Communications in Mathematics **30** (2022), no. 3, 25–40
DOI: https://doi.org/10.46298/cm.10456

Transitive irreducible Lie superalgebras of vector fields

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Abstract. Let $\mathfrak{d}$ be the Lie superalgebra of superderivations of the sheaf of sections of the exterior algebra of the homogeneous vector bundle E over the flag variety G/P, where G is a simple finite-dimensional complex Lie group and P its parabolic subgroup. Then, $\mathfrak{d}$ is transitive and irreducible whenever E is defined by an irreducible P-module V such that the highest weight of V^* is dominant. Moreover, $\mathfrak{d}$ is simple; it is isomorphic to the Lie superalgebra of vector fields on the superpoint, i.e., on a $0|n$-dimensional supervariety.

Preface of the editor

The manuscript of this paper was deposited in VINITI 12.06.86 No 4329-B which is inaccessible. There still appear papers with references to some parts of this inaccessible deposition, see, e.g., [IO*], [OVs*] which contains references to this preprint and its continuation [On*] preprinted in the proceedings of the "Seminar on Supersymmetries", see `http://staff.math.su.se/mleites/sos.html`. So I decided to make it available, together with [On*], by translating these texts and updating the references; the ones I added are endowed with an asterisk. The abstract, footnotes, and comments are due to me.

For a comprehensive description of simple Lie superalgebras of vector fields over algebraically closed fields of any characteristic, see [BGLLS*]. *D. Leites*

1 Basics. Introduction

Recall that a filtered Lie superalgebra

$$\mathfrak{a} = \mathfrak{a}_{(-d)} \supset \cdots \supset \mathfrak{a}_{(-1)} \supset \mathfrak{a}_{(0)} \supset \mathfrak{a}_{(1)} \supset \dots, \quad \text{where } d \in \mathbb{N} := \{1, 2, \dots\}$$

is said to be *transitive*, see [Dr*], if

$$\mathfrak{a}_{(p+1)} = \{x \in \mathfrak{a}_{(p)} \mid [x, \mathfrak{a}] \subset \mathfrak{a}_{(p)}\} \text{ for all } p \geq 0. \tag{1}$$

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{b} = \oplus_{i \geq -d} \mathfrak{b}_i$ is said to be *transitive* if for all $p \geq 0$ we have

$$\{x \in \mathfrak{b}_p \mid [x, \mathfrak{b}_-] = 0\} = 0, \quad \text{where } \mathfrak{b}_- := \oplus_{p<0} \mathfrak{b}_p. \tag{2}$$

MSC 2020: Primary 32C11,17B20
Keywords: Lie superalgebra, homogeneous supermanifold.
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As is well-known, see [K*], [Sch], condition (1) is satisfied if and only if the graded Lie algebra $\mathfrak{b} := \operatorname{gr}\mathfrak{a}$ satisfies conditions (2).

We say that a graded Lie superalgebra $\mathfrak{b}$ is *irreducible* if the **adjoint** representation of $\mathfrak{b}_0$ in $\mathfrak{b}_{-1}$ is irreducible. (The term *irreducible* is usually used to denote the image of the (Lie) algebra or superalgebra $\mathfrak{b}$ in any irreducible representation. *D.L.*)

Simple, and even primitive[1], Lie (super)algebras have a filtration in which the associated graded Lie (super)algebra is both transitive and irreducible. For the classification of **simple** finite-dimensional Lie superalgebras over $\mathbb{C}$, see [K*], [Sch]. (Actually, no complete proof of the classification of simple finite-dimensional Lie superalgebras over $\mathbb{C}$ is known to this day; e.g., the proof of completeness of the list of examples of known deformations with odd parameters is not published. Such deformations were not even mentioned in [K*], [Sch]. *D.L.*)

Hereafter I assume that the $\mathbb{Z}$-grading of every Lie superalgebra considered is compatible with parity.

In this paper, I show that such Lie superalgebras naturally appear in the study of homogeneous vector bundles over complex homogeneous flag varieties G/P, where P is a parabolic subgroup of a complex semisimple Lie group G. Namely, to each homogeneous vector bundle $E \longrightarrow M$ there is associated a *split* complex supermanifold $(M, \Lambda(\mathcal{E}))$, where $\mathcal{E}$ is the sheaf of holomorphic sections of E, and $\Lambda(\mathcal{E})$ is the Grassmann algebra of $\mathcal{E}$.

Here, I prove that the graded Lie superalgebra $\mathfrak{d} = \Gamma(M, \operatorname{Der}\Lambda(\mathcal{E}))$ of vector fields on this supermanifold is transitive and irreducible if E is defined by an irreducible representation φ of P satisfying certain natural requirements. The main point here is that the highest weight of the representation φ^* should be *dominant*. This guarantees the existence of sufficiently many holomorphic sections of "the odd tangent bundle" E^* over the supermanifold $(M, \Lambda(\mathcal{E}))$.

The Bott–Borel–Weil theorem, along with other standard methods, enables one to explicitly compute the Lie superalgebra $\mathfrak{d}$ in many cases.

In this paper, this computation is carried out in the cases where E is the cotangent bundle over M, and G is simple.

2 Vector fields on supermanifolds

The term "supermanifold" is meant in the sense of Berezin-Leites, in the complex–analytic situation, see [L], [MaG*], [Va*]. Let us recall the definition.

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[1]Let a Lie algebra $\mathcal{L}$ contain a subalgebra $\mathcal{L}_0$ which does not contain any nonzero ideal of $\mathcal{L}$. Let

$$\mathcal{L}_1 := \{t \in \mathcal{L}_0 \mid [t, \mathcal{L}] \subset \mathcal{L}_0\}.$$

Then, $\mathcal{L}$ is called *primitive* (see [O*]) if

$$\mathcal{L}_0 \text{ is a maximal subalgebra of } \mathcal{L} \text{ and } \mathcal{L}_1 \neq \{0\}. \tag{3}$$

Unlike primitive Lie algebras (both finite-dimensional and $\mathbb{Z}$-graded of polynomial growth) which do not differ much from the simple Lie algebras, the classification of primitive Lie **super**algebras over $\mathbb{C}$ announced by Kac, see [K*], is a wild problem, as shown in [ALSh*]. Compare with [CaKa*].

Let (n, m) be a pair of non-negative integers. Let a *model superspace* of dimension $n|m$ be the ringed space $(\mathbb{C}^n, \widetilde{\mathcal{O}})$, where $\widetilde{\mathcal{O}} := \Lambda_{\mathcal{O}}[\xi]$ for $\xi = (\xi_1, \dots, \xi_m)$, and $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. A topological space M equipped with a sheaf of supercommutative superalgebras $\widetilde{\mathcal{O}}$, which as a ringed space is locally isomorphic to a model superspace of dimension $n|m$ is said to be a(n almost) *complex supermanifold* and $\widetilde{\mathcal{O}}$ is its *structure sheaf*. (For details, e.g., which of almost complex supermanifolds are complex, and for phenomena indigenous **not only** to the "super" world — *real–complex (super)manifolds*, see [BGLS*]. *D.L.*)

Let us introduce *parity* in $\widetilde{\mathcal{O}}$ by declaring all of the ξ_i to be odd; so $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}_{\bar 0} \oplus \widetilde{\mathcal{O}}_{\bar 1}$ is the sum of its homogeneous components even and odd.

Let $\mathcal{I} \subset \widetilde{\mathcal{O}}$ be the subsheaf of ideals generated by subsheaf $\widetilde{\mathcal{O}}_{\bar 1}$ and let $\widetilde{\mathcal{O}}_{\mathrm{rd}} := \widetilde{\mathcal{O}}/\mathcal{I}$. Clearly, $(M, \widetilde{\mathcal{O}}_{\mathrm{rd}})$ is a(n almost) complex-analytic manifold.

The simplest supermanifolds are the *split* ones which are described as follows. Let $(M, \mathcal{O})$ be a(n almost) complex-analytic manifold of dimension n and $\mathcal{F}$ a locally free analytic sheaf of rank m on M. Set $\widetilde{\mathcal{O}} := \Lambda_{\mathcal{O}}(\mathcal{F})$. Clearly, $(M, \widetilde{\mathcal{O}})$ is a supermanifold, and $\widetilde{\mathcal{O}}_{\mathrm{rd}}$ is naturally isomorphic to the subsheaf $\mathcal{O} \subset \widetilde{\mathcal{O}}$. In what follows I will identify $\widetilde{\mathcal{O}}_{\mathrm{rd}}$ with $\mathcal{O}$ in this situation and will briefly write $\widetilde{\mathcal{O}} = \Lambda_{\mathcal{O}}(\mathcal{F})$. The structure sheaf of the split supermanifold is endowed with a $\mathbb{Z}$-grading

$$\widetilde{\mathcal{O}} = \bigoplus_{0 \leq p \leq m} \widetilde{\mathcal{O}}_p, \quad \text{where } \widetilde{\mathcal{O}}_p = \Lambda^p_{\mathcal{O}}(\mathcal{F}).$$

Observe that **every C^∞ of analytic supermanifold is locally split**.[2]

In general, using the subsheaf $\mathcal{I}^1 := \mathcal{I}$ we construct the following filtration of the structure sheaf:

$$\widetilde{\mathcal{O}} = \mathcal{I}^0 \supset \mathcal{I}^1 \supset \mathcal{I}^2 \supset \ldots \supset \mathcal{I}^m \supset \mathcal{I}^{m+1} = 0,$$

and the associated $\mathbb{Z}$-graded sheaf of superalgebras

$$\operatorname{gr} \widetilde{\mathcal{O}} = \bigoplus_{0 \leq p \leq m} \operatorname{gr}_p \widetilde{\mathcal{O}}, \quad \text{where } \operatorname{gr}_p \widetilde{\mathcal{O}} = \mathcal{I}^p/\mathcal{I}^{p+1}.$$

Here $\operatorname{gr}_0 \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}_{\mathrm{rd}}$, and $\operatorname{gr}_1 \widetilde{\mathcal{O}} = \mathcal{F}$ is a locally free sheaf of $\widetilde{\mathcal{O}}_{\mathrm{rd}}$-modules. Hence, it is clear that $\operatorname{gr} \widetilde{\mathcal{O}} = \Lambda_{\widetilde{\mathcal{O}}_{\mathrm{rd}}}(\mathcal{F})$.

Thus, to every supervariety $(M, \widetilde{\mathcal{O}})$ there is a corresponding split supervariety $(M, \operatorname{gr} \widetilde{\mathcal{O}})$.

Denote by $\operatorname{Der} \widetilde{\mathcal{O}}$ the sheaf of germs of derivations of the structure sheaf $\widetilde{\mathcal{O}}$. Its stalk at point $w \in M$ is the Lie superalgebra $\operatorname{Der}_{\mathbb{C}} \widetilde{\mathcal{O}}_w$. The sections of the sheaf $\operatorname{Der} \widetilde{\mathcal{O}}$ are called *vector fields* on the supermanifold $(M, \widetilde{\mathcal{O}})$. The space $\mathfrak{d} := \Gamma(M, \operatorname{Der} \widetilde{\mathcal{O}})$ is naturally endowed with a Lie superalgebra structure over $\mathbb{C}$.

[2]This is not true for algebraic supervarieties and supeschemes, as follows from [AD*]: over a contractible paracompact real set all vector bundles are trivial, but this is not necessarily true over other fields and non-affine schemes. The first example of non-split supervariety is due to P. Green ([Gre*]), see also [Pa*] submitted only 2 month later than [Gre*] and expounded in [B1*, Ch.4, §4, Sections 6–9] reproduced in [B2*, Ch.3, Th.2, p.126], as well as in [MaG*, Ch.4, §2, Prop. 9, p. 191].

If $(M,\widetilde{\mathcal{O}})$ is split, then $\mathrm{Der}\,\widetilde{\mathcal{O}}$ is a graded sheaf of Lie superalgebras whose homogeneous components are

$$\mathrm{Der}_p\,\widetilde{\mathcal{O}} = \{\delta \in \mathrm{Der}\,\widetilde{\mathcal{O}} \mid \delta\widetilde{\mathcal{O}}_q \subset \widetilde{\mathcal{O}}_{q+p} \quad \text{for all} \quad q \in \mathbb{Z}\}.$$

Therefore, $\mathfrak{d}$ is a graded Lie superalgebra. In the general case we endow $\mathrm{Der}\,\widetilde{\mathcal{O}}$ with the following filtration

$$\begin{aligned} &\mathrm{Der}_{(-1)}\,\widetilde{\mathcal{O}} = \mathrm{Der}\,\widetilde{\mathcal{O}},\\ &\mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}} = \{\delta \in \mathrm{Der}\,\widetilde{\mathcal{O}} \mid \delta\widetilde{\mathcal{O}} \subset \mathcal{I}^p, \ \ \delta\mathcal{I} \subset \mathcal{I}^{p+1}\} \quad \text{for all} \ \ p > 0. \end{aligned} \tag{4}$$

2.1. Lemma. 1) *The formulas* (4) *show that* $\mathrm{Der}\,\widetilde{\mathcal{O}}$ *can be naturally considered as a sheaf of filtered Lie superalgebras.*

2) *The sheaves* $\mathrm{gr}\,\mathrm{Der}\,\widetilde{\mathcal{O}}$ *and* $\mathrm{Der}\,\mathrm{gr}\,\widetilde{\mathcal{O}}$ *are naturally isomorphic.*

Proof. 1) It is easy to verify that

$$\delta\mathcal{I}^r \subset \mathcal{I}^{r+p} \quad \text{for any } \delta \in \mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}} \text{ and } r \in \mathbb{Z}. \tag{5}$$

This implies that $[\mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}}, \mathrm{Der}_{(q)}\,\widetilde{\mathcal{O}}] \subset \mathrm{Der}_{(p+q)}\,\widetilde{\mathcal{O}}$.

2) To prove this, observe that every element $\delta \in \mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}}$ determines, thanks to eq. (5), linear mappings

$$\tilde{\delta}_r : \mathcal{I}^r/\mathcal{I}^{r+1} \longrightarrow \mathcal{I}^{r+p}/\mathcal{I}^{r+p+1} \ \ \text{for all } r.$$

It is subject to a direct verification that the mappings $\tilde{\delta}_r$ form a derivation $\tilde{\delta} \in \mathrm{Der}_p\,\mathrm{gr}\,\widetilde{\mathcal{O}}$. The sheaf $\mathrm{Der}_{(p+1)}\,\widetilde{\mathcal{O}}$ is the kernel of the mapping $\delta \mapsto \tilde{\delta}$. Thus we obtain an injective sheaf homomorphism

$$\mathrm{gr}_p\,\mathrm{Der}\,\widetilde{\mathcal{O}} \longrightarrow \mathrm{Der}_p\,\mathrm{gr}\,\widetilde{\mathcal{O}}.$$

Using local splitness of supermanifolds, it is not difficult to show that this homomorphism is, moreover, surjective. Finally, a direct verification shows that this is an isomorphism of sheaves of graded Lie superalgebras. □

2.2. Corollary. 1) *The Lie superalgebra* $\mathfrak{d} := \Gamma(M, \mathrm{Der}\,\widetilde{\mathcal{O}})$ *is filtered:*

$$\mathfrak{d} = \mathfrak{d}_{(-1)} \supset \mathfrak{d}_{(0)} \supset \mathfrak{d}_{(1)} \supset \dots, \quad \textit{where} \ \ \mathfrak{d}_{(p)} = \Gamma(M, \mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}}).$$

2) *The following injective homomorphism of graded Lie superalgebras is well-defined:*

$$\mathrm{gr}\,\mathfrak{d} \longrightarrow \tilde{\mathfrak{d}} := \Gamma(M, \mathrm{Der}\,\mathrm{gr}\,\widetilde{\mathcal{O}}).$$

If M *is compact, then* $\dim_{\mathbb{C}} \mathfrak{d} < \infty$.

Proof. 1) The exact sequence

$$0 \longrightarrow \mathrm{Der}_{(p+1)}\,\widetilde{\mathcal{O}} \longrightarrow \mathrm{Der}_{(p)}\,\widetilde{\mathcal{O}} \longrightarrow \mathrm{Der}_p\,\mathrm{gr}\,\widetilde{\mathcal{O}} \longrightarrow 0,$$

existing thanks to Lemma 2.1, implies existence of the exact sequence

$$0 \longrightarrow \mathfrak{d}_{(p+1)} \longrightarrow \mathfrak{d}_{(p)} \longrightarrow \Gamma(M, \mathrm{Der}_p\,\mathrm{gr}\,\widetilde{\mathcal{O}}). \tag{6}$$

2) The sequence (6) implies the existence of an injection

$$\mathrm{gr}\,\mathfrak{d} \longrightarrow \Gamma(M, \mathrm{Der}\,\mathrm{gr}\,\widetilde{\mathcal{O}}).$$ □

3 Split supermanifolds

Let $\widetilde{\mathcal{O}} = \Lambda(\mathcal{F})$, where $\mathcal{F}$ is a locally free analytic sheaf of rank m on an n-dimensional complex manifold $(M, \mathcal{O})$. Let $\mathcal{T}$ denote the tangent sheaf $\operatorname{Der}\mathcal{O}$ on this manifold.

3.1. Theorem. 1) *The sheaves* $\operatorname{Der}_p \widetilde{\mathcal{O}}$ *are locally free analytic. We have*

$$\operatorname{Der}_{-1} \widetilde{\mathcal{O}} \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \widetilde{\mathcal{O}}) = \mathcal{F}^*. \tag{7}$$

2) *For* $p \geq 0$, *there is an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}^* \otimes \Lambda^{p+1}(\mathcal{F}) \longrightarrow \operatorname{Der}_p \widetilde{\mathcal{O}} \xrightarrow{\alpha} \mathcal{T} \otimes \Lambda^p(\mathcal{F}) \longrightarrow 0. \tag{8}$$

Proof. 1) It suffices to consider the case of the model supermanifold of dimension $n|m$. Clearly, in this case $\operatorname{Der}\widetilde{\mathcal{O}}$ is a free sheaf of $\widetilde{\mathcal{O}}$-modules, a basis of its sections consisting of

$$\partial_{x_i} \quad (i = 1, \ldots, n), \qquad \partial_{\xi_j} \quad (j = 1, \ldots, m),$$

where $x_1, \ldots, x_n$ are coordinates in $\mathbb{C}^n$, and $\{\xi_1, \ldots, \xi_m\}$ is any local basis of sections of the sheaf $\mathcal{F}$. Therefore, $\operatorname{Der}_p \widetilde{\mathcal{O}}$ is a free sheaf of $\mathcal{O}$-modules with a basis of its sections being formed by

$$\begin{gathered} \xi_{i_1} \ldots \xi_{i_p} \partial_{x_i} \quad (i_1 < \ldots < i_p, \quad i = 1, \ldots, n) \\ \xi_{j_1} \ldots \xi_{j_{p+1}} \partial_{\xi_j} \quad (j_1 < \ldots < j_{p+1}, \quad j = 1, \ldots, m) \end{gathered}$$

Let $\delta \in \operatorname{Der}_p \widetilde{\mathcal{O}}$. Since the sheaf $\widetilde{\mathcal{O}}$ is generated by its subsheaves $\mathcal{O}$ and $\mathcal{F}$, the derivation δ is completely determined by its restrictions

$$\delta_0 := \alpha(\delta) = \delta|_{\mathcal{O}} \quad \text{and} \quad \delta_1 := \beta(\delta) = \delta|_{\mathcal{F}}.$$

Obviously,

$$\delta_0 \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \Lambda^p_{\mathcal{O}}(\mathcal{F})) \quad \text{and} \quad \delta_1 \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{F}, \Lambda^{p+1}_{\mathcal{O}}(\mathcal{F})) \tag{9}$$

and

$$\begin{aligned} &\delta_0(\varphi\psi) = (\delta_0\varphi)\psi + \varphi(\delta_0\psi) \\ &\delta_1(\varphi s) = (\delta_0\varphi)s + \varphi(\delta_1 s) \qquad \text{for any } \varphi, \psi \in \mathcal{O}, s \in \mathcal{F}. \end{aligned} \tag{10}$$

The other way around, for any pair (δ_0, δ_1), see eq. (9), which satisfies conditions (10), there exists a $\delta \in \operatorname{Der}_p \widetilde{\mathcal{O}}$ such that

$$\alpha(\delta) = \delta_0 \quad \text{and} \quad \beta(\delta) = \delta_1.$$

For $p = -1$, we have $\alpha(\delta) = 0$ and conditions (10) yields

$$\beta(\delta) \in \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) = \mathcal{F}^*.$$

This implies the existence of the isomorphism (7).

2) For $p \geq 0$, consider the sheaf homomorphism

$$\alpha : \operatorname{Der}_p \widetilde{\mathcal{O}} \longrightarrow \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}, \Lambda^p_{\mathcal{O}}(\mathcal{F})).$$

Obviously, $\operatorname{Ker} \alpha = (\operatorname{Der}_p \widetilde{\mathcal{O}}) \cap \operatorname{Der}_{\mathcal{O}} \widetilde{\mathcal{O}}$, and therefore β defines a sheaf isomorphism

$$\operatorname{Ker} \alpha \longrightarrow \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \Lambda^{p+1}_{\mathcal{O}}(\mathcal{F})) \cong \mathcal{F}^* \otimes \Lambda^{p+1}_{\mathcal{O}}(\mathcal{F}).$$

To compute $\operatorname{Im} \alpha$ we use an isomorphism

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \mathcal{G}) \cong (\operatorname{End}_{\mathbb{C}} \mathcal{O}) \otimes \mathcal{G} \tag{11}$$

for any locally free analytic sheaf $\mathcal{G}$. Let $\{g_1, \ldots, g_q\}$ be a basis of sections of $\mathcal{G}$ in a neighborhood of a point $w \in M$. If $h \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \mathcal{G})$, then $h(\varphi) = \sum\limits_{1 \leq i \leq q} \varphi_i g_i$, where $\varphi \in \mathcal{O}_w$ and $\varphi_i \in \mathcal{O}_w$ for all i. Setting $h_i(\varphi) = \varphi_i$ for all i, we see that $h_i \in (\operatorname{End}_{\mathbb{C}} \mathcal{O})_w$. The correspondence $h \mapsto \sum\limits_{1 \leq i \leq q} h_i \otimes g_i$ does not depend on the basis $(g_i)_{i=1}^q$ and is the desired isomorphism (11).

If $\mathcal{G} = \Lambda^p_{\mathcal{O}}(\mathcal{F})$, then, for any given basis $\{g_i\}_{i=1}^q$, we can take local sections $\xi_{i_1} \ldots \xi_{i_p}$ for $i_1 < \cdots < i_p$. To the element $h \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \mathcal{G})$ the isomorphism (11) assigns

$$\sum_{i_1 < \ldots < i_p} h_{i_1 \ldots i_p} \otimes \xi_{i_1} \ldots \xi_{i_p},$$

where $h_{i_1 \ldots i_p} \in \operatorname{End}_{\mathbb{C}} \mathcal{O}$ is defined by the formula

$$h(\varphi) = \sum_{i_1 < \ldots < i_p} h_{i_1 \ldots i_p}(\varphi) \xi_{i_1} \ldots \xi_{i_p} \quad \text{for any } \varphi \in \mathcal{O}. \tag{12}$$

Conditions (10) imply that if $h = \alpha(\delta)$, where $\delta \in \operatorname{Der}_p \mathcal{O}$, then $h_{i_1 \ldots i_p} \in \operatorname{Der} \widetilde{\mathcal{O}}$.

Thus, $\operatorname{Im} \alpha \subset \mathcal{T} \otimes \Lambda^p_{\mathcal{O}}(\mathcal{F})$, if we identify

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \Lambda^p_{\mathcal{O}}(\mathcal{F})) \simeq \operatorname{End}_{\mathbb{C}} \mathcal{O} \otimes \Lambda^p_{\mathcal{O}}(\mathcal{F})$$

by means of isomorphism (11).

Conversely, in formula (12), let $h \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{O}, \Lambda^p_{\mathcal{O}}(\mathcal{F}))$ and $h_{i_1 \ldots i_p} \in \mathcal{T}$. Then, in local coordinates $x_1, \ldots, x_n$ on M, we have

$$h_{i_1 \ldots i_p} = \sum_{1 \leq i \leq n} u^i_{i_1 \ldots i_p} \partial_{x_i}, \quad \text{where} \quad u^i_{i_1 \ldots i_p} \in \mathcal{O}.$$

Clearly,

$$\alpha(\delta) = h \quad \text{for} \quad \delta = \sum_{1 \leq i \leq n} \sum_{i_1 < \ldots < i_p} \xi_{i_1} \ldots \xi_{i_p} \partial_{x_i} \in \operatorname{Der}_p \mathcal{O}.$$

Observe that for $p = 0$ the sequence (8) is of the form

$$0 \longrightarrow \operatorname{End} \mathcal{F} \longrightarrow \operatorname{Der}_0 \mathcal{O} \xrightarrow{\alpha} \mathcal{T} \longrightarrow 0. \tag{13}$$

In particular, $\operatorname{End}\mathcal{F}$ is a sheaf of ideals in $\operatorname{Der}_0\mathcal{O}$. The sequence of sheaves leads to the exact sequence of Lie algebras

$$0 \longrightarrow \operatorname{End}F \longrightarrow \mathfrak{d}_0 \longrightarrow \Gamma(M,\mathcal{T}), \tag{14}$$

where $\operatorname{End}F = \Gamma(M,\operatorname{End}\mathcal{F})$ is the Lie algebra of endomorphisms of the vector bundle F.

Let $\varepsilon \in \mathfrak{d}_0$ be the element corresponding to the identity automorphism of F. Clearly, ε is a grading element of $\mathfrak{d}$, i.e., $[\varepsilon,\delta] = p\delta$ for any $\delta \in \mathfrak{d}_p$. □

3.1.1 Example. Consider the split supermanifold (M,Ω), where

$$\Omega = \bigoplus_{0\leq p\leq n} \Omega^p = \Lambda_{\mathcal{O}}(\mathcal{T}^*)$$

is the sheaf of holomorphic forms on $(M,\mathcal{O})$. If $\mathcal{F} = \mathcal{T}^*$, then the sheaves which appear in Theorem 3.1 coincide with $\Omega^p_{\mathcal{T}} = \mathcal{T}\otimes_{\mathcal{O}}\Omega^p$ — sheaves of vector-valued holomorphic p-forms, and we obtain the exact sequence which introduces maps i_p and α_p

$$0 \longrightarrow \Omega^{p+1}_{\mathcal{T}} \xrightarrow{i_p} \operatorname{Der}_p\Omega \xrightarrow{\alpha_p} \Omega^p_{\mathcal{T}} \longrightarrow 0. \tag{15}$$

As was proved in [Fr] (in the C^∞ case, but this is inessential), this sequence splits. Therefore, we obtain the following theorem which I prove for completeness.

3.2. Theorem. *For any $p \geqslant -1$, we have*

$$\operatorname{Der}_p\Omega \cong \Omega^p_{\mathcal{T}} \oplus \Omega^{p+1}_{\mathcal{T}}.$$

Proof. Let $\omega \in \Omega^p_{\mathcal{T}}$. Consider $\delta_\omega := i_{p-1}(\omega) \in \operatorname{Ker}\alpha_{p-1}$, and set $\delta := [\delta_\omega, d]$, where $d:\Omega \longrightarrow \Omega$ is the exterior differential, a section of the sheaf $\operatorname{Der}_1\Omega$. Then, $\alpha_p(\delta) = \omega$.

Indeed, let $\omega = \sum\limits_i \delta_i\omega_i$, where $\delta_i \in \mathcal{T}$ and $\omega_i \in \Omega^p$. Then, for any $\varphi \in \mathcal{O}$, we have

$$\delta\varphi = \delta_\omega(d\varphi) = \sum_i (d\varphi)(\delta_i)\otimes\omega_i = \sum_i \delta_i(\varphi)\otimes\omega_i = \omega(\varphi).$$

Therefore, the mapping $\omega \mapsto \delta = [\delta_\omega, d]$ splits the sequence (15). It is easy to see that the subsheaf of $\operatorname{Der}_p\Omega$ defined by the splitting, which is complementary to $\Omega^{p+1}_{\mathcal{T}}$, coincides with the centralizer of d. (The first description of the centralizer of d in terms of Lie superalgebras and invariant differential operators is due to Grozman; for details, see the arXiv version of [Gro*] reproduced in this Special Volume *D.L.*). □

3.3. Corollary. *There exist isomorphisms*

$$\mathfrak{d} = \Gamma(M,\operatorname{Der}_p\Omega) \cong \Gamma(M,\Omega^p_{\mathcal{T}}) \oplus \Gamma(M,\Omega^{p+1}_{\mathcal{T}}).$$

In particular, $\mathfrak{d}_{-1} \cong \Gamma(M,\mathcal{T})$.

The latter isomorphism is described as follows: every vector field $v \in \Gamma(M, \mathcal{T})$ determines a derivation $i_v \in \mathfrak{d}_{-1}$, called *inner derivation by* (or a *convolution with*) v. Further, we have a decomposition into a semidirect sum of Lie algebras

$$\mathfrak{d}_0 \cong \Gamma(M, \mathcal{T}) \oplus \operatorname{End} T,$$

where $\operatorname{End} T$ is an ideal, and the Lie subalgebra $\Gamma(M, \mathcal{T})$ is embedded into $\mathfrak{d}_0$ by means of injective homomorphism $v \mapsto \theta_v$, where θ_v is the *Lie derivative along the field* v. Finally,

$$\mathfrak{d}_1 \cong \operatorname{End} T \oplus \Gamma(M, \Omega^2_{\mathcal{T}}).$$

Under this isomorphism the identity automorphism $\varepsilon \in \operatorname{End} T$ corresponds to the exterior differential $d \in \mathfrak{d}_1$.

Consider the graded subspace $\hat{\mathfrak{d}} = \hat{\mathfrak{d}}_{-1} \oplus \hat{\mathfrak{d}}_0 \oplus \hat{\mathfrak{d}}_1 \subset \mathfrak{d}$, where

$$\hat{\mathfrak{d}}_{-1} = i_{\Gamma(M,\mathcal{T})} = \mathfrak{d}_{-1}, \quad \hat{\mathfrak{d}}_0 = \theta_{\Gamma(M,\mathcal{T})} \oplus \mathbb{C}\varepsilon \subset \mathfrak{d}_0, \quad \hat{\mathfrak{d}}_1 = \mathbb{C}d \subset \mathfrak{d}_1.$$

The classical relations (here $p(i_v) = \bar{1}$, i.e., i_v is odd for all vector fields v)

$$\begin{gathered}[i_v, i_w] = 0, \quad [\theta_v, \theta_w] = \theta_{[v,w]}, \quad [\theta_v, i_w] = i_{[v,w]}, \\ [d, i_v] = \theta_v, \quad [d, \theta_v] = 0 \quad \text{for any } v, w \in \Gamma(M, \mathcal{T}), \\ [d, d] = 0, \quad [\varepsilon, \delta] = p\delta \quad \text{for any } \delta \in \mathfrak{d}_p\end{gathered}$$

immediately imply that $\hat{\mathfrak{d}}$ is a Lie subsuperalgebra in $\mathfrak{d}$.

4 Transitive Lie superalgebras

Let us deduce a sufficient condition for the Lie superalgebra of vector fields $\Gamma(M, \operatorname{Der} \widetilde{\mathcal{O}})$ on the supermanifold $(M, \widetilde{\mathcal{O}})$ to be transitive. As before let $\mathcal{J}$ be the subsheaf of ideals generated by odd elements. The locally free sheaf $\mathcal{F}^* = (\mathcal{J}/\mathcal{J}^2)^*$ on the complex manifold $(M, \widetilde{\mathcal{O}}_{\mathrm{rd}})$ will be called *odd tangent sheaf* and denoted $\mathcal{T}_{\bar{1}}$. Let $T_{\bar{1}}$ be the corresponding holomorphic vector bundle over $(M, \widetilde{\mathcal{O}}_{\mathrm{rd}})$ called the *odd tangent bundle*.

4.1 Split case, i.e., $\widetilde{\mathcal{O}} = \Lambda(\mathcal{F})$. By Theorem 3.1, $\mathfrak{d}_{-1} \cong \Gamma(M, \mathcal{T}_{\bar{1}})$.

4.1. Lemma. *Let $(M, \widetilde{\mathcal{O}})$ be a split supermanifold. Suppose a global holomorphic section passes through every point of any fiber of the bundle $T_{\bar{1}}$.*

Then, the graded Lie superalgebra $\mathfrak{d} = \Gamma(M, \operatorname{Der} \widetilde{\mathcal{O}})$ satisfies condition (2) for all $p > 0$.

If this condition holds for $p = 0$ as well, i.e., if the adjoint representation of $\mathfrak{d}_0$ in $\mathfrak{d}_{-1}$ is exact, then $\mathfrak{d}$ is transitive.

Proof. Let $\delta \in \mathfrak{d}_p$ for $p > 0$ and $\gamma \in \mathfrak{d}_{-1}$. Then,

$$\begin{gathered}[\gamma, \delta](\varphi) = \gamma(\delta\varphi) \quad \text{for any } \varphi \in \widetilde{\mathcal{O}}, \qquad (16) \\ [\gamma, \delta](s) = \gamma(\delta s) + (-1)^{p+1}\delta(\gamma s) \quad \text{for any } s \in \mathcal{F}.\end{gathered}$$

□

If $[\gamma, \delta] = 0$ for all $\gamma \in \mathfrak{d}_{-1}$, then $\gamma(\delta\varphi) = 0$ for all $\varphi \in \mathcal{O}$. This follows from (16). Since $\delta\varphi \in \Lambda^p(\mathcal{F})$, then under Lemma's hypothesis for $p > 0$, we have $\delta\varphi = 0$ for all $\varphi \in \mathcal{O}$. But then Eq. (□) implies that $\gamma(\delta s) = 0$ for all $\gamma \in \mathfrak{d}_{-1}$ and $s \in \mathcal{F}$, so $\delta s = 0$ for all $s \in \mathcal{F}$; i.e., $\delta = 0$.

Observe that, in general, the Lie superalgebra $\mathfrak{d}$ does not satisfy condition (2) for $p = 0$. Under assumptions of the Lemma we can only claim that the action of the ideal $\operatorname{End} F = \operatorname{End} T_{\bar{1}}$, see (14), on $\mathfrak{d}_{-1} = \Gamma(M, \mathcal{T}_{\bar{1}})$ is exact.

4.2 General case. Let $\mathfrak{d} = \Gamma(M, \widetilde{\mathcal{O}})$. Consider the graded Lie superalgebra

$$\tilde{\mathfrak{d}} := \Gamma(M, \operatorname{Der} \operatorname{gr} \widetilde{\mathcal{O}}) \simeq \Gamma(M, \operatorname{gr} \operatorname{Der} \widetilde{\mathcal{O}}).$$

By item 2) of Corollary 2.2, there is an injective homomorpshism $\operatorname{gr} \mathfrak{d} \longrightarrow \tilde{\mathfrak{d}}$.

In particular, $\operatorname{gr}_{-1} \mathfrak{d} = \mathfrak{d}/\mathfrak{d}_{(0)}$ is identified with $\tilde{\mathfrak{d}}_{-1} = \Gamma(M, \mathcal{T}_{\bar{1}})$.

4.2. Lemma. *For the Lie superalgebra $\mathfrak{d} := \Gamma(M, \operatorname{Der} \widetilde{\mathcal{O}})$ to be transitive the following two conditions are sufficient:*

a) *Given any point of any fiber of the bundle $T_{\bar{1}}$, there is a section of the subspace*

$$\mathfrak{d}_{-1} = \mathfrak{d}/\mathfrak{d}_{(0)} \subset \Gamma(M, \mathcal{T}_{\bar{1}})$$

whose image contains it.

b) *The adjoint action of the Lie algebra $\mathfrak{d}_0 := \mathfrak{d}_{(0)}/\mathfrak{d}_{(1)}$ in $\mathfrak{d}_{-1}$ is exact.*

Proof. By repeating almost verbatim the proof of Lemma 4.1 we see that the subalgebra $\operatorname{gr} \mathfrak{d}$ of the Lie superalgebra $\tilde{\mathfrak{d}}$ satisfies conditions (2) for any $p > 0$. The case $p = 0$ is handled by condition of item b). □

5 Homogeneous vector bundles

Let $p : E \longrightarrow M$ be a holomorphic vector bundle over a complex manifold $(M, \mathcal{O})$. A fiber-wise linear biholomorphic mapping $E \longrightarrow E$ is said to be an *automorphism* of the bundle E. Let $A(E)$ be the group of all automorphisms of E. Obviously, every automorphism a of the bundle E determines an automorphism $p(a)$ of the base M of E. We obtain an exact sequence of groups

$$e \longrightarrow \operatorname{Aut} E \longrightarrow A(E) \xrightarrow{p} \operatorname{Aut} M, \tag{17}$$

where $\operatorname{Aut} E \subset A(E)$ is the normal subgroup consisting of the automorphisms sending every fiber into itself. If M is compact, then the sequence (17) consists of complex Lie groups and their homomorphisms, see [Mo].

Observe that it is possible to describe the automorphisms of the bundle E in terms of the corresponding sheaf $\mathcal{E}$. Namely, every $a \in A(E)$ determines an automorphism $\tilde{a}$ of the sheaf $\mathcal{E}$ over the automorphism $p(a)$ of the base M, i.e., determines an isomorphism of sheaves $\tilde{a} : \mathcal{E} \longrightarrow p(a)^*\mathcal{E}$. The latter isomorphism is given on local sections s by the formula

$$\tilde{a}(s)(w) = a^{-1}(s(p(a)(w))) \text{ for any } w \in M. \tag{18}$$

Consider the sheaf $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$ corresponding to the dual bundle E^*. The space of sections $\Gamma(U, \mathcal{E}^*)$ over any open set $U \subset M$ can be identified with the subspace in $\Gamma(p^{-1}(U), \mathcal{O}_E)$ consisting of the functions linear on fibers. This gives an embedding $\mathcal{E}^* \subset p_* \mathcal{O}_E$.

A vector field on E with a projection to M and sending $\mathcal{E}^*$ into itself will be called an *infinitesimal automorphism of the bundle E*. The infinitesimal automorphisms determine a sheaf of complex Lie algebras $\mathcal{A}(E)$ on M. Projection to the base yields a sheaf homomorphism $\pi : \mathcal{A}(E) \longrightarrow \mathcal{T}$, where $\mathcal{T}$ is the tangent sheaf on M. If M is compact, then the Lie algebra $\mathfrak{a}(E) = \Gamma(M, \mathcal{A}(E))$ is tangent to the Lie group $A(E)$, and the homomorphism $\pi : \mathfrak{a}(E) \longrightarrow \Gamma(M, \mathcal{T})$ coincides with dp.

5.1. Lemma. *The sheaf $\mathcal{A}(E)$ is naturally isomorphic to the sheaf $\mathrm{Der}_0 \widetilde{\mathcal{O}}$ of degree 0 derivations of $\widetilde{\mathcal{O}} := \Lambda(\mathcal{E}^*)$. Under this isomorphism π turns into the homomorphism α from sequence (13).*

Proof. Every $\delta \in \mathcal{A}(E)$ determines a pair

$$\delta_0 = \pi(\delta) \in \mathcal{T} \text{ and } \delta_1 = \delta|_{\mathcal{E}^*} \in \mathrm{End}_{\mathbb{C}} \mathcal{E}^*,$$

so that conditions (10) are satisfied. Let $\hat{\delta} \in \mathrm{Der}_0 \widetilde{\mathcal{O}}$ be the derivation corresponding to the pair (δ_0, δ_1), see proof of Theorem 3.1. It is easy to see that the mapping $\delta \mapsto \hat{\delta}$ is an injective homomorphism of sheaves of Lie algebras.

To prove surjectivity of the mapping $\delta \mapsto \hat{\delta}$, consider $\hat{\delta} \in (\mathrm{Der}_0 \widetilde{\mathcal{O}})_w$ determined by a pair (δ_0, δ_1) satisfying conditions (10). Let $U \subset M$ be an open neighborhood of any point $w \in M$ over which E admits trivialization $p^{-1}(M) = U \times \mathbb{C}^m$. Set $\delta\varphi = \delta_0\varphi$, where $\varphi \in \mathcal{O}|_U$, and $\delta\ell = \delta_1\ell$, where $\ell \in (\mathbb{C}^m)^*$.These conditions completely determine an element $\delta \in \mathcal{A}(E)_w$ which turns into $\hat{\delta}$ under this mapping $\widehat{\ }$. □

Now let $(M, \mathcal{O})$ be a homogeneous space of a connected complex Lie group G, i.e., let there be given a transitive analytic G-action on M. Fix a point $w_0 \in M$, and let P be the stationary subgroup of w_0 in G.

As is well-known, M can be naturally identified with the manifold of the left cosets G/P. Let M be compact.

Recall that a holomorphic vector bundle $p : E \to M$ is *homogeneous* (under a G-action) if there is an analytic G-action by automorphisms of the bundle E whose projection is a given G-action on M. Equivalently, there exists an analytic homomorphism $\hat{t} : G \to A(E)$ such that $p \circ \hat{t} = t$ is a homomorphism $G \to \mathrm{Aut}\ M$ which determines the given G-action on M.

Associated to a homogeneous bundle E there is a linear representation $\varphi : g \mapsto \hat{t}(g)|_{E_{w_0}}$ of P on the space E_{w_0}. This representation completely determines the bundle E together with the G-action on E. For the role of φ one can take any finite-dimensional linear analytic representation of P. We write E_φ to denote the homogeneous vector bundle over M corresponding to the representation φ. The corresponding locally free sheaves $\mathcal{E}_\varphi$ are also called *homogeneous*.

The G-action on E_φ determines an analytic linear representation $\Phi: G \to GL(\Gamma(M, \mathcal{E}_\varphi))$ given by the formula

$$(\Phi(g)s)(w) = gs(g^{-1}w) \quad \text{for any } g \in G, \quad w \in M.$$

The representation Φ of G is called *induced* by the representation φ of P.

5.2. Example. The tangent bundle T over M is endowed with a natural G-action, and therefore is homogeneous. It corresponds to the linear representation $\tau : P \to GL(T_{w_0}(M))$ given by the formula

$$\tau(g) = dt(g)_{w_0} \quad \text{for any } g \in P.$$

In what follows we consider the case where M is the homogeneous flag variety, i.e., where G is a connected semisimple (or reductive) complex Lie group and P is its parabolic subgroup. In this case, the space of sections of the homogeneous vector bundle and the induced representation are described by the famous Bott–Borel–Weil theorem, see [B]. Let us recall it.

Let T be a *maximal torus* in a *Borel subgroup* B of G and let $\mathfrak{t}$ and $\mathfrak{b}$ be the respective Lie algebras. Let R be the *root system* with respect to T (or $\mathfrak{t}$); let R^+ be the *set of positive roots* corresponding to $\mathfrak{b}$ and Π the *set of simple roots* and let

$$\left\{h_\alpha := \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in R\right\} \subset \mathfrak{t}$$

be the set of *coroots*, i.e., the system of roots dual to R. The weight $\lambda \in \mathcal{I}^*$ is called *dominant* if $\lambda(h_\alpha) \geqslant 0$ for all $\alpha \in \Pi$.

5.3 Theorem (Bott–Borel–Weil). *Let P be a parabolic subgroup of G containing the opposite to B Borel group B^-, and let $M = G/P$. Let φ be an irreducible analytic representation of P with highest weight λ.*

Then, $\Gamma(M, \mathcal{E}_\varphi) \neq 0$ if and only if λ is dominant. In this case, the induced representation Φ in $\Gamma(M, \mathcal{E}_\varphi)$ is irreducible with highest weight λ.

This theorem is applicable only if φ is irreducible or completely reducible. However, one often encounters homogeneous bundles arising from representations which are not completely reducible. The following Lemma is useful for studying them.

5.4. Lemma. *Let $M = G/P$ be a homogeneous flag variety, φ a holomorphic linear representation of P. If the induced representation Φ of G contains an irreducible representation with highest weight λ of multiplicity t, then λ is contained in the set of weights of φ of multiplicity $\geqslant t$.*

Proof. Let us perform induction by the length $\ell(\varphi)$ of the Jordan-Hoelder series of φ. If $\ell(\varphi) = 1$, then λ is the highest weight of φ by Theorem 5.3. Let the Lemma be proved for representations φ' such that $\ell(\varphi') < \ell(\varphi)$. Assuming that φ is reducible, consider a proper subrepresentation φ_1 of φ and the quotient representation φ_2. We obtain an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E}_{\varphi_1} \longrightarrow \mathcal{E}_{\varphi} \longrightarrow \mathcal{E}_{\varphi_2} \longrightarrow 0,$$

which defines the exact sequence of G-modules

$$0 \longrightarrow \Gamma(M, \mathcal{E}_{\varphi_1}) \longrightarrow \Gamma(M, \mathcal{E}_{\varphi}) \longrightarrow \Gamma(M, \mathcal{E}_{\varphi_2}).$$

Since every finite-dimensional analytic representation of G is completely reducible, the irreducible representation with highest weight λ is contained in $\Gamma(M, \mathcal{E}_{\varphi_1})$ and $\Gamma(M, \mathcal{E}_{\varphi_2})$ with total multiplicity $\geqslant t$. By induction hypothesis the statement of the Lemma follows because the set of weights of φ is the union of the sets of weights of φ_1 and φ_2. □

5.5. Corollary. *If φ has no dominant weights, then the induced representation Φ in $\Gamma(M, \mathcal{E}_{\varphi})$ is trivial. Moreover, $\Gamma(M, \mathcal{E}_{\varphi}) = 0$.*

6 Split homogeneous supervarieties

In this section we consider a particular class of split supervarieties associated with homogeneous flag varieties. Let G be a connected semisimple (or reductive) complex Lie group, P a complex Lie subgroup, and E_{φ} a homogeneous vector bundle over $M = G/P$ determined by a representation φ of P. Consider a split supermanifold $(M, \widetilde{\mathcal{O}})$, where $\widetilde{\mathcal{O}} = \Lambda(\mathcal{E}_{\varphi})$. The sheaf $\widetilde{\mathcal{O}}$ corresponds to the homogeneous vector bundle $\Lambda(E_{\varphi}) = E_{\Lambda(\varphi)}$, i.e., $\widetilde{\mathcal{O}} = \mathcal{E}_{\Lambda(\varphi)}$.

The sheaf $\operatorname{Der} \widetilde{\mathcal{O}}$ is also homogeneous. Indeed, every element $g \in G$ corresponds to an automorphism $\widetilde{\hat{t}(g)}$ of $\widetilde{\mathcal{O}}$ over an automorphism $t(g) \in \operatorname{Aut} M$, see eq. (18). By setting

$$g\delta = \widetilde{\hat{t}(g)} \circ \delta \circ \widetilde{\hat{t}(g)}^{-1} \quad \text{for any } \delta \in \operatorname{Der} \widetilde{\mathcal{O}} \tag{19}$$

we obtain the desired G-action on the locally free sheaf $\operatorname{Der} \widetilde{\mathcal{O}}$. Also notice that the sheaves in the exact sequence (8) correspond to the homogeneous vector bundles $E_{\varphi}^* \otimes \Lambda^{p+1}(E_{\varphi})$ and $T \otimes \Lambda^p(E_{\varphi})$, while the homomorphisms in this sequence are G-equivariant.

By Lemma 5.1 we can identify the Lie algebra $\mathfrak{d}_0 = \Gamma(M, \operatorname{Der}_0 \widetilde{\mathcal{O}})$ with $\mathfrak{a}(E_{\varphi}^*)$. Since the bundle E_{φ}^* is also homogeneous, we have a Lie algebra homomorphism $d\hat{t} : \mathfrak{g} \longrightarrow \mathfrak{d}_0$. On the other hand, the G-action (19) on $\widetilde{\mathcal{O}}$ determines a representation $\Psi : G \longrightarrow GL(\mathfrak{d})$.

6.1. Lemma. *Let $u \in \mathfrak{g}$ and $\psi = d\Psi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{d})$. Then, $[d\hat{t}(u), \delta] = \psi(u)\delta$.*

Proof. This follows directly from (19): instead of g, substitute in eq. (19) the curve $g(t) \in G$ with tangent vector u at $t = 0$ passing through $e := g(0)$, and differentiate both parts with respect to t at $t = 0$. □

A split supermanifold $(M, \Lambda(\mathcal{E}_{\varphi}))$ will be called *homogeneous* if $T_{\bar{1}} = E_{\varphi}^* = E_{\varphi^*}$ has "sufficiently many" holomorphic sections. This means that every point of every fiber of this bundle is contained in the image of some section.

In what follows G is semisimple and P is its parabolic subgroup.

6.2. Lemma. *Let φ be irreducible and λ the highest weight of representation φ^*. The following properties are equivalent:*

$$\begin{array}{ll} i) & (M, \Lambda(\mathcal{E}_\varphi)) \ \ \textit{is homogeneous}; \\ ii) & \lambda \ \ \textit{is dominant}; \\ iii) & \mathfrak{d}_{-1} = \Gamma(M, \mathcal{E}_{\varphi^*}) \neq 0. \end{array}$$

Under these conditions the induced representation Ψ_{-1} of G in $\mathfrak{d}_{-1}$ is irreducible.

Proof. By Theorem 5.3, it remains to prove that if λ is dominant, then $T_{\bar{1}}$ has sufficiently many holomorphic sections. Obviously, the restriction map $r_{w_0} : \Gamma(M, \mathcal{E}_{\varphi^*}) \to (E_{\varphi^*})_{w_0}$ is a P-module homomorphism. Therefore, either r_{w_0} is surjective or $r_{w_0} = 0$. But for any $g \in G$ and $s \in \Gamma(M, \mathcal{E}_{\varphi^*})$, we have

$$r_{gw_0}(s) = g r_{w_0}(\Phi(g^{-1})s), \tag{20}$$

where Φ is the representation induced by φ^*. Therefore, $rw_0 = 0$ implies $r_w = 0$ for all $w \in M$, which is impossible. Thus r_w is surjective for all $w \in M$. □

Consider the following commutative diagram whose upper line is the exact sequence (14) for the bundle $F = E^*_\varphi$:

$$\begin{array}{ccccccc} 0 \longrightarrow \operatorname{End} E_\varphi \longrightarrow & \mathfrak{d} & \xrightarrow{\alpha} & \Gamma(M, \mathcal{T}) \\ & {\scriptstyle d\hat{t}} \nwarrow & & \nearrow {\scriptstyle dt} \\ & & \mathfrak{g} & \end{array} \tag{21}$$

6.3. Theorem. *Let $M = G/P$ be a flag variety of a semisimple Lie group G. Let φ be an irreducible finite-dimensional analytic linear representation of P; let $\widetilde{\mathcal{O}} = \Lambda(\mathcal{E}_\varphi)$ and $\alpha(\mathfrak{d}_0) = dt(\mathfrak{g})$, where α is defined in eq. (21).*

Let $G = G_1 \dots G_r$ be a decomposition of G into simple factors, and let $\Pi = \Pi_1 \sqcup \dots \sqcup \Pi_r$ be the corresponding decomposition of the system of simple roots. Let the highest weight λ of the representation φ^* be dominant, and for every i assume that there exists a $\beta \in \Pi_i$ such that $\lambda(h_\beta) > 0$.

Then, the graded Lie superalgebra $\mathfrak{d} = \Gamma(M, \operatorname{Der} \mathcal{O})$ is transitive and irreducible.

Proof. By Lemma 6.2 and since λ is dominant, the condition of Lemma 4.1 is satisfied. Therefore, it remains to show that the adjoint representation of $\mathfrak{d}_0$ in $\mathfrak{d}_{-1} = \Gamma(M, \mathcal{E}_{\varphi^*})$ is irreducible and exact.

The irreducibility easily follows from Lemmas 6.1 and 6.2.

Observe that the homomorphism $d\hat{t}$ is injective. Indeed, otherwise $\mathfrak{g}_i \subset \operatorname{Ker} d\hat{t}$ for some i. Hence, $dt(\mathfrak{g}_i) = 0$ which implies that $G_i \subset P$. Furthermore, $\mathfrak{g}_i \subset \operatorname{Ker} \psi$ by Lemma 6.1.

Since $r_{w_0} : \Gamma(M, \mathcal{E}_{\varphi^*}) \to (E_{\varphi^*})_{w_0}$ is surjective (Lemma 6.2), it follows that $G_i \subset \operatorname{Ker} \varphi^*$, which contradicts the hypothesis. □

Let us identify $\mathfrak{g}$ with $d\hat{t}(\mathfrak{g})$ by means of $d\hat{t}$. Form eq. (21) we see that $dt = \alpha|_{\mathfrak{g}}$ and $\alpha(\mathfrak{d}_0) = \alpha(\mathfrak{g})$.

Consider the ideal $\mathfrak{g}_0 = \mathfrak{g} \cap \operatorname{End} E_\varphi$ of $\mathfrak{g}$. We see that $\mathfrak{g} = \mathfrak{g}_0 \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is also an ideal. Obviously, our assumptions imply that there is a decomposition into a semi-direct sum

$$\mathfrak{d}_0 = \operatorname{End} E_\varphi \ltimes \tilde{\mathfrak{g}},$$

where $\operatorname{End} E_\varphi$ is an ideal and $\tilde{\mathfrak{g}}$ a subalgebra. Let us show that this sum is actually direct.

Consider the induced representation ψ of the Lie algebra $\mathfrak{g}$ in $\operatorname{End} E_\varphi$. The weights of the induced representation $\varphi \otimes \varphi^*$ are of the form $\mu = \nu_1 - \nu_2$, where ν_1, ν_2 are weights of φ. Since φ is irreducible, μ can be expressed in terms of the set $\Pi_P \subset \Pi$ corresponding to the semisimple part of P. But the roots of any subset Π_i should enter the expression of any dominant weight in terms of Π with either positive or zero coefficients, see [H, §13, Exc. 8] and [R].

Since $(dt)|_{\tilde{\mathfrak{g}}}$ is injective, $G_i \not\subset P$ for all i such that $\mathfrak{g}_i \subset \tilde{\mathfrak{g}}$. Therefore, if μ is dominant, then $\mu(h_\beta) = 0$ for all $\beta \in \Pi_i$ and any such i.

Since $\varphi \otimes \varphi^*$ is completely reducible, we can apply Theorem 5.3 which implies that $\psi(\tilde{\mathfrak{g}}) = 0$, and $[\tilde{\mathfrak{g}}, \operatorname{End} E_\varphi] = 0$ thanks to Lemma 6.1.

The homogeneity of the supermanifold $(M, \widetilde{\mathcal{O}})$ proved above implies that the action of $\operatorname{End} E_\varphi = \operatorname{End} E_{\varphi^*}$ in $\mathfrak{d}_{-1}$ is exact. The radical $\mathfrak{r}$ of the Lie algebra $\mathfrak{d}_0$ is contained in $\operatorname{End} E_\varphi$ and is non-zero since $\mathbb{C}\varepsilon \subset \mathfrak{r}$, see § 2. Further, the Lie subalgebra $\operatorname{ad}_{\mathfrak{d}_0} \subset \mathfrak{gl}(\mathfrak{d}_{-1})$ is irreducible, and the dimension of its radical should be at most 1. Hence, $\mathfrak{r} = \mathbb{C}\varepsilon$.

Therefore, $\operatorname{End} E_\varphi = \mathbb{C}\varepsilon \oplus \mathfrak{h}$ and $\mathfrak{d}_0 = \operatorname{End} E_\varphi \oplus \tilde{\mathfrak{g}}$ are reductive Lie algebras with 1-dimensional centers. Any ideal of $\mathfrak{d}_0$ is contained either in $\operatorname{End} E_\varphi$ or in $\tilde{\mathfrak{g}}$. Thanks to our assumption on λ and Theorem 5.3, the action of the Lie algebra $\tilde{\mathfrak{g}}$ in $\mathfrak{d}_{-1}$ is exact, and so is the $\mathfrak{d}_0$-action.

Observe that the assumptions of Theorem 6.3 are satisfied, e.g., if G is simple and coincides with $\operatorname{Aut}^\circ M$, where $\circ$ singles out the connected component of the unit, and φ is a non-trivial irreducible representation such that the highest weight of φ^* is dominant.

On the other hand, the version of the theorem proved above, which does not presuppose that G-action on M is faithful, is useful in applications. As is well-known, the cases where the inclusions between Lie algebras $\alpha(\mathfrak{d}_0)$, $(dt)(\mathfrak{g})$ and $\Gamma(M, \mathcal{T})$ are strict do occur very seldom.

Observe that, if φ is irreducible and the G-action on M is locally faithful, then we have $\operatorname{End} E_\varphi = \mathbb{C}\varepsilon$, see [R].

In conclusion, let us compute the Lie superalgebras $\mathfrak{d}$ for the supermanifolds (M, Ω) of Examples 3.1.1 and 5.2, where M is a flag variety. Obviously, such a supermanifold is homogeneous for any compact complex homogeneous manifold M, although the representation φ that determines (M, Ω) is seldom completely reducible.

6.4. Theorem. *Let $M = G/P$ be a flag variety of the simple Lie group G. Then,* $\operatorname{End} T = \mathbb{C}\varepsilon$ *and $\Gamma(M, \Omega^p_{\mathcal{T}}) = 0$ for $p > 1$; the Lie superalgebra $\mathfrak{d} = \Gamma(M, \operatorname{Der} \Omega)$ coincides with its subsuperalgebra $\hat{\mathfrak{d}}$, see §2.*

Proof. The equality $\mathrm{End}\, T = \mathbb{C}\varepsilon$ is proved in [I]. It can also be deduced from Theorem 5.3 by using certain facts of Lie algebra theory.

Let $p > 1$. Then, the weights of the representation $\tau \otimes \Lambda^p(\tau^*)$ which determines the sheaf $\Omega^p_{\mathcal{T}}$ are of the form

$$\alpha - \beta_{i_1} - \ldots - \beta_{i_p}, \quad \text{where} \quad \alpha, \beta_{i_1}, \ldots, \beta_{i_p} \in \Pi^+$$

Obviously, such a weight can not be dominant. Applying item 1) of Corollary 2.2, we see that $\Gamma(M, \Omega^p_{\mathcal{T}}) = 0$.

The Lie superalgebra $\mathfrak{d}$ spoken about in Theorem 6.4 is isomorphic to $\mathfrak{vect}(0|m)$. $\square$

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[3]Now I can recommend the 2nd edition of this book and its extensions, see [L*], [L2*] and [B1*], respectively, and Bernstein's lectures in [Del*].