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On a sum of a multiplicative function linked to the divisor function over the set of integers B-multiple of 5

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Abstract. Let $d(n)$ and $d^*(n)$ be the numbers of divisors and the numbers of unitary divisors of the integer $n \geq 1$. In this paper, we prove that

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}}}\frac{d(n)}{d^*(n)} = \frac{16\pi^2}{123}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x + \mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right),\ \ (x \geqslant 1,\ \varepsilon > 0)\,,$$

where $\mathcal{B}$ is the set which contains any integer that is not a multiple of 5, but some permutations of its digits is a multiple of 5.

1 Introduction and main result

A positive integer is called $\mathcal{A}$-multiple of 5 if a permutation of its digits is a multiple of 5, comprising the identity permutation (for example 50, 55, 505, 5505, ...). A positive integer is called $\mathcal{B}$-multiple of 5 if it is not a multiple of 5, but some permutations of its digits is multiples of 5 (for example 51, 53, 107, 151, ...). For practical reasons, $\mathcal{A}$ represents the set of all integers $\mathcal{A}$-multiple of 5, and $\mathcal{B}$ represents the set of all integers $\mathcal{B}$-multiple of 5.

In this paper, we will use deep analytic methods to give an asymptotic formula to the following sum

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}}}\frac{d(n)}{d^*(n)}, \tag{1}$$

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where

$$d(n)=\sum_{d|n}1 \quad \text{and} \quad d^*(n)=\sum_{\substack{d|n\\(d,n|d)=1}}1,\ \ (n \text{ denotes a strictly positive integer}).$$

In order to estimate the sum (1) by noting that the function $\frac{d(n)}{d^*(n)}$ is multiplicative, we first recall by the following two concepts of the Riemann zeta function:

We have for all $s\in\mathbb{C}$, such that $\mathrm{Re}(s)>1$,

$$\zeta(s)=\prod_p(1-\frac{1}{p^s})^{-1},$$

and

$$\zeta(s)=1+\frac{1}{s-1}-s\int_1^\infty\frac{\{t\}}{t^s+1}dt,$$

where $\{t\}$ denotes the fractional part of the real t.

Recall that according to this last form, the function ζ extends to a meromorphic function on $Re(s)>0$, which has a simple pole at $s=1$ with residue 1 and no other poles. Moreover, if a is a strictly positive constant, we have, in the region of the plane defined by the inequalities $\sigma\geq\frac{1}{2}$, $\sigma\geq 1-a/\log|t|$, and $\sigma\leq 2$, the following majoration

$$\zeta(\sigma+it)\ll\mathcal{O}\left(\log|t|\right), \text{ for } |t| \text{ large enough (see [2}, p.\ 54-55]).$$

Secondly, we present the first effective formula of Perron (see [3, p. 147]): Let

$$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

be the Dirichlet series of finite absolute convergence abscissa σ_a. Then, if $x\geq 1$, $T\geq 1$ and $c>\max(0,\sigma_a)$, we have the following asymptotic formula

$$\sum_{n\leq x}a(n)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}f(s)\frac{x^s}{s}ds+\mathcal{O}\left(x^c\sum_{n\geq 1}\frac{|a(n)|}{n^c(1+T\left|\ln(x/n)\right|}\right).$$

In the following, we will present the main result that has been proven:

Theorem 1.1. *For any real $x\geq 1$, we have the following asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}}}\frac{d(n)}{d^*(n)}=\frac{16\pi^2}{123}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x+\mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right),$$

where

$$\frac{\pi^2}{6}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})\simeq 1.4276565\cdots,\ \textit{and } \varepsilon>0.$$

The proof of the theorem is based on the following lemmas:

Lemma 1.2. *Let q be a prime number or $q = 1$. So for any real number $x \geq 1$, we have the following asymptotic formula*

$$\sum_{n\leq x}\mathcal{D}(qn) = \frac{2q^2-q}{2q^2-2q+1}\frac{\pi^2}{6}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x+\mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

where $\mathcal{D}(n) = \dfrac{d(n)}{d^*(n)}$, *and ε denotes a positive real number.*

Proof. For a prime number q and a complex number s such that $\mathrm{Re}(s) > 1$, we put

$$f(s) = \sum_{n=1}^{\infty}\frac{\mathcal{D}(qn)}{n^s},$$

Then, by the product formula Eulerian [1, *p*.230], we get

$$\begin{aligned}
f(s) &= \sum_{\alpha=0}^{\infty}\sum_{\substack{n_1=1\\(n_1,q)=1}}^{\infty}\frac{\mathcal{D}(q^{\alpha+1}n_1)}{q^{\alpha s}n_1^s}\\
&= \sum_{\alpha=0}^{\infty}\frac{\alpha+2}{2q^{\alpha s}}\sum_{\substack{n_1=1\\(n_1,q)=1}}^{\infty}\frac{\mathcal{D}(n_1)}{n_1^s}\\
&= \frac{1}{2}\sum_{\alpha=0}^{\infty}\left(\frac{\alpha+1}{q^{\alpha s}}+\frac{1}{q^{\alpha s}}\right)\prod_{\substack{p\\(p,q)=1}}\left(1+\sum_{k=1}^{\infty}\frac{\mathcal{D}\left(p^k\right)}{p^{ks}}\right),
\end{aligned}$$

then

$$\begin{aligned}
f(s) &= \frac{1}{2}\left(\left(\sum_{\alpha=0}^{\infty}\frac{1}{q^{\alpha s}}\right)^2+\sum_{\alpha=0}^{\infty}\frac{1}{q^{\alpha s}}\right)\prod_p\left(1+\sum_{k=1}^{\infty}\frac{\mathcal{D}\left(p^k\right)}{p^{ks}}\right)\frac{1}{1+\sum_{k=1}^{\infty}\frac{\mathcal{D}\left(q^k\right)}{q^{ks}}}\\
&= \frac{1}{2}\left(\frac{1}{\left(1-\frac{1}{q^s}\right)^2}+\frac{1}{1-\frac{1}{q^s}}\right)\zeta(s)\zeta(2s)\prod_p\left(1-\frac{1}{2p^{2s}}+\frac{1}{2p^{3s}}\right)\frac{2\left(q^s-1\right)^2}{2q^{2s}-2q^s+1}\\
&= \left(\frac{2q^{2s}-q^s}{2q^{2s}-2q^s+1}\right)\zeta(s)\zeta(2s)\prod_p\left(1-\frac{1}{2p^{2s}}+\frac{1}{2p^{3s}}\right).
\end{aligned}$$

We notice that the function $f(s)$, is convergent if $\mathrm{Re}(s) > \frac{1}{2}$, where we recall here that $\zeta(s) = 1+\dfrac{1}{s-1}-s\displaystyle\int_1^{\infty}\frac{\{t\}}{t^s+1}dt$. According to Perron's formula, for all $x \geq 1$ and

$T \geq 1,$ we getting

$$\sum_{n \leq x} \mathcal{D}(qn) = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} f(s) \frac{x^s}{s} ds + \mathcal{O}(\frac{x^{\frac{3}{2}+\varepsilon}}{T}), \tag{2}$$

such that ε is a positive real.

Now, if we choose a linear contour integral of $s = \dfrac{3}{2} \pm iT$ to $s = \dfrac{1}{2} \pm iT$, in this case the function $F(s) = f(s)\dfrac{x^s}{s}$, admits a simple pole in $s = 1,$ then

$$\frac{1}{2\pi i} \left(\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \right) f(s) \frac{x^s}{s} ds = \operatorname{Re} s \left[f(s) \frac{x^s}{s}, 1 \right].$$

Note that $\lim_{s \to 1} \zeta(s)(s-1) = 1$, and we can get immediately

$$\operatorname{Re} s \left[f(s) \frac{x^s}{s}, 1 \right] = \left(\frac{2q^2 - q}{2q^2 - 2q + 1} \right) \frac{\pi^2}{6} \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3} \right) x,$$

such that

$$\frac{\pi^2}{6} \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3} \right) \simeq 1.4276565 \ldots$$

By taking $T = x$, and $f(s) = \zeta(s)R(s)$, where

$$R(s) = \left(\frac{2q^{2s} - q^s}{2q^{2s} - 2q^s + 1} \right) \zeta(2s) \prod_p \left(1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}} \right),$$

we obtain

$$\begin{aligned} &\left| \frac{1}{2\pi i} \left(\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} \right) \zeta(s) R(s) \frac{x^s}{s} ds \right| \\ &\ll \int_{\frac{1}{2}}^{\frac{3}{2}} \left| \zeta(\sigma + iT) R(s) \frac{x^{\frac{3}{2}}}{T} \right| d\sigma \\ &\ll \frac{x^{\frac{3}{2}+\varepsilon}}{T} = x^{\frac{1}{2}+\varepsilon}, \end{aligned}$$

and

$$\left| \frac{1}{2\pi i} \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \zeta(s) R(s) \frac{x^s}{s} ds \right| \ll \int_0^T \left| \zeta(\frac{1}{2} + it) R(s) \frac{x^{\frac{1}{2}}}{t} \right| dt \ll x^{\frac{1}{2}+\varepsilon}.$$

So by estimate

$$\left| \frac{1}{2\pi i} \left(\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \right) f(s) \frac{x^s}{s} ds \right| \ll x^{\frac{1}{2}+\varepsilon},$$

and from the formula (2), we get

$$\sum_{n\le x}\mathcal{D}(qn)=\left(\frac{2q^2-q}{2q^2-2q+1}\right)\left(\frac{\pi^2}{6}\right)\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x+\mathcal{O}(x^{\frac{1}{2}+\varepsilon}).\qquad\square$$

Lemma 1.3. *For any real $x\ge 1$, we have the following asymptotic formula*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\mathcal{D}(n)=\frac{\pi^2}{6}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x+\mathcal{O}(x^{\frac{\ln 8}{\ln 10}+\varepsilon}).\tag{3}$$

where $\mathcal{D}(n)=\dfrac{d(n)}{d^*(n)}$, and ε denotes a positive real number.

Proof. For any real $x\ge 1$, there exists a positive integer k such that $10^k\le x\le 10^{k+1}$. Consequently, $k\le\log x\le k+1$. According to the definition of the set $\mathcal{A}$, we know that the number of integers ($\le x$) that is not in $\mathcal{A}$ is 8^{k+1}. Indeed, there are 8 integers composed of a single number, they are $1,2,3,4,6,7,8,9$; there are 8^2 integers composed of two digits; and the number of integers composed of k digits is 8^k. Since

$$8^k\le 8^{\log x}=x^{\frac{\ln 8}{\ln 10}},$$

we get

$$\sum_{\substack{n\le x\\ n\notin\mathcal{A}}}1\le 8+8^2+8^3+...+8^{k+1}\le\frac{8^{k+2}}{7}\le\frac{64}{7}8^k\le\frac{64}{7}x^{\frac{\ln 8}{\ln 10}},$$

Note that for any $\varepsilon>0$ and for all $n\ge 1,$ we have $d(n)\ll n^{\varepsilon}$, and since $\dfrac{d(n)}{d^*(n)}\le d(n)$, we get $\dfrac{d(n)}{d^*(n)}\ll n^{\varepsilon}$. Now we apply the lemma 2 with $q=1$, we get

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\mathcal{D}(n)&=\sum_{n\le x}\mathcal{D}(n)-\sum_{\substack{n\le x\\ n\notin\mathcal{A}}}\mathcal{D}(n)\\&=\sum_{n\le x}\mathcal{D}(n)+\mathcal{O}\left(\sum_{\substack{n\le x\\ n\notin A}}x^{\varepsilon}\right)\\&=\sum_{n\le x}\mathcal{D}(n)+\mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right)\\&=\frac{\pi^2}{6}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x+\mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right).\end{aligned}$$

This proves Lemma 3. $\square$

2 Proof of Theorem 1.1

In this section, we complete the proof of Theorem. From the definition of the set $\mathcal{A}$ and set $\mathcal{B}$, we know the relation between them. Therefore

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in\mathcal{B}}}\mathcal{D}(n) &= \sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\mathcal{D}(n) - \sum_{5n\leq x}\mathcal{D}(5n)\\ &= \sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\mathcal{D}(n) - \sum_{n\leq \frac{x}{5}}\mathcal{D}(5n).\end{aligned}$$

Now we use the two results of Lemmas 2 and 3, we get

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in\mathcal{B}}}\frac{d(n)}{d^*(n)} &= \frac{\pi^2}{6}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x - \frac{3\pi^2}{82}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x + \mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right)\\ &= \frac{16\pi^2}{123}\prod_p(1-\frac{1}{2p^2}+\frac{1}{2p^3})x + \mathcal{O}\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right).\end{aligned}$$

This completes the proof of the theorem.

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