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On the Diophantine equation $B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$

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Abstract. In this study we find all solutions of the Diophantine equation

$$B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$$

in positive integer variables $(n_1, n_2, a_1, a_2, a_3)$, where B_n denotes the n-th balancing number.

1 Introduction

Balancing sequence $\{B_n\}_{n\geq 1}$ is originated from a simple Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

introduced by Behera and Panda [1]. Here, r is called a balancer corresponding to a balancing number n. The balancing sequence satisfies the binary recurrence

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1$$

with seeds $B_0 = 0$ and $B_1 = 1$. The Binet's formula for $\{B_n\}_{n\geq 1}$ is given by

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}},$$

where $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$ are the zeros of the polynomial $f(x) = x^2 - 6x + 1$. Clearly, $\beta^{-1} = \alpha$. It can be easily seen that

$$\alpha^{n-1} < B_n < \alpha^n, \quad \text{for} \quad n > 1. \tag{1}$$

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Diophantine equations involving powers and binary recurrence sequences have been extensively studied by many researchers in recent past. For example, Bravo and Luca [2] found all solutions of the equation $F_n + F_m = 2^a$, where F_n is the n-th Fibonacci number. Later, Bravo and Bravo [3] extended this work and found all positive integer solutions of the Diophantine equation $F_n + F_m + F_l = 2^a$. In [9], Şiar and Keskin solved the same type equation, instead of taking sum, they considered the difference of two Fibonacci numbers and found solutions to the equation $F_n - F_m = 2^a$. Chim and Ziegler [5] considered the equations $F_{n_1} + F_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$ and $F_{m_1} + F_{m_2} + F_{m_3} = 2^{t_1} + 2^{t_2}$ and proved that $\max\{n_1, n_2, a_1, a_2, a_3\} \le 18$ and $\max\{m_1, m_2, m_3, t_1, t_2\} \le 16$, respectively.

The authors used lower bounds for linear forms in logarithms and a version of Baker-Davenport reduction method as their main tools to solve all the problems stated above. A natural question arises: What will be the solution if we replace Fibonacci numbers by balancing numbers? Therefore, in this note, we look at the Diophantine equation

$$B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}, \tag{2}$$

where B_n is the n-th balancing number with $n_1 \ge n_2 \ge 0$ and $a_1 \ge a_2 \ge a_3 \ge 0$ and try to find all solutions using the same techniques.

The main result of this article is the following.

Theorem 1.1. *All non-negative integer solutions* $(n_1, n_2, a_1, a_2, a_3)$ *of the equation* (2) *are given by*

$$(n_1, n_2, a_1, a_2, a_3) \in \big\{(2,0,1,1,1), (2,0,2,0,0), (2,1,2,1,0), (2,2,2,2,2), (2,2,3,1,1),\\ (3,0,5,1,0), (3,1,4,4,2), (3,1,5,1,1), (3,2,5,3,0), (3,3,6,2,1)\big\}.$$

For the proof of Theorem 1.1, we run a program in *Mathematica* and search all solutions $(n_1, n_2, a_1, a_2, a_3)$ with $n_1 < 100$ to the equation (2). Then, we take $n_1 > 100$ and write (2) in six different ways. We apply lower bounds for linear forms in logarithms to obtain an upper bound on $n_1 = \max\{n_1, n_2, a_1, a_2, a_3\}$. This is done in the following seven steps:

Step 1: We find an upper bound

$$\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} < 8.22 \cdot 10^{12}(1 + \log n_1).$$

So, we divide into two cases:

Case 1: $\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} = (a_1 - a_2)\log 2$

Case 2: $\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} = (n_1 - n_2)\log\alpha$.

Step 2: We consider case 1 and show that

$$\min\{(a_1 - a_3)\log 2, (n_1 - n_2)\log\alpha\} < 4 \cdot 10^{25}(1 + \log n_1)^2.$$

We further divide case 1 into two following sub-cases:

Case 1A: $\min\{(a_1 - a_3)\log 2, (n_1 - n_2)\log\alpha\} = (a_1 - a_3)\log 2$

Case 1B: $\min\{(a_1 - a_3)\log 2, (n_1 - n_2)\log\alpha\} = (n_1 - n_2)\log\alpha$.

Step 3: We consider case 1A and show that

$$(n_1 - n_2) \log \alpha < 2 \cdot 10^{38}(1 + \log n_1)^3.$$

Step 4: We consider case 1B and show that

$$(a_1 - a_3) \log 2 < 9.96 \cdot 10^{37}(1 + \log n_1)^3.$$

Step 5: We consider case 2 and show that

$$(a_1 - a_2) \log 2 < 2 \cdot 10^{25}(1 + \log n_1)^2.$$

Step 6: We continue to consider case 2 and show that

$$(a_1 - a_3) \log 2 < 9.96 \cdot 10^{37}(1 + \log n_1)^3.$$

Step 7: Using the upper bounds $(a_1 - a_2) \log 2$, $(a_1 - a_3) \log 2$, $(n_1 - n_2) \log \alpha$, we obtain an absolute upper bound for n_1 as

$$n_1 < 7.9 \cdot 10^{59}.$$

We repeat all seven steps after finding an upper bound for n_1, but instead of lower bounds for linear forms in logarithms, we apply the Baker-Davenport reduction method. As a result, we have small absolute bounds and get to $n_1 < 86$, a contradiction. In this way, we complete the proof of our main result.

In order to prove Theorem 1.1, we need some preliminary results which are discussed in the next section.

2 Preliminaries

Baker's theory of linear forms in logarithms of algebraic numbers plays an important role while solving various Diophantine equations. Here, we use several times the same to solve the equation (2), but before that, we recall some basic notations and results from algebraic number theory.

Let η be an algebraic number with minimal primitive polynomial

$$f(X) = a_0(X - \eta^{(1)}) \dots (X - \eta^{(k)}) \in \mathbb{Z}[X],$$

where $a_0 > 0$, and $\eta^{(i)}$'s are conjugates of η. Then, the *logarithmic height* of η is defined by

$$h(\eta) = \frac{1}{k} \left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log |\eta^{(j)}|\} \right).$$

If $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$. The following are some known properties of logarithmic height function:

$$h(\eta + \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$
$$h(\eta^k) = |k| h(\eta), \quad k \in \mathbb{Z}.$$

The following theorem is a modified version of a result of Matveev (see [8] or [4, Theorem 9.4]) which provides a large upper bound for n_1 in (2).

Theorem 2.1. *Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ be positive real numbers and $b_1, b_2, \ldots, b_l$ be nonzero integers. If $\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1$ is not zero, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \ldots A_l,$$

where $D \geq \max\{|b_1|, |b_2|, \ldots, |b_l|\}$ and $A_1, A_2, \ldots, A_l$ are positive real numbers such that

$$A_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l.$$

We use the following method of Baker-Davenport due to Dujella and Pethő [6] to reduce the bound on n_1.

Lemma 2.2 ([6]). *Let M be a positive integer and p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|.\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following results will also be used to prove Theorem 1.1.

Lemma 2.3 ([7]). *Let $r \geq 1$ and $H > 0$ be such that $H > (4r^2)^r$ and $H > L/(\log L)^r$. Then*

$$L < 2^r H (\log H)^r.$$

Lemma 2.4. *All solutions of (2) satisfy $(n_1 - 1) < \frac{\log 3}{\log \alpha} + a_1 \frac{\log 2}{\log \alpha}$ and $n_1 > (a_1 - 1)\frac{\log 2}{\log \alpha}$.*

Proof. From (1) and (2) we have

$$\alpha^{n_1-1} < B_{n_1} \leq B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3} \leq 3 \cdot 2^{a_1}.$$

Taking logarithm on both sides, we get

$$(n_1 - 1)\log\alpha < \log 3 + a_1 \log 2,$$

which implies

$$(n_1 - 1) < \frac{\log 3}{\log\alpha} + a_1 \frac{\log 2}{\log\alpha}.$$

On the other hand, $2\alpha^{n_1} > 2B_{n_1} \geq B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3} > 2^{a_1}$. Taking logarithm on both sides, we get

$$\log 2 + n_1 \log\alpha > a_1 \log 2,$$

which implies

$$n_1 > (a_1 - 1)\frac{\log 2}{\log\alpha}.$$

□

3 Proof of Theorem 1.1

Consider the Diophantine equation

$$B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}.$$

First, we search the solutions to the above equation using *Mathematica* for $n_1 \leq 100$. Using Lemma 2.4, we calculate $a_1 \leq 256$. By *Mathematica*, for $0 \leq n_2 \leq n_1 \leq 100$ and $0 \leq a_3 \leq a_2 \leq a_1 \leq 256$, we find all the solutions that are listed in Theorem 1.1. Now, assume that $n_1 > 100$.

3.1 An upper bound on n_1

Using Binet's formula (2) can be written as

$$\frac{\alpha^{n_1} - \beta^{n_1}}{4\sqrt{2}} + \frac{\alpha^{n_2} - \beta^{n_2}}{4\sqrt{2}} = 2^{a_1} + 2^{a_2} + 2^{a_3}. \tag{3}$$

We write (3) in the following six different ways and examine each one to prove our result.

$$\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} = 2^{a_2} + 2^{a_3} + \frac{\beta^{n_1}}{4\sqrt{2}} - \frac{\alpha^{n_2} - \beta^{n_2}}{4\sqrt{2}}. \tag{4}$$

$$\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} = 2^{a_3} + \frac{\beta^{n_1}}{4\sqrt{2}} - \frac{\alpha^{n_2} - \beta^{n_2}}{4\sqrt{2}} \tag{5}$$

$$\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} - 2^{a_3} = \frac{\beta^{n_1}}{4\sqrt{2}} - \frac{\alpha^{n_2} - \beta^{n_2}}{4\sqrt{2}} \tag{6}$$

$$\frac{\alpha^{n_1}}{4\sqrt{2}} + \frac{\alpha^{n_2}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} = 2^{a_3} + \frac{\beta^{n_1}}{4\sqrt{2}} + \frac{\beta^{n_2}}{4\sqrt{2}} \tag{7}$$

$$\frac{\alpha^{n_1}}{4\sqrt{2}} + \frac{\alpha^{n_2}}{4\sqrt{2}} - 2^{a_1} = 2^{a_2} + 2^{a_3} + \frac{\beta^{n_1}}{4\sqrt{2}} + \frac{\beta^{n_2}}{4\sqrt{2}} \tag{8}$$

$$\frac{\alpha^{n_1}}{4\sqrt{2}} + \frac{\alpha^{n_2}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} - 2^{a_3} = \frac{\beta^{n_1}}{4\sqrt{2}} + \frac{\beta^{n_2}}{4\sqrt{2}} \tag{9}$$

Step 1: First, we consider (4). Here, we assume n_1 and a_1 to be large and collect the large terms involving n_1 and a_1 on the left side. Taking absolute values on both sides of (4), we get

$$\begin{aligned}\left|\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1}\right| &< 2^{a_2+1} + \frac{\alpha^{n_2}}{4\sqrt{2}} + 0.1 \\ &< 2.5 \max\left\{2^{a_2}, \alpha^{n_2}\right\}.\end{aligned}$$

Dividing both sides by 2^{a_1}, we get

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} 2^{-a_1} - 1\right| < \max\left\{2.5 \cdot 2^{a_2-a_1}, \frac{2.5\alpha^{n_2}}{2^{a_1}}\right\} < \max\left\{2.5 \cdot 2^{a_2-a_1}, \frac{7.5\alpha^{n_2}}{\alpha^{n_1-1}}\right\}.$$

Hence, we obtain

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} 2^{-a_1} - 1\right| < 43.72 \max\left\{2^{a_2-a_1}, \alpha^{n_2-n_1}\right\}. \tag{10}$$

Put

$$\Gamma = \frac{\alpha^{n_1}}{4\sqrt{2}} 2^{-a_1} - 1. \tag{11}$$

Suppose $\Gamma = 0$, then $\alpha^{2n_1} \in \mathbb{Q}$ which is not possible for any $n_1 > 0$. Therefore, $\Gamma \neq 0$. To apply Theorem 2.1 in (11), let

$$\eta_1 = \alpha,\ \eta_2 = 2,\ \eta_3 = 4\sqrt{2},\ b_1 = n_1,\ b_2 = -a_1,\ b_3 = -1,\ l = 3,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\alpha)$ and $b_1,\ b_2,\ b_3 \in \mathbb{Z}$. The degree $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is 2.

Since $n_1 > a_1 > 1$, therefore $D = \max\{1, n_1, |a_2|\} = n_1$. We calculate the logarithmic heights of η_1, η_2, η_3 as follows:

$$h(\eta_1) = h(\alpha) = \frac{\log \alpha}{2},\ h(\eta_2) = \log 2 \text{ and } h(\eta_3) = \log(4\sqrt{2}).$$

Thus, we can take

$$A_1 = \log \alpha, \quad A_2 = 2\log 2 \quad \text{and} \quad A_3 = 2\log(4\sqrt{2}).$$

Applying Theorem 2.1 we find

$$\log|\Gamma| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n_1)(\log\alpha)(2\log 2)(2\log(4\sqrt{2})).$$

Comparing the above inequality with (10) gives

$$\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} < 8.22 \cdot 10^{12}(1 + \log n_1).$$

Now, we divide into two cases.

Case 1: $\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} = (a_1 - a_2)\log 2$.

Case 2: $\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} = (n_1 - n_2)\log\alpha$.

Step 2: First, we consider case 1 and assume that

$$\min\{(a_1 - a_2)\log 2, (n_1 - n_2)\log\alpha\} = (a_1 - a_2)\log 2 < 8.22 \cdot 10^{12}(1 + \log n_1). \tag{12}$$

Assuming n_1, a_1 and a_2 to be large and collecting large terms on the left hand side, we consider (5). Taking absolute values on both sides of (5), we have

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2}\right| = \left|2^{a_3} + \frac{\beta^{n_1}}{4\sqrt{2}} - \frac{\alpha^{n_2} - \beta^{n_2}}{4\sqrt{2}}\right|,$$

which implies

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2}\right| < 2^{a_3} + \frac{\alpha^{n_2}}{4\sqrt{2}} + 0.1 < 1.2\max\{2^{a_3}, \alpha^{n_2}\}.$$

Dividing both sides by $\frac{\alpha^{n_1}}{4\sqrt{2}}$, we obtain

$$\begin{aligned}\left|1 - \alpha^{-n_1}2^{a_2}4\sqrt{2}(2^{a_1-a_2} + 1)\right| &< \max\left\{\frac{(1.2)(4\sqrt{2})}{\alpha^{n_1}} \cdot 2^{a_3}, (1.2)(4\sqrt{2})\alpha^{n_2-n_1}\right\} \\ &\leq \max\left\{\frac{(1.2)(4\sqrt{2})}{2^{a_1-1}} \cdot 2^{a_3}, (1.2)(4\sqrt{2})\alpha^{n_2-n_1}\right\}.\end{aligned}$$

Hence, we obtain

$$\left|1 - \alpha^{-n_1}2^{a_2}4\sqrt{2}(2^{a_1-a_2} + 1)\right| < 13.57\max\left\{2^{a_3-a_1}, \alpha^{n_2-n_1}\right\}. \tag{13}$$

Put

$$\Gamma_1 = 1 - \alpha^{-n_1}2^{a_2}4\sqrt{2}(2^{a_1-a_2} + 1).$$

By similar arguments as before we can show that $\Gamma_1 \neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \alpha,\ \eta_2 = 2,\ \eta_3 = 4\sqrt{2}(2^{a_1-a_2} + 1),\ b_1 = -n_1,\ b_2 = a_2,\ b_3 = 1,\ l = 3.$$

Since $a_2 < n_1$, we take $D = n_1$. As before, we have the same logarithmic heights for η_1 and η_2. Thus A_1 and A_2 remain unchanged. Computing the height of η_3, we have

$$\begin{aligned} h(\eta_3) &= h(4\sqrt{2}(2^{a_1-a_2}+1)) \\ &\le h(4\sqrt{2}) + h(2^{a_1-a_2}+1) \\ &\le \log(4\sqrt{2}) + (a_1-a_2)\log 2 + \log 2. \end{aligned}$$

Hence, from (12), we get

$$h(\eta_3) < 8.23 \cdot 10^{12}(1+\log n_1).$$

So, we take

$$A_3 = 16.46 \cdot 10^{12}(1+\log n_1).$$

Using all these values in Theorem 2.1, we have

$$\log|\Gamma_1| > -1.4\cdot 30^6 \cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log n_1)(\log\alpha)(2\log 2)(16.46\cdot 10^{12}(1+\log n_1)).$$

Comparing the above inequality with (13) gives

$$\min\{(a_1-a_3)\log 2, (n_1-n_2)\log\alpha\} < 4\cdot 10^{25}(1+\log n_1)^2.$$

Now, we divide this into two sub-cases.

Case 1A: $\min\{(a_1-a_3)\log 2, (n_1-n_2)\log\alpha\} = (a_1-a_3)\log 2$.

Case 1B: $\min\{(a_1-a_3)\log 2, (n_1-n_2)\log\alpha\} = (n_1-n_2)\log\alpha$.

Step 3: Assume the first sub-case, that is

$$\min\{(a_1-a_3)\log 2, (n_1-n_2)\log\alpha\} = (a_1-a_3)\log 2 < 4\cdot 10^{25}(1+\log n_1)^2. \tag{14}$$

In this step, we consider n_1, a_1, a_2 and a_3 to be large. By collecting large terms on the left side, we consider (6), that is

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} - 2^{a_3}\right| = \left|\frac{\beta^{n_1}}{4\sqrt{2}} - \frac{\alpha^{n_2}-\beta^{n_2}}{4\sqrt{2}}\right|,$$

which implies

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}} - 2^{a_1} - 2^{a_2} - 2^{a_3}\right| < \frac{\alpha^{n_2}}{4\sqrt{2}} + 0.1 < 0.3\alpha^{n_2}.$$

Dividing both sides by $\frac{\alpha^{n_1}}{4\sqrt{2}}$, we obtain

$$\left|1-\alpha^{-n_1}2^{a_1}4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})\right| < 0.3\alpha^{n_2}\left(\frac{4\sqrt{2}}{\alpha^{n_1}}\right) = 1.7\alpha^{n_2-n_1}. \tag{15}$$

Put

$$\Gamma_A = 1 - \alpha^{-n_1} 2^{a_1} 4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1}).$$

We can show that $\Gamma_A \neq 0$. Take

$$\eta_1 = \alpha,\ \eta_2 = 2,\ \eta_3 = 4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1}),\ b_1 = -n_1,\ b_2 = a_1,\ b_3 = 1.$$

Computing the logarithmic height of η_3, we get

$$\begin{aligned} h(\eta_3) &= h(4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1})) \\ &\leq h(4\sqrt{2}) + h(1 + 2^{a_2-a_1} + 2^{a_3-a_1}) \\ &\leq \log(4\sqrt{2}) + (a_1 - a_2)\log 2 + (a_1 - a_3)\log 2 + 2\log 2. \end{aligned}$$

Hence, from (12) and (14), we get

$$h(\eta_3) < 4.1 \cdot 10^{25}(1 + \log n_1)^2.$$

So, we take

$$A_3 = 8.2 \cdot 10^{25}(1 + \log n_1)^2.$$

The parameters A_1 and A_2 remain unchanged as before. Using all these values in Theorem 2.1, we have

$$\log|\Gamma_A| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n_1)(\log\alpha)(2\log 2)(8.2 \cdot 10^{25}(1 + \log n_1)^2).$$

Comparing the above inequality with (15) gives

$$(n_1 - n_2)\log\alpha < 2 \cdot 10^{38}(1 + \log n_1)^3.$$

Step 4: Now, we consider the second sub-case, that is

$$\min\{(a_1 - a_3)\log 2, (n_1 - n_2)\log\alpha\} = (n_1 - n_2)\log\alpha < 4 \cdot 10^{25}(1 + \log n_1)^2. \qquad (16)$$

Equation (7) implies

$$\left|\frac{\alpha^{n_2}(1 + \alpha^{n_1-n_2})}{4\sqrt{2}} - 2^{a_2}(2^{a_1-a_2} + 1)\right| < 1.1 \cdot 2^{a_3}.$$

Dividing both sides by $2^{a_2}(2^{a_1-a_2} + 1)$, we obtain

$$\left|\alpha^{n_2} 2^{-a_2} \frac{(1 + \alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2} + 1)} - 1\right| < 1.1 \cdot 2^{a_3-a_1}. \qquad (17)$$

Take

$$\Gamma_B = \alpha^{n_2} 2^{-a_2} \frac{(1 + \alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2} + 1)} - 1,$$

with $\eta_1 = \alpha,\ \eta_2 = 2,\ \eta_3 = \frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2}+1)},\ b_1 = n_2,\ b_2 = -a_2,\ b_3 = 1$. Since $a_2 < n_2 < n_1$, $D = n_1$. The height of η_3 is calculated as

$$\begin{aligned} h(\eta_3) &= h\left(\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2}+1)}\right) \\ &\leq h(1+\alpha^{n_1-n_2}) + h(4\sqrt{2}(2^{a_1-a_2}+1)) \\ &\leq (n_1-n_2)h(\alpha) + h(4\sqrt{2}) + (a_1-a_2)h(2) + 2\log 2 \\ &= (n_1-n_2)\frac{\log\alpha}{2} + \log(4\sqrt{2}) + (a_1-a_2)\log 2 + 2\log 2. \end{aligned}$$

Hence, from (12) and (16), we get

$$h(\eta_3) < 2.1\cdot 10^{25}(1+\log n_1)^2.$$

So, we take

$$A_3 = 4.2\cdot 10^{25}(1+\log n_1)^2.$$

Applying Theorem 2.1, we have

$$\log|\Gamma_B| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log n_1)(\log\alpha)(2\log 2)(4.2\cdot 10^{25}(1+\log n_1)^2).$$

Comparing the above inequality with (17) gives

$$(a_1-a_3)\log 2 < 9.96\cdot 10^{37}(1+\log n_1)^3.$$

Step 5: Now, we consider case 2, that is

$$\min\{(a_1-a_2)\log 2, (n_1-n_2)\log\alpha\} = (n_1-n_2)\log\alpha < 8.22\cdot 10^{12}(1+\log n_1). \tag{18}$$

Equation (8) implies

$$\left|\frac{\alpha^{n_2}(1+\alpha^{n_1-n_2})}{4\sqrt{2}} - 2^{a_1}\right| < 2.2\cdot 2^{a_2}.$$

Dividing both sides by 2^{a_1}, we obtain

$$\left|\alpha^{n_2}2^{-a_1}\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}} - 1\right| < 2.2\cdot 2^{a_2-a_1}. \tag{19}$$

Put

$$\Gamma_2 = \alpha^{n_2}2^{-a_1}\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}} - 1.$$

We can show that $\Gamma_2 \neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \alpha,\ \eta_2 = 2,\ \eta_3 = \frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}},\ b_1 = n_2,\ b_2 = -a_1,\ b_3 = 1.$$

Since $a_2 < n_2 < n_1$, $D = n_1$. Computing the logarithmic height of η_3, we get

$$\begin{aligned} h(\eta_3) &= h\left(\frac{1+\alpha^{n_1-n_2}}{4\sqrt{2}}\right) \\ &\leq h(1+\alpha^{n_1-n_2}) + h(4\sqrt{2}) \\ &\leq (n_1-n_2)h(\alpha) + h(4\sqrt{2}) + \log 2 \\ &= (n_1-n_2)\frac{\log\alpha}{2} + \log(4\sqrt{2}) + \log 2. \end{aligned}$$

Hence, from (18), we obtain

$$h(\eta_3) < 4.12 \cdot 10^{12}(1+\log n_1).$$

So, we take

$$A_3 = 8.24 \cdot 10^{12}(1+\log n_1).$$

The value of A_1 and A_2 remain same as before. Applying Theorem 2.1, we have

$$\log|\Gamma_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log n_1)(\log\alpha)(2\log 2)(8.24 \cdot 10^{12}(1+\log n_1)).$$

Comparing the above inequality with (19) gives

$$(a_1 - a_2)\log 2 < 2 \cdot 10^{25}(1+\log n_1)^2. \tag{20}$$

Step 6: We apply Theorem 2.1 once more to obtain an upper bound for $(a_1 - a_3)\log 2$. The derivation is similar to case 1B. By the similar derivation as in step 4, we obtain

$$\left|\alpha^{n_2}2^{-a_2}\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2}+1)} - 1\right| < 1.1 \cdot 2^{a_3-a_1}. \tag{21}$$

We estimate the height of η_3 as

$$\begin{aligned} h(\eta_3) &= h\left(\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2}+1)}\right) \\ &\leq (n_1-n_2)h(\alpha) + h(4\sqrt{2}) + (a_1-a_2)h(2) + 2\log 2 \\ &= (n_1-n_2)\frac{\log\alpha}{2} + \log(4\sqrt{2}) + (a_1-a_2)\log 2 + 2\log 2. \end{aligned}$$

Hence, from (18) and (20), we get

$$h(\eta_3) < 2.1 \cdot 10^{25}(1+\log n_1)^2.$$

So, we take

$$A_3 = 4.2 \cdot 10^{25}(1+\log n_1)^2.$$

Applying Theorem 2.1, we have

$$\log|\Gamma_B| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log n_1)(\log\alpha)(2\log 2)(4.2\cdot 10^{25}(1+\log n_1)^2).$$

Comparing the above inequality with (21) gives

$$(a_1-a_3)\log 2 < 9.96\cdot 10^{37}(1+\log n_1)^3.$$

We summarize our results obtained so far in the following table.

Upper bound of	Case 1A	Case 1B	Case 2
$(a_1-a_2)\log 2$	$8.22\cdot 10^{12}(1+\log n_1)$	$8.22\cdot 10^{12}(1+\log n_1)$	$2\cdot 10^{25}(1+\log n_1)^2$
$(a_1-a_3)\log 2$	$4\cdot 10^{25}(1+\log n_1)^2$	$9.96\cdot 10^{37}(1+\log n_1)^3$	$9.96\cdot 10^{37}(1+\log n_1)^3$
$(n_1-n_2)\log\alpha$	$2\cdot 10^{38}(1+\log n_1)^3$	$4\cdot 10^{25}(1+\log n_1)^2$	$8.22\cdot 10^{12}(1+\log n_1)$

Step 7: Lastly, we consider (9), that is

$$\frac{\alpha^{n_1}}{4\sqrt{2}}+\frac{\alpha^{n_2}}{4\sqrt{2}}-2^{a_1}-2^{a_2}-2^{a_3}=\frac{\beta^{n_1}}{4\sqrt{2}}+\frac{\beta^{n_2}}{4\sqrt{2}}.$$

Taking absolute values on both sides, we have

$$\left|\frac{\alpha^{n_1}(1+\alpha^{n_2-n_1})}{4\sqrt{2}}-2^{a_1}(1+2^{a_2-a_1}+2^{a_3-a_1})\right|<0.1.$$

Dividing both sides by $\frac{\alpha^{n_1}(1+\alpha^{n_2-n_1})}{4\sqrt{2}}$ gives

$$\left|1-\alpha^{-n_1}2^{a_1}\frac{4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})}{(1+\alpha^{n_2-n_1})}\right|<0.6\cdot\alpha^{-n_1}. \tag{22}$$

Put

$$\Gamma_3=\left|1-\alpha^{-n_1}2^{a_1}\frac{4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})}{(1+\alpha^{n_2-n_1})}\right|.$$

Using similar arguments as before we can show that $\Gamma_3\neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1=\alpha,\ \eta_2=2,\ \eta_3=\frac{4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})}{(1+\alpha^{n_2-n_1})},\ b_1=-n_1,\ b_2=a_1,\ b_3=1.$$

Since $a_1<n_1$, $D=n_1$. Computing the logarithmic height of η_3, we get

$$\begin{aligned}
h(\eta_3)&=h\left(\frac{4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})}{(1+\alpha^{n_2-n_1})}\right)\\
&\le h(4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1}))+h(1+\alpha^{n_2-n_1})\\
&\le h(4\sqrt{2})+(a_1-a_2)h(2)+(a_1-a_3)h(2)+(n_1-n_2)h(\alpha)+3\log 2\\
&=\log(4\sqrt{2})+(a_1-a_2)\log 2+(a_1-a_3)\log 2+(n_1-n_2)\frac{\log\alpha}{2}+3\log 2.
\end{aligned}$$

Hence, we get

$$h(\eta_3) < 9.97 \cdot 10^{37}(1 + \log n_1)^3.$$

So, we take

$$A_3 = 19.95 \cdot 10^{37}(1 + \log n_1)^3.$$

Applying Theorem 2.1, we have

$$\log|\Gamma_3| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n_1)(\log\alpha)(2\log 2)(19.95 \cdot 10^{37}(1 + \log n_1)^3).$$

Comparing the above inequality with (22) gives

$$n_1 \log\alpha < 4.73 \cdot 10^{50}(1 + \log n_1)^4.$$

With the notation of Lemma 2.3, we take $r = 4$, $L = n$ and $H = \frac{4.73 \cdot 10^{50}}{\log\alpha}$. Applying the lemma, we have

$$\begin{aligned} n_1 &< 2^4\left(\frac{4.73 \cdot 10^{50}}{\log\alpha}\right)\left(\log\left(\frac{4.73 \cdot 10^{50}}{\log\alpha}\right)\right)^4 \\ &< 7.9 \cdot 10^{59}. \end{aligned}$$

The bound on n_1 is too large. So, in the next subsection, we reduce this bound using Lemma 2.2.

3.2 Bound Reduction

To reduce the bound on n_1, we use the following steps.
Step 1: Put

$$\Lambda = n_1 \log\alpha - a_1 \log 2 - \log\left(4\sqrt{2}\right).$$

The inequality (10) can be written as

$$\left|\frac{\alpha^{n_1}}{4\sqrt{2}}2^{-a_1} - 1\right| = |e^{\Lambda} - 1| < 43.72 \max\left\{2^{a_2 - a_1}, \alpha^{n_2 - n_1}\right\}.$$

Observe that $\Lambda \neq 0$ as $e^{\Lambda} - 1 = \Gamma \neq 0$. Assuming $\min\{a_1 - a_2, n_1 - n_2\} \geq 7$, the right-hand side in the above inequality is at most $\frac{1}{2}$. The inequality $|e^z - 1| < y$ for real values of z and y implies $z < 2y$. Thus, we get

$$|\Lambda| < 87.44 \max\left\{2^{a_2 - a_1}, \alpha^{n_2 - n_1}\right\},$$

which implies that

$$\left|n_1 \log\alpha - a_1 \log 2 - \log\left(4\sqrt{2}\right)\right| < 87.44 \max\left\{2^{a_2 - a_1}, \alpha^{n_2 - n_1}\right\}.$$

Dividing both sides by $\log 2$ gives

$$\left| n_1 \left(\frac{\log \alpha}{\log 2}\right) - a_1 + \frac{\log\left(1/4\sqrt{2}\right)}{\log 2} \right| < \max\left\{ \frac{87.44}{\log 2} \cdot 2^{a_2-a_1}, \frac{87.44}{\log 2}\alpha^{n_2-n_1} \right\}$$
$$< \max\left\{127 \cdot 2^{-(a_1-a_2)}, 127\alpha^{-(n_1-n_2)}\right\}.$$

We let

$$u = n_1,\ \tau = \left(\frac{\log \alpha}{\log 2}\right),\ v = a_1,\ \mu = \frac{\log\left(1/4\sqrt{2}\right)}{\log 2},\ \text{with}$$
$$(A, B, w) = (127, 2, (a_1 - a_2)) \text{ or } (127, \alpha, (n_1 - n_2)).$$

Choose $M = 7.9 \cdot 10^{59}$. We find q_{126} exceeds $6M$ with $\varepsilon = \|\mu q_{126}\| - M\|\tau q_{126}\| = 0.5$. By virtue of Lemma 2.2, we get $a_1 - a_2 \leq 214$ or $n_1 - n_2 \leq 84$. Now, we divide this into two cases.

Case 1: $a_1 - a_2 \leq 214$

Case 2: $n_1 - n_2 \leq 84$

Step 2: First, we consider case 1. Let

$$\Lambda_1 = -n_1 \log \alpha + a_2 \log 2 + \log\left(4\sqrt{2}(1 + 2^{a_1-a_2})\right).$$

The inequality (13) can be written as

$$\left|e^{\Lambda_1} - 1\right| = |\Gamma_1| < 13.57 \max\left\{2^{a_3-a_1}, \alpha^{n_2-n_1}\right\}.$$

Assuming $\min\{a_1 - a_3, n_1 - n_2\} \geq 5$, the right-hand side in the above inequality at most $\frac{1}{2}$. Thus, we get

$$\left| n_1 \log \alpha - a_2 \log 2 + \log\left(1/(4\sqrt{2}(1 + 2^{a_1-a_2}))\right) \right| < 27.14 \max\left\{2^{a_3-a_1}, \alpha^{n_2-n_1}\right\}.$$

Dividing both sides by $\log 2$ gives

$$\left| n_1 \left(\frac{\log \alpha}{\log 2}\right) - a_2 + \frac{\log\left(1/(4\sqrt{2}(1 + 2^{a_1-a_2}))\right)}{\log 2} \right| < \max\left\{ \frac{27.52}{\log 2} \cdot 2^{a_3-a_1}, \frac{27.52}{\log 2}\alpha^{n_2-n_1} \right\}$$
$$< \max\left\{40 \cdot 2^{-(a_1-a_3)}, 40\alpha^{-(n_1-n_2)}\right\}.$$

Let

$$u = n_1,\ \tau = \left(\frac{\log \alpha}{\log 2}\right),\ v = a_2,\ \mu = \frac{\log\left(1/(4\sqrt{2}(1 + 2^{a_1-a_2}))\right)}{\log 2},$$

with $(A, B, w) = (40, 2, (a_1 - a_3))$ or $(40, \alpha, (n_1 - n_2))$. With the same M, we find q_{124} exceeds $6M$ with $\varepsilon > 0.00179287$. By virtue of Lemma 2.2 for $(a_1 - a_2) \leq 214$, we get $a_1 - a_3 \leq 218$ or $n_1 - n_2 \leq 86$.

Again, we divide case 1 into two sub-cases.

Case 1A: $a_1 - a_3 \leq 218$

Case 1B: $n_1 - n_2 \leq 86$

Step 3: We consider 1A. Put

$$\Lambda_A = -n_1 \log\alpha + a_1 \log 2 + \log\left(4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1})\right).$$

Then, inequality (15) can be written as

$$\left|e^{\Lambda_A} - 1\right| = |\Gamma_A| < 1.7\alpha^{n_2-n_1}.$$

Assuming $(n_1 - n_2) \geq 1$, we get

$$\left|n_1 \log\alpha - a_1 \log 2 + \log\left(1/(4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1}))\right)\right| < 3.4\alpha^{n_2-n_1},$$

which implies

$$\begin{aligned}\left|n_1\left(\frac{\log\alpha}{\log 2}\right) - a_1 + \frac{\log\left(1/(4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1}))\right)}{\log 2}\right| &< \frac{3.4}{\log 2}\alpha^{n_2-n_1}\\ &< 5\alpha^{-(n_1-n_2)}.\end{aligned}$$

Let

$$u = n_1,\ \tau = \left(\frac{\log\alpha}{\log 2}\right),\ v = a_1,\ \mu = \frac{\log\left(1/(4\sqrt{2}(1 + 2^{a_2-a_1} + 2^{a_3-a_1}))\right)}{\log 2},$$

with $(A, B, w) = (5, \alpha, (n_1 - n_2))$. With the same M, we estimate $\varepsilon > 0.0000354843$. Applying Lemma 2.2 for $(a_1 - a_2) \leq 214$ and $(a_1 - a_3) \leq 218$, we get $n_1 - n_2 \leq 87$.

Step 4: We consider the case 1B. Put

$$\Lambda_B = n_2 \log\alpha - a_2 \log 2 + \log\frac{(1 + \alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2} + 1)}.$$

The inequality (17) can be written as

$$\left|e^{\Lambda_B} - 1\right| = |\Gamma_B| < 1.1 \cdot 2^{a_3-a_1}.$$

Assuming $(a_1 - a_3) \geq 2$, we get

$$\left|n_2 \log\alpha - a_2 \log 2 + \log\frac{(1 + \alpha^{n_1-n_2})}{4\sqrt{2}(2^{a_1-a_2} + 1)}\right| < 2.2 \cdot 2^{-(a_1-a_3)},$$

which implies

$$\begin{aligned}\left|n_2\left(\frac{\log\alpha}{\log 2}\right) - a_2 + \frac{\log\left((1 + \alpha^{n_1-n_2})/(4\sqrt{2}(2^{a_1-a_2} + 1))\right)}{\log 2}\right| &< \frac{2.2}{\log 2} \cdot 2^{a_3-a_1}\\ &< 3.1 \cdot 2^{-(a_1-a_3)}.\end{aligned}$$

Let

$$u = n_2,\ \tau = \left(\frac{\log\alpha}{\log 2}\right),\ v = a_2,\ \mu = \frac{\log\left((1+\alpha^{n_1-n_2})/(4\sqrt{2}(2^{a_1-a_2}+1))\right)}{\log 2},$$

with $(A, B, w) = (3.1, 2, (a_1 - a_3))$. With the same M we find $\varepsilon > 0.0000119685$. Applying Lemma 2.2 for $(a_1 - a_2) \leq 214$ and $(n_1 - n_2) \leq 86$, we get $a_1 - a_3 \leq 222$.

Step 5: Now, consider case 2. Take

$$\Lambda_2 = n_2 \log\alpha - a_1 \log 2 + \log\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}}.$$

The inequality (19) can be written as

$$\left|e^{\Lambda_2} - 1\right| = |\Gamma_2| < 2.2 \cdot 2^{a_2-a_1}.$$

Assuming $(a_1 - a_2) \geq 3$, we get

$$\left|n_2 \log\alpha - a_1 \log 2 + \log\frac{(1+\alpha^{n_1-n_2})}{4\sqrt{2}}\right| < 4.4 \cdot 2^{-(a_1-a_2)}.$$

Dividing both sides by $\log 2$ gives

$$\begin{aligned}\left|n_2\left(\frac{\log\alpha}{\log 2}\right) - a_1 + \frac{\log\left((1+\alpha^{n_1-n_2})/4\sqrt{2}\right)}{\log 2}\right| &< \frac{4.4}{\log 2} \cdot 2^{-(a_1-a_2)} \\ &< 6.3 \cdot 2^{-(a_1-a_2)}.\end{aligned}$$

Let

$$u = n_2,\ \tau = \left(\frac{\log\alpha}{\log 2}\right),\ v = a_1,\ \mu = \frac{\log\left((1+\alpha^{n_1-n_2})/4\sqrt{2}\right)}{\log 2},$$

with $(A, B, w) = (6.3, 2, (a_1 - a_2))$. We calculate $\varepsilon > 0.00225968$. Applying Lemma 2.2 for $(n_1 - n_2) \leq 84$, we get $a_1 - a_2 \leq 215$.

Step 6: We continue case 2. We have that $a_1 - a_2 \leq 215$ and $n_1 - n_2 \leq 84$. Applying similar steps as in case 1B, we obtain $a_1 - a_3 \leq 222$. We summarize our results obtained so far in the following table.

Upper bound of	Case 1A	Case 1B	Case 2
$(a_1 - a_2)$	214	214	215
$(a_1 - a_3)$	218	222	222
$(n_1 - n_2)$	87	86	84

Step 7: Now, under the assumption that $n_1 - n_2 \leq 87, a_1 - a_2 \leq 215, a_1 - a_3 \leq 222$, put

$$\Lambda_3 = -n_1 \log\alpha + a_1 \log 2 + \log\frac{4\sqrt{2}(1+2^{a_2-a_1}+2^{a_3-a_1})}{(1+\alpha^{n_2-n_1})}.$$

The inequality (22) can be written as

$$\left|e^{\Lambda_3} - 1\right| = |\Gamma_3| < 0.6\alpha^{-n_1}.$$

which implies that

$$\left|n_1 \log \alpha - a_1 \log 2 + \log \frac{(1 + \alpha^{n_2 - n_1})}{4\sqrt{2}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_1})}\right| < 1.2\alpha^{-n_1}.$$

Dividing both sides by $\log 2$ gives

$$\begin{aligned}\left|n_1 \left(\frac{\log \alpha}{\log 2}\right) - a_1 + \frac{\log\left((1 + \alpha^{n_2 - n_1})/(4\sqrt{2}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_1}))\right)}{\log 2}\right| &< \frac{1.2}{\log 2}\alpha^{-n_1} \\ &< 1.7\alpha^{-n_1}.\end{aligned}$$

Let

$$u = n_1,\ \tau = \left(\frac{\log \alpha}{\log 2}\right),\ v = a_1,\ \mu = \frac{\log\left((1 + \alpha^{n_2 - n_1})/(4\sqrt{2}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_1}))\right)}{\log 2},$$

with $(A, B, w) = (1.7, \alpha, n_1)$. With the same M, we find $\varepsilon > 0.00001$. Applying Lemma 2.2 for $n_1 - n_2 \leq 87, a_1 - a_2 \leq 215$ and $a_1 - a_3 \leq 222$, we get $n_1 \leq 86$, which is a contradiction. Hence, the theorem is proved.

As a consequence of Theorem 1.1 we obtain the following corollaries.

Theorem 3.1. *All non-negative integer solutions* (n_1, n_2, a_1, a_2) *of the equation*

$$B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2},$$

with $n_1 \geq n_2 \geq 0$ *and* $a_1 \geq a_2 \geq 0$ *are given by*

$$(n_1, n_2, a_1, a_2) \in \{(1, 1, 0, 0), (2, 0, 2, 1), (2, 2, 3, 2), (3, 1, 5, 2)\}.$$

Theorem 3.2. *All non-negative integer solutions* (n_1, n_2, a_1) *of the equation*

$$B_{n_1} + B_{n_2} = 2^{a_1},$$

with $n_1 \geq n_2 \geq 0$ *and* $a_1 \geq 0$ *are given by*

$$(n_1, n_2, a_1) \in \{(1, 1, 0), (1, 1, 1)\}.$$

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