

Action of vectorial Lie superalgebras on some split supermanifolds

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Abstract. The “curved” super Grassmannian is the supervariety of sub-supervarieties of purely odd dimension k in a supervariety of purely odd dimension n , unlike the “usual” super Grassmannian which is the supervariety of linear sub-superspaces of purely odd dimension k in a superspace of purely odd dimension n . The Lie superalgebras of all and Hamiltonian vector fields on the superpoint are realized as Lie superalgebras of derivations of the structure sheaves of certain “curved” super Grassmannians.

Preface of the editor

The manuscript of this paper appeared as a preprint in proceedings of the “Seminar on Supersymmetries”, see <http://staff.math.su.se/mleites/sos.html>, and in Russian, see [52] in the list of references in the jubilee paper [AVGDZKLST*]. I updated the references; the ones I added are endowed with an asterisk. The abstract and comments are due to me. For a comprehensive description of simple Lie superalgebras of vector fields over algebraically closed fields of any characteristic, see [BGLLS*]. *D. Leites*

1 Introduction

The Lie superalgebra $W_n := \text{Der } \Lambda_{\mathbb{C}}[\xi_1, \dots, \xi_n]$ consisting of all vector fields on the superpoint $\mathcal{C}^{0,n}$ is isomorphic, as shown in [S1], to the Lie superalgebra of vector fields, i.e., the global derivations of the structure sheaf, see [O], of a split complex supermanifold $\mathcal{C}\mathcal{G}r_{n-1}^n$ determined by the tautological vector bundle of rank $n - 1$ on the complex Grassmann manifold Gr_{n-1}^n .

The Lie superalgebra H_n , the subsuperalgebra of W_n consisting of Hamiltonian vector fields on the superpoint $\mathcal{C}^{0,n}$, is isomorphic to the Lie superalgebra of vector fields on a split complex supermanifold $\mathcal{C}\mathcal{Q}^{n-2}$ associated with a vector bundle of rank $n - 1$ orthogonal to the tautological bundle on the quadric $Q^{n-2} \subset \mathbb{C}P^{n-1}$.

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However, the method used in [S1], [S2] does not allow one to indicate explicitly these isomorphisms. In this paper I explicitly construct the W_n - and H_n -actions on the supermanifolds $\mathcal{CG}r_{n-1}^n$ and \mathcal{CQ}^{n-2} ; I give a new version of the proof of the above results.

D. Leites told me that the supermanifolds $\mathcal{CG}r_{n-1}^n$ and \mathcal{CQ}^{n-2} are the simplest examples of what Manin called “curved” Grassmannians and “curved” quadrics, see [Ma*]. They were introduced in [KL*]. (Compare with the Grassmannians of **linear** subspaces in a linear superspace, see [Ma*, Ch.4, § 3]. For the complete list of homogeneous superdomains associated with the known Lie superalgebras of polynomial growth, see [L*]. *D.L.*)

2 Superization of a construction due to Serre

In this section, I superize a construction Serre introduced in [Se]. This enables us to interpret elements of a Lie superalgebra as Hamiltonian vector fields on the superpoint.

Let V be a purely odd vector space over \mathbb{C} , and \bar{V} a second copy of the same space considered as purely even. The change of parity $V \longrightarrow \bar{V}$ will be denoted by $x \mapsto \bar{x}$ on every non-zero $x \in V$.

Construct the Koszul complex of the \mathbb{Z} -graded algebra

$$A := S(\bar{V} \oplus V) = S(\bar{V}) \otimes \Lambda(V)$$

which can be naturally considered as a free supercommutative superalgebra. There exists a unique derivation $d \in \text{Der}_{-1} A$ such that $dx = \bar{x}$ and $d\bar{x} = 0$ for any $x \in V$. Obviously, $d^2 = 0$.

Consider the \mathbb{Z} -graded Lie superalgebra $W(V) = \text{Der } \Lambda(V)$. Any element $\delta \in W(V)$ can be uniquely extended to a derivation $\tilde{\delta} \in \text{Der } A$ such that $[\tilde{\delta}, d] = 0$, see [K]. The correspondence $\delta \mapsto \tilde{\delta}$ is a faithful linear representation of the Lie superalgebra $W(V)$ in A .

Let $\omega \in S^2(\bar{V})$ be a nondegenerate bilinear form. Set

$$H(\omega) := \{\delta \in W(V) \mid \tilde{\delta}(\omega) = 0\}.$$

Then $H(\omega)$ is a \mathbb{Z} -graded subalgebra in $W(V)$ called the *Lie superalgebra of Hamiltonian vector fields*. Set

$$DH(\omega) := \{\delta \in W(V) \mid \tilde{\delta}(\omega) = \varphi\omega \text{ for some } \varphi \in A\}.$$

Clearly, $DH(\omega)$ is a \mathbb{Z} -graded subalgebra of $W(V)$, and $H(\omega)$ is its ideal.

Hereafter we assume that $\dim V = n$. Set $W_n := W(V)$, $H_n := H(\omega)$, $DH_n := DH(\omega)$ since these algebras are determined, up to an isomorphism, by $\dim V$. In V , select a basis ξ_1, \dots, ξ_n . Then the elements $x_i = \bar{\xi}_i = d\xi_i$, where $i = 1, \dots, n$, constitute a basis in \bar{V} and

$$A = \mathbb{C}[x_1, \dots, x_n] \otimes \Lambda[\xi_1, \dots, \xi_n].$$

Obviously, $d = \sum_{1 \leq i \leq n} x_i \partial_{\xi_i}$. Notice that $\partial_{\xi_i} \in \text{Der}_{-1} A$ coincides with the extension $\tilde{\partial}_{\xi_i}$ of $\partial_{\xi_i} \in W(V)_{-1}$.

- 2.1. Lemma.** 1) If $\tilde{\delta}(\omega) = \varphi\omega$ for some $\delta \in W(V)$, $\varphi \in A$, then $\varphi \in \mathbb{C}$.
 2) There is the following semidirect sum decomposition:

$$DH(\omega) = H(\omega) \ltimes \mathbb{C}E, \quad \text{where } E = \sum_{1 \leq i \leq n} \xi_i \partial_{\xi_i}.$$

Proof. 1) We may assume that $(x_i)_{i=1}^n$ is a basis in which ω is the form

$$\omega = \sum_{1 \leq i \leq n} x_i^2 = \sum_{1 \leq i \leq n} (d\xi_i)^2.$$

Then we have

$$\tilde{\delta}\omega = 2 \sum_{1 \leq i \leq n} x_i \tilde{\delta}x_i = (-1)^k 2 \sum_{1 \leq i \leq n} x_i d(\delta\xi_i) \quad \text{for any } \delta \in W(V)_k.$$

Setting $h_i = \delta\xi_i$ we get

$$\tilde{\delta}\omega = (-1)^k 2 \sum_{1 \leq i, j \leq n} x_i x_j \frac{\partial h_j}{\partial \xi_i}, \quad \text{where } \frac{\partial h_j}{\partial \xi_i} \in \Lambda^k(V).$$

If δ satisfies the conditions of Lemma 2.1, then $\varphi = (-1)^k 2 \frac{\partial h_i}{\partial \xi_i}$ for any $i = 1, \dots, k$. In particular, $\varphi \in \Lambda^k(V)$. Further, for any i , we have $\frac{\partial \varphi}{\partial \xi_i} = (-1)^k 2 \frac{\partial^2 h_i}{\partial \xi_i^2} = 0$. Therefore, $\varphi \in \mathbb{C}$.

2) Observe that $E(\omega) = 2\omega$, and so $E \in DH(\omega)_0$. Further, if $\delta \in DH(\omega)$, then, due to the proved above, $\tilde{\delta}\omega = c\omega$, where $c \in \mathbb{C}$. Hence, $(\tilde{\delta} - \frac{1}{2}cE)(\omega) = 0$, i.e., $\delta = \delta_0 + \frac{1}{2}cE$, where $\delta_0 \in H(\omega)$. \square

3 Vector bundles over $\mathbb{C}P^{n-1}$ and Q^{n-2} and supermanifolds

Define some special supermanifolds associated with vector bundles over $Q^{n-2} \subset \mathbb{C}P^{n-1}$ and $\mathbb{C}P^{n-1}$.

Let $\dim V = n$, and $P(V)$ the corresponding projective space. Let us assume that the nonzero elements of V^* are odd and those of \bar{V}^* are even. In V , select a basis e_1, \dots, e_n , and consider the dual bases ξ_1, \dots, ξ_n and x_1, \dots, x_n of V^* and \bar{V}^* , respectively. In the notation of § 2 (applied to V^* and \bar{V}^*) we have $x_i = \bar{\xi}_i$.

The elements x_1, \dots, x_n are homogeneous coordinates on $P(V)$; this means that the stalk \mathcal{F}_z^a of the structure sheaf \mathcal{F}^a of the algebraic variety $P(V)$ at $z \in P(V)$ is a subring of the field $\mathbb{C}(V) = \mathbb{C}(x_1, \dots, x_n)$ consisting of elements of the form f/g , where f, g are homogeneous polynomials of the same degree in $\mathbb{C}[x_1, \dots, x_n] = S(\bar{V}^*)$ and $g(z) \neq 0$.

Consider the trivial vector bundle $P(V) \times V^*$ and its subbundle $E \subset P(V) \times V^*$ consisting of the pairs $(\mathbb{C}x, y)$, where $x \in V \setminus \{0\}$ and $y \in \text{Ann } x = \{\alpha \in V^* \mid \alpha(x) = 0\}$. Clearly, E is an algebraic vector bundle of rank $n - 1$ over $P(V)$ with the fiber

$$E_{\mathbb{C}x} = \text{Ann } x, \quad \text{where } x \in V \setminus \{0\}.$$

The map $\mathbb{C}x \mapsto \text{Ann } x$ identifies $P(V)$ with the Grassmann variety $\text{Gr}_{n-1}(V^*)$, which consists of $(n-1)$ -dimensional subspaces in V^* , and E is the tautological bundle over this Grassmann variety.

Let $\mathcal{E}^a \subset \mathcal{F}^a \otimes V^*$ be the sheaf of germs of the polynomial sections of E . The variety $P(V)$ can be endowed with two structures of a split complex algebraic supervariety: one is determined by the sheaf $\hat{\mathcal{O}}^a = \mathcal{F}^a \otimes \Lambda(V^*)$, and the other by its subsheaf $\mathcal{O}^a = \Lambda(\mathcal{E}^a)$.

Besides, $\hat{\mathcal{O}} = \mathcal{F} \otimes \Lambda(V^*)$ and $\mathcal{O} = \Lambda(\mathcal{E}) \subset \hat{\mathcal{O}}$, where \mathcal{F} is the sheaf of holomorphic functions on $P(V)$ and \mathcal{E} is the sheaf of germs of holomorphic sections of E , determine two structures of a split complex analytic supermanifold associated with $P(V)$.

Consider the superalgebra

$$A := S(\bar{V}^*) \otimes \Lambda(V^*) = \mathbb{C}[x_1, \dots, x_n] \otimes \Lambda[\xi_1, \dots, \xi_n]$$

and the derivation $d \in \text{Der}_{-1} A$ constructed in § 2 (with V replaced by V^* in these constructions). Let

$$B := \mathbb{C}(V) \otimes \Lambda(V^*) = \mathbb{C}(x_1, \dots, x_n) \otimes \Lambda[\xi_1, \dots, \xi_n]$$

be the localization of A with respect to the multiplicative system $S(\bar{V}^*) \setminus \{0\}$. The algebra B has a natural supercommutative superalgebra structure, and d can be uniquely extended to an odd derivation of B , which we will denote also by d . Clearly,

$$\hat{\mathcal{O}}_z^a = \mathcal{F}_z^a \otimes \Lambda(V^*) \subset B \quad \text{for any } z \in P(V).$$

3.1. Lemma. *We have $\mathcal{O}_z^a = \hat{\mathcal{O}}_z^a \cap \text{Ker } D$ for any $z \in P(V)$.*

Proof. For any $x \in V \setminus \{0\}$, consider the derivation

$$d_x = \sum_{1 \leq i \leq n} x_i \partial_{\xi_i} \in \text{Der}_{-1} \Lambda(V^*)$$

uniquely determined by the condition $d_x(\xi) = \xi(x)$ for any $\xi \in V^*$. Clearly,

$$\text{Ker } d_x = \Lambda(\text{Ann } x) = \Lambda(E_{\mathbb{C}x}).$$

Let $u := \sum_{1 \leq i \leq r} \varphi_i v_i \in B$, where $\varphi_i \in \mathcal{F}^a$ and $v_i \in \Lambda(V^*)$. Considering u as a function on $V \setminus \{0\}$ with values in $\Lambda(V^*)$ we see that $u(x) = \sum_{1 \leq i \leq r} \varphi_i(\mathbb{C}x) v_i$ at any point x from its domain implying

$$d_x u(x) = \sum_{1 \leq i \leq r} \varphi_i(\mathbb{C}x) d_x v_i = \sum_{1 \leq i \leq r} \varphi_i(\mathbb{C}x) \sum_{1 \leq j \leq n} x_j \frac{\partial v_i}{\partial \xi_j}.$$

On the other hand,

$$du = \sum_{1 \leq i \leq r} \varphi_i dv_i = \sum_{1 \leq i \leq r} \varphi_i \sum_{1 \leq j \leq n} x_j \frac{\partial v_i}{\partial \xi_j}.$$

Therefore, $(du)(x) = d_x u(x)$ on a Zariski open subset of V . Hence, $du = 0$ if and only if $u(x) \in \Lambda(E_{\mathbb{C}x})$ for all $x \in V$ from the domain of u . \square

Now, define a subsupermanifold of $(P(V), \mathcal{O}^a)$ whose underlying manifold is the quadric $Q \subset P(V)$. Let ω be a non-degenerate quadratic function on V ; it can be considered as an element of $S^2(\bar{V}^*)$. Let Q be the quadric in $P(V)$ given by the equation $\omega = 0$ and $E' = E|_Q$ the restriction of E to Q . If we identify V^* with V with the help of the non-degenerate symmetric bilinear form corresponding to ω , then E' is identified with the subbundle of $Q \times V$ orthogonal to the tautological line bundle over Q .

Let \mathcal{F}'^a and \mathcal{E}'^a (resp. \mathcal{F}' and \mathcal{E}') be the sheaves of polynomial (resp. holomorphic) functions on Q and polynomial (resp. holomorphic) sections of \mathcal{E}' , respectively. Set $\mathcal{O}'^a := \Lambda(\mathcal{E}'^a)$ and $\mathcal{O}' := \Lambda(\mathcal{E}')$.

Let us give a description of \mathcal{O}'^a similar to the above description of \mathcal{O}^a . For this, consider the superalgebra

$$A' := A/\omega A = \mathbb{C}[\hat{Q}] \otimes \Lambda(V^*),$$

where $\mathbb{C}[\hat{Q}] := S(\bar{V}^*)/\omega S(\bar{V}^*)$ is the algebra of polynomial functions on the cone $\hat{Q} \subset V$ given by the equation $\omega = 0$; the localization of A' is $B' := \mathbb{C}(\hat{Q}) \otimes \Lambda(V^*)$. The algebras \mathcal{F}'^a_z and \mathcal{O}'^a_z , where $z \in Q$, are embedded into B' . The derivation d transforms ωA into itself, and therefore determines an odd derivation d' of A' and B' . Lemma 3.1 implies that

$$\mathcal{O}'^a_z = (\mathcal{F}'^a_z \otimes \Lambda(V^*)) \cap \text{Ker } d' \quad \text{for any } z \in Q.$$

4 Several remarks on vector fields on algebraic and analytic supervarieties

Let M be a nonsingular complex algebraic variety, E an algebraic vector bundle over M . Denote by \mathcal{F}^a , \mathcal{T}^a , \mathcal{E}^a the structure sheaf on M , the tangent sheaf on M , and the locally free algebraic sheaf corresponding to E , respectively. We denote by the same letters without the superscript a the corresponding analytic sheaves on M .

In particular, (M, \mathcal{F}) is the complex analytic manifold corresponding to the algebraic variety M . The sheaves $\mathcal{O}^a = \Lambda_{\mathcal{F}^a}(\mathcal{E}^a)$ and $\mathcal{O} = \Lambda_{\mathcal{F}}(\mathcal{E})$ rig M with structures of a split algebraic and a split analytic supervariety, respectively. We call the sheaves of \mathbb{Z} -graded Lie superalgebras $\text{Der } \mathcal{O}^a$ and $\text{Der } \mathcal{O}$ the *sheaves of vector fields* on these supervarieties.

4.1. Lemma. *There exists a natural injective homomorphism of sheaves of \mathbb{Z} -graded Lie superalgebras $\text{Der } \mathcal{O}^a \rightarrow \text{Der } \mathcal{O}$.*

Proof. As shown in [O], for any $k \in \mathbb{Z}$, every $\gamma \in \text{Der}_k \mathcal{O}$ can be identified with a pair (γ_0, γ_1) , where

$$\gamma_0 \in \text{Hom}_{\mathcal{F}}(\mathcal{E}, \Lambda^{k+1}(\mathcal{E})) \text{ and } \gamma_1 \in \text{Hom}_{\mathbb{C}}(\mathcal{F}, \Lambda^k(\mathcal{E})) = \mathcal{T} \otimes_{\mathcal{F}} \Lambda^k(\mathcal{E}) \quad (1)$$

such that

$$\begin{aligned} \gamma_0(\varphi s) &= \gamma_1(\varphi)s + \varphi\gamma_0(s) \\ \gamma_1(\varphi\psi) &= \gamma_1(\varphi)\psi + \varphi\gamma_1(\psi) \end{aligned} \quad \text{for any } s \in \mathcal{E} \text{ and } \varphi, \psi \in \mathcal{F}. \quad (2)$$

A similar statement holds also for $\text{Der } \mathcal{O}^a$.

Since the sections of the sheaves

$$\mathrm{Hom}_{\mathcal{F}^a}(\mathcal{E}^a, \Lambda^{k+1}(\mathcal{E}^a)) \quad (\text{resp. } \mathrm{Hom}_{\mathcal{F}}(\mathcal{E}, \Lambda^{k+1}(\mathcal{E})))$$

are algebraic (resp. holomorphic) homomorphisms of vector bundles $E \rightarrow \Lambda^{k+1}(E)$, there is a natural embedding

$$\mathrm{Hom}_{\mathcal{F}^a}(\mathcal{E}^a, \Lambda^{k+1}(\mathcal{E}^a)) \rightarrow \mathrm{Hom}_{\mathcal{F}}(\mathcal{E}, \Lambda^{k+1}(\mathcal{E})).$$

Further, the natural embedding $\mathcal{T}^a \rightarrow \mathcal{T}$ induces the embedding

$$\mathcal{T}^a \otimes_{\mathcal{F}^a} \Lambda^k(\mathcal{E}^a) \rightarrow \mathcal{T} \otimes_{\mathcal{F}} \Lambda^k(\mathcal{E}).$$

Therefore, to every pair $(\gamma_0, \gamma_1) \in \mathrm{Hom}_{\mathcal{F}^a}(\mathcal{E}^a, \Lambda^{k+1}(\mathcal{E}^a)) \times \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}^a, \Lambda^k(\mathcal{E}^a))$ that satisfies conditions similar to (2), there corresponds a pair, see (1), satisfying (2), as is easy to verify.

This is the desired embedding $\mathrm{Der} \mathcal{O}^a \rightarrow \mathrm{Der} \mathcal{O}$. \square

4.2. Corollary. *There exists an injective homomorphism of \mathbb{Z} -graded Lie superalgebras of vector fields $\mathfrak{d}^a \rightarrow \mathfrak{d}$, where $\mathfrak{d}^a = \Gamma(M, \mathrm{Der} \mathcal{O}^a)$ and $\mathfrak{d} = \Gamma(M, \mathrm{Der} \mathcal{O})$.*

The results of [Se] imply that the homomorphism $\mathfrak{d}^a \rightarrow \mathfrak{d}$ is an isomorphism for any projective variety M .

5 W_n and DH_n as vectorial Lie superalgebras

Now we are able to determine the action of W_n and DH_n on the supermanifolds constructed in § 3. Retaining the notation of § 3 let us first prove the following statement.

5.1. Lemma. *Let $\gamma \in \mathrm{Der} B$ be such that $\gamma V \subset \Lambda(V)$, so that*

$$\gamma x_i = \sum_{1 \leq j \leq n} v_{ij} x_j, \quad \text{where } i = 1, \dots, n \text{ and } v_{ij} \in \Lambda(V).$$

Then, $\gamma(\hat{\mathcal{O}}_z^a) \subset \hat{\mathcal{O}}_z^a$ for all $z \in P(V)$.

Proof. Let $u = \varphi v$, where $\varphi \in \mathcal{F}_z^a$ and $v \in \Lambda(V)$. Then,

$$\gamma(u) = \gamma(\varphi)v + \varphi\gamma(v).$$

Therefore, it suffices to verify that $\gamma(\varphi) \in \hat{\mathcal{O}}_z^a$. Let us express φ in the form f/g , where $f, g \in \mathbb{C}[x_1, \dots, x_n]$ are homogeneous polynomials of the same degree and $g(z) \neq 0$. Our statement follows easily from the hypothesis on γ and the identity

$$\gamma(\varphi) = \frac{1}{g^2} (g\gamma(f) - f\gamma(g)).$$

Now, let $\delta \in W_n = W(V^*)$. As we have seen in § 2, δ can be uniquely extended to a derivation $\tilde{\delta}$ of $A = S(\overline{V}^*) \otimes \Lambda(V^*)$ such that $[\tilde{\delta}, d] = 0$. Obviously, the action of $\tilde{\delta}$ can be uniquely extended to the localization B of A , so (see Proof of Lemma 2.1)

$$\tilde{\delta}x_i = \tilde{\delta}d\xi_i = d\delta\xi_i = \pm \sum_{1 \leq j \leq n} x_j \frac{\partial h_i}{\partial \xi_j}, \quad \text{where } h_i = \delta\xi_i \in \Lambda(V).$$

By Lemma 5.1 $\tilde{\delta}$ transforms all the algebras $\hat{\mathcal{O}}_z^a$, where $z \in P(V)$, into themselves. Since $\tilde{\delta}$ transforms $\text{Ker } d$ into itself, then Lemma 3.1 implies that $\delta(\mathcal{O}_z^a) \subset \mathcal{O}_z^a$ for all $z \in P(V)$. Therefore, δ determines a global derivation of \mathcal{O}^a . We have constructed a map

$$W_n \longrightarrow \mathfrak{d}^a = \Gamma(P(V), \text{Der } \mathcal{O}^a).$$

As is easy to see, this map is an injective homomorphism of \mathbb{Z} -graded Lie superalgebras. By Lemma 4.1 we also have an injective homomorphism $W_n \longrightarrow \mathfrak{d} = \Gamma(P(V), \text{Der } \mathcal{O})$. \square

We similarly construct an injective homomorphism

$$DH_n \longrightarrow \mathfrak{d}' = \Gamma(Q, \text{Der } \mathcal{O}').$$

If $\delta \in DH_n$, then $\tilde{\delta} \in \text{Der } A$ transforms ωA into itself, and therefore determines a derivation of A' from § 3 which extends to B' and yields a derivation of \mathcal{O}'^a .

5.2. Theorem. *The above-constructed homomorphisms $W_n \longrightarrow \mathfrak{d}$ and $DH_n \longrightarrow \mathfrak{d}'$ are isomorphisms if $n \geq 2$ and $n \geq 5$, respectively.*

Proof. By the proved above we may assume that the finite-dimensional \mathbb{Z} -graded Lie superalgebras \mathfrak{d} and \mathfrak{d}' contain W_n and DH_n , respectively, as subalgebras. Therefore, it suffices to prove that \mathfrak{d} and \mathfrak{d}' are transitive and irreducible and $\mathfrak{d}_k = (W_n)_k$ and $\mathfrak{d}'_k = (DH_n)_k$ for $k = -1, 0$ (see [K, Theorem 4]). A proof of these statements is contained in [S1], [S2]. This proof essentially depends on Lemma 4.1.1 in [O] and actually reduces to the calculation of \mathfrak{d}_0 and \mathfrak{d}'_0 and also \mathfrak{d}_0 -module \mathfrak{d}_{-1} and \mathfrak{d}'_0 -module \mathfrak{d}'_{-1} with the help of Bott's theorem. \square

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