} = \frac{\varepsilon_{i_k}}{q_{i_k}}$ as $k \rightarrow \infty$. That is, here $x_{i_k} = \sigma^{i_k-1}(x)$.

Lemma 4.17 ([31]). *A necessary and sufficient condition that x given by convergent series (1), where q_k and ε_k are integers, shall be irrational is that for every integer $p \in \mathbb{N}$ we can find an integer $r \in \mathbb{Z}$ and a subsequence (i_k) such that*

$$\frac{r}{p} < x_{i_k} < \frac{r+1}{p}, \quad k = 1, 2, 3, \dots$$

Finally, in this section, we note necessary and sufficient conditions for a rational number to be representable by certain types of Cantor series which were investigated by P. Erdős and E. G. Straus in [7].

Theorem 4.18 ([7]). *Let (ε_n) be a sequence of integers and (q_k) be a sequence of positive integers with $q_k > 1$ for all large k and*

$$\lim_{k \rightarrow \infty} \frac{|\varepsilon_k|}{q_{k-1}q_k} = 0.$$

Then $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} \in \mathbb{Q}$ if and only if there exist a positive integer B and a sequence of integers (c_k) such that for all large k we have

$$B\varepsilon_k = c_k q_k - c_{k+1}, \quad |c_{k+1}| < \frac{q_k}{2}.$$

Theorem 4.19 ([7]). *Let p_k be the k th prime and let (q_k) be a monotonic sequence of positive integers satisfying*

$$\lim_{k \rightarrow \infty} \frac{p_k}{q_k^2} = 0, \quad \liminf_{k \rightarrow \infty} \frac{q_k}{p_k} = 0.$$

Then $\sum_{k=1}^{\infty} \frac{p_k}{q_1 q_2 \cdots q_k} \in \mathbb{I}$.

5 Certain approaches to investigations of expansions of rational numbers

We considered mainly the shift operator. Now one can consider another approaches to investigations of expansions of rational numbers but some of them are related with the shift operator.

In [26], the probabilistic approach is used and the attention is given to Cantor series (1) for which $\varepsilon_k \neq q_k - 1$ infinitely often. Irrationality of numbers having a property of a certain type of the normality is investigated.

Definition 5.1 ([26, p.45]). A number $x \in [0, 1)$ is called *Q-distribution normal* if the sequence

$$X = (x \pmod{1}, q_1x \pmod{1}, q_1q_2x \pmod{1}, q_1q_2 \cdots q_kx \pmod{1}, \dots)$$

is uniformly distributed in $[0, 1)$.

Theorem 5.2 ([26, p. 264]). *A number $x \in [0, 1)$ is irrational if and only if there exists a basic sequence $Q = (q_k)$ such that x is Q-distribution normal.*

In the paper [22], the subspace theorem is used for proving conditions for a transcendental number to be representable by positive Cantor series. Such conditions were formulated in terms of blocks of digits ε_k and in terms of tuples of digits for expansion (1).

Finally, one approach based on the notion of cylinders of Cantor expansions gives an opportunity to model rational numbers. However, we have necessary and sufficient conditions for the case of the positive Cantor series and a necessary condition for the sign-variable Cantor series. Let us consider the following two theorems.

Theorem 5.3 ([45]). *A number $x = \Delta_{\varepsilon_1\varepsilon_2\dots\varepsilon_n\dots} \in (0, 1)$ represented by series (1) is a rational number $\frac{p}{r}$, where $p, r \in \mathbb{N}$, $(p, r) = 1$, and $p < r$, if and only if the condition*

$$\varepsilon_n = \left[\frac{q_n(\Delta_{n-1} - r\varepsilon_{n-1})}{r} \right]$$

holds for all $1 < n \in \mathbb{N}$, where $\Delta_1 = pq_1$, $\varepsilon_1 = \left[\frac{\Delta_1}{r} \right]$, and $[a]$ is the integer part of a .

For fullness, we give some examples of rational numbers from [45]. Really, suppose

$$x = \Delta_{\varepsilon_1\varepsilon_2\dots\varepsilon_n\dots}^{(2n+1)} = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}.$$

Then

$$\frac{1}{4} = \Delta_{035229[11]4\dots}^{(2n+1)}, \quad \frac{3}{8} = \Delta_{104341967\dots}^{(2n+1)}.$$

Theorem 5.4 ([46]). *If $x = \Delta_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}^{(\pm Q, N_B)} = \frac{p}{r}$, where $p \in \mathbb{Z}, r \in \mathbb{N}, (|p|, |r|) = 1$, and $|p| < r$, then the condition*

$$\varepsilon_k = \left\lfloor \left[\frac{q_k(\Delta_{k-1}^{(g)} - (-1)^{\rho_{k-1}} r \varepsilon_{k-1})}{r} + s_n \right] \right\rfloor$$

holds for all $1 < k \in \mathbb{N}$. Here $\Delta_1^{(g)} = pq_1$, $\varepsilon_1 = \left\lfloor \left[\frac{\Delta_1^{(g)}}{r} + s_1 \right] \right\rfloor$, and $[a]$ is the integer part of a . Also,

$$s_1 = \sum_{1 < k \in N_B} \frac{q_k - 1}{q_2 q_3 \cdots q_k}, \quad s_k = \begin{cases} q_k s_{k-1} & \text{whenever } k \notin N_B \\ q_k s_{k-1} - (q_k - 1) & \text{whenever } k \in N_B. \end{cases}$$

One can note that the two last statements are related to the shift operator. Really, in the case of positive Cantor series [45], we have $\sigma^n(x) = \{\frac{\Delta_n}{r}\}$ and $\varepsilon_n = [\frac{\Delta_n}{r}]$. In the general case of sign-variable Cantor series (i.e., there is no number k_0 such that any $k \in N_B$ or any $k \notin N_B$ for all $k > k_0$), we obtain [46] the following:

$$\left\{ \frac{\Delta_n^{(g)}}{r} \right\} = \begin{cases} \sigma^n(x) & \text{whenever } \sigma^n(x) \geq 0 \\ 1 - \sigma^n(x) & \text{whenever } \sigma^n(x) < 0, \end{cases}$$

where $\{a\}$ is the fractional part of a (i.e., $a = [a] + \{a\}$).

In this survey, we have demonstrated the main conditions for a rational number to be representable by positive, alternating, and sign-variable Cantor series. Connections among some of them are described. An important role of the notion of the shift operator in investigations in this topic, is noted.

References

- [1] Beros Achilles A. and Beros Kostantinos A.: Normal numbers and limit computable Cantor series. <http://arxiv.org/pdf/1404.2178v1.pdf>.
- [2] Cantor G.: Ueber die einfachen Zahlensysteme. *Z. Math. Phys.* 14 (1869) 121–128.
- [3] Diananda P. H. and Oppenheim A.: Criteria for irrationality of certain classes of numbers II. *Amer. Math. Monthly* 62 (4) (1955) 222–225.
- [4] Erdős P. and Rényi A.: Some further statistical properties of the digits in Cantor's series. *Acta Math. Acad. Sci. Hungar.* 10 (1959) 21–29.
- [5] Erdős P. and Rényi A.: On Cantor's series with convergent $\sum \frac{1}{q_n}$. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* 2 (1959) 93–109.
- [6] Erdős P. and Straus E. G.: On the irrationality of certain Ahmes series. *J. Indian. Math. Soc.* 27 (1968) 129–133.
- [7] Erdős P. and Straus E. G.: On the irrationality of certain series. *Pacific J. Math.* 55 (1) (1974) 85–92.

- [8] Galambos J.: Representations of real numbers by infinite series. In: *Lecture Notes in Mathematics* 502. Springer, 1976, 148 pp.
- [9] Galambos J.: Uniformly distributed sequences $\pmod{1}$ and Cantor's series representation. *Czech. Math. J.* 26 (4) (1976) 636–641.
- [10] Hančl J.: A note to the rationality of infinite series I. *Acta Math. Inf. Univ. Ostr.* 5 (1) (1997) 5–11.
- [11] Hančl J.: A note on a paper of Oppenheim and Šalát concerning series of Cantor type. *Acta Math. Inf. Univ. Ostr.* 10 (1) (2002) 35–41.
- [12] Hančl J. and Rucki P.: A note to the transcendence of special infinite series. *Math. Slov.* 56 (4) (2006) 409–414.
- [13] Hančl J. and Tijdeman R.: On the irrationality of Cantor series. *J. Reine Angew. Math.* 571 (2004) 145–158.
- [14] Hančl J. and Tijdeman R.: On the irrationality of Cantor and Ahmes series. *Publ. Math. Debrecen* 65 (3-4) (2004) 371–380.
- [15] Hančl J. and Tijdeman R.: On the irrationality of factorial series. *Acta Arith.* 118 (2005) 383–401.
- [16] Hančl J. and Tijdeman R.: On the irrationality of factorial series III. *Indag. Mathem., N. S.* 20 (4) (2009) 537–549.
- [17] Hančl J. and Tijdeman R.: On the irrationality of factorial series II. *J. Number Theory* 130 (3) (2010) 595–607.
- [18] Hofer R., Pillichshammer F., and Piršic G.: Distribution properties of sequences generated by \mathbb{Q} -additive functions with respect to Cantor representations of integers. *Acta Arith.* 138 (2) (2009) 179–200.
- [19] Iommi G. and Skorulski B.: Hausdorff dimension of Cantor series.
http://www.mat.uc.cl/~giommi/new_cantor_series091118-3.pdf.
- [20] Kirschenhofer P. and Tichy R. F.: On the distribution of digits in Cantor representations of integers. *J. Number Theory* 18 (1984) 121–134.
- [21] Kolouch Ondřej and Novotný Lukáš: Diophantine approximations of infinite series and products. *Communications in Mathematics* 24 (1) (2016) 71–82.
- [22] Kumar Veekesh: A transcendence criterion for Cantor series. *Acta Arith.* 188 (3) (2019) 269–287.
- [23] Komatsu T., Laohakosol V., Prugsapitak S., and Rattanaoong J.: Independence of complex Cantor series and Cantor products.
<http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1898-05.pdf>.
- [24] Kuhapatanakul P. and Laohakosol V.: Irrationality of some series with rational terms. *Kas. J. (Nat. Sci.)* 35 (2001) 205–209.
- [25] Mance B. and Li B.: Number theoretic applications of a class of Cantor series fractal functions, II. *Int. J. Number Theory* 11 (2) (2015) 407–435.
- [26] Mance B.: *Normal numbers with respect to the Cantor series expansion, Dissertation*. The Ohio State University (2010).
- [27] Mance B.: Cantor series constructions of sets of normal numbers.
<http://arxiv.org/pdf/1010.2782v2.pdf>.

- [28] Mance B.: On the Hausdorff dimension of countable intersections of certain sets of normal numbers. <http://arxiv.org/pdf/1302.7064v2.pdf>.
- [29] Mance B.: Number theoretic applications of a class of Cantor series fractal functions, I. <http://arxiv.org/pdf/1310.2377v2.pdf>.
- [30] Marques D.: A geometric proof to Cantor's theorem and an irrationality measure for some Cantor's series. *Ann. Math. et Inf.* 36 (2009) 117–121.
- [31] Oppenheim A.: Criteria for irrationality of certain classes of numbers. *Amer. Math. Mon.* 61 (4) (1954) 235–241.
- [32] Paštéka M.: The Cantor series of polyadic numbers. *Acta Math Inf. Univ. Ostr.* 4 (1) (1996) 75–82.
- [33] Rényi A.: On a new axiomatic theory of probability. *Acta Math. Acad. Sci. Hungar.* 6 (1955) 285–335.
- [34] Rényi A.: On the distribution of the digits in Cantor's series. *Hung. Mat. Lap.* 7 (1956) 77–100.
- [35] Rényi A.: Representations for real numbers and their ergodic properties. *Acta. Math. Acad. Sci. Hungar.* 8 (1957) 477–493.
- [36] Rényi A.: Probabilistic methods in number theory. *Shux. Jin.* 4 (1958) 465–510.
- [37] Serbenyuk S. O.: Functions, that defined by functional equations systems in terms of Cantor series representation of numbers. *Naukovi Zapysky NaUKMA* 165 (2015) 34–40.
- [38] Serbenyuk S.: Nega- \tilde{Q} -representation as a generalization of certain alternating representations of real numbers. *Bull. Taras Shev. Natl. Univ. Kyiv Math. Mech.* 35 (1) (2016) 32–39.
- [39] Serbenyuk Symon: Representation of real numbers by the alternating Cantor series. *Integ.* 17 (2017) 27.
- [40] Serbenyuk S. O.: Continuous Functions with Complicated Local Structure Defined in Terms of Alternating Cantor Series Representation of Numbers. *Zh. Mat. Fiz. Anal. Geom.* 13 (1) (2017) 57–81.
- [41] Serbenyuk S.: Rational numbers in terms of positive Cantor series. *Bull. Taras Shev. Natl. Univ. Kyiv Math. Mech. (in Ukrainian)* 38 (2) (2017) 11–15.
- [42] Serbenyuk S.: Non-differentiable functions defined in terms of classical representations of real numbers. *Zh. Mat. Fiz. Anal. Geom.* 14 (2) (2018) 197–213.
- [43] Serbenyuk S.: Cantor series and rational numbers. <https://arxiv.org/pdf/1702.00471.pdf>.
- [44] Serbenyuk S.: Generalizations of certain representations of real numbers. *Tatra Moun. Math. Publ.* 77 (2020) 59–72.
- [45] Serbenyuk S.: A note on expansions of rational numbers by certain series. *Tatra Moun. Math. Publ.* 77 (2020) 53–58.
- [46] Serbenyuk S.: Rational numbers represented by sign-variable Cantor series. *Acta Math. Hungar.* 164 (2021) 580–592.
- [47] Sondow J.: A geometric proof that e is irrational and a new measure of its irrationality. *Amer. Math. Monthly* 113 (2006) 637–641.
- [48] Tijdeman Robert and Yuan Pingzhi: On the rationality of Cantor and Ahmes series. *Indag. Math. (N.S.)* 13 (3) (2002) 407–418.
- [49] Turán P.: On the distribution of “digits” in Cantor systems. *Mat. Lapok* 7 (1956) 71–76.

- [50] Wang Yi, Wen Zhixiong , and Xi Lifeng: Some fractals associated with Cantor expansions. *J. Math. Anal. Appl.* 354 (2009) 445–450.
- [51] Waterman M. S.: Cantor’s series for vectors. *Amer. Math. Monthly* 82 (6) (1975) 622–625.
- [52] Wegmann H.: Die Hausdorffsche Dimension von Mengen reeller Zahlen, die durch Zifferneigenschaften einer Cantorentwicklung charakterisiert sind. *Czech. Math. J.* 18 (4) (1968) 622–632.
- [53] Wen Liu: A nowhere differentiable continuous function constructed using Cantor series. *Math. Magaz.* 74 (5) (2001) 400–402.

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