

Results on K_1 of general quadratic groups

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Abstract. In the first part of this article, we discuss the relative cases of Quillen–Suslin’s local-global principle for the general quadratic (Bak’s unitary) groups, and its applications for the (relative) stable and unstable K_1 -groups. The second part is dedicated to the graded version of the local-global principle for the general quadratic groups and its application to deduce a result for Bass’ nil groups.

1 Introduction

In [7] and [8], the first author has discussed many results in classical K-theory for the absolute cases, related to the Serre’s problem on projective modules. In this article, following the tricks used in [10], we are going to consider some problems for the relative cases; *viz* Quillen–Suslin’s local-global principle of the transvection subgroups, K_1 -stabilization and the structure of unstable K_1 -groups of the general quadratic (Bak’s unitary) groups over associative rings which are finite over the center. For previous results on these problems we refer to the works of Bak–Basu–Rao–Khanna for the local-global principle (L-G principle) in [9], [4], [10], Bak–Petrov–Tang for K_1 -stabilization in [5], and Bak–Harzat–Vavilov for solvability of unstable K_1 -groups in [2], [19], [18].

For the linear case, the graded version of L-G principle was studied by Chouinard in [13], and by Gubeladze in [14], [15]. In [11], the first author and M.K. Singh deduced an analog for the traditional classical groups. In the second part of the article, we have deduced an analog for the transvection subgroups of the general quadratic groups over graded rings. As an application, by using a recent result on Higman linearization due to V. Kepoeiko (cf. [21], [22]), we could revisit a problem on absence of torsion in Bass’ nil

MSC 2020: 11E57, 13A02, 13C10, 15A63, 19B14, 19D35, 20F18

Keywords: Graded rings, Form rings, Bak’s unitary groups, Bass’ nil groups

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group of the general quadratic groups, and its graded analog. We refer to [23], [12], [24], [25], and [7] for previous results in this direction.

2 Preliminaries

Let us recall some necessary definitions and the key lemmas.

Definition 2.1 (cf. [1]). Let R be an (not necessarily commutative) associative ring with identity, and with involution $- : R \rightarrow R$, $a \mapsto \bar{a}$. Let $\lambda \in C(R) =$ center of R be an element with the property $\lambda\bar{\lambda} = 1$. We define additive subgroups of R :

$$\Lambda_{max} = \{a \in R \mid a = -\lambda\bar{a}\} \text{ and } \Lambda_{min} = \{a - \lambda\bar{a} \mid a \in R\}.$$

One checks that Λ_{max} and Λ_{min} are closed under the conjugation operation $a \mapsto \bar{x}ax$ for any $x \in R$. A λ -form parameter on R is an additive subgroup Λ of R such that $\Lambda_{min} \subseteq \Lambda \subseteq \Lambda_{max}$, and $\bar{x}\Lambda x \subseteq \Lambda$ for all $x \in R$. A pair (R, Λ) is called a *form ring*.

Remark 2.2. For a form ring (R, Λ) , we can extend the involution $- : R \rightarrow R$ to an involution $- : R[X] \rightarrow R[X]$ by setting $\overline{X} = X$.

Lemma 2.3. *Let (R, Λ, λ) be a form ring. Then $(R[X], \Lambda[X], \lambda)$ is also a form ring with respect to the involution obtained by extending, as in the previous remark.*

Proof. Let us consider the subgroups $\Lambda_{min}(R[X])$ and $\Lambda_{max}(R[X])$. It can be checked that $\Lambda_{min}(R[X]) = \Lambda_{min}(R)[X]$, and $\Lambda_{max}(R[X]) = \Lambda_{max}(R)[X]$. Hence it follows

$$\Lambda_{min}(R[X]) \subseteq \Lambda[X] \subseteq \Lambda_{max}(R[X]).$$

To prove that it is closed under conjugation, let us consider elements $a(X) \in R[X]$ and $b(X) \in \Lambda[X]$. Using double induction on the degrees of $a(X)$ and $b(X)$, we will show that

$$\overline{a(X)}b(X)a(X) \in \Lambda[X].$$

We first show that the statement is true for $\deg(b(X)) = 0$. Assume $b(X) = b \in \Lambda$. Hence we need to prove $\overline{a(X)}ba(X) \in \Lambda[X]$ for all $a(X) \in R[X]$. Clearly, this is obvious when $\deg(a(X)) = 0$. For a linear polynomial $a(X) = a_0 + a_1X$,

$$\begin{aligned} \overline{a(X)}ba(X) &= (\bar{a}_0 + \bar{a}_1X)b(a_0 + a_1X) \\ &= \bar{a}_0ba_0 + (\bar{a}_0ba_1 + \bar{a}_1ba_0)X + \bar{a}_1ba_1X^2 \\ &= \bar{a}_0ba_0 + \overline{(a_0 + a_1)}b(a_0 + a_1)X - (\bar{a}_0ba_0 + \bar{a}_1ba_1)X + \bar{a}_1ba_1X^2. \end{aligned}$$

Hence the statement is true for $\deg(b(X)) = 0$ and $\deg(a(X)) = 1$. Assume the statement for $\deg(b(X)) = 0$ is true for all $a(X)$ with $\deg(a(X)) < m$. Suppose $\deg(a(X)) = m$. Then, $a(X) = a_{m-1}(X) + a_mX^m$, where $a_{m-1}(X)$ is a polynomial of degree at most $m-1$, and $a_m \in R$. Hence, we have

$$\begin{aligned}\overline{a(X)}ba(X) &= (\overline{a_{m-1}(X)} + \overline{a_m}X^m)b(a_{m-1}(X) + a_mX^m) \\ &= \overline{a_{m-1}(X)}ba_{m-1}(X) + \overline{a_{m-1}(X)}ba_mX^m + \overline{a_m}ba_{m-1}(X)X^m + \overline{a_m}ba_mX^{2m}.\end{aligned}$$

Now

$$\begin{aligned}& \overline{a_{m-1}(X)}ba_m + \overline{a_m}ba_{m-1}(X) = \\ &= (\overline{a_{m-1}(X)} + \overline{a_m})b(a_{m-1}(X) + a_m) - (\overline{a_{m-1}(X)}ba_{m-1}(X) + \overline{a_m}ba_m).\end{aligned}$$

Hence by induction it follows that $\overline{a(X)}ba(X) \in \Lambda[X]$. Assume the statement is true for $\deg(b(X)) < n$. Let $b'(X)$ be a polynomial in $\Lambda[X]$ of degree n . Then we can write $b'(X) = b''(X) + b_nX^n$, where $b''(X)$ is a polynomial in $\Lambda[X]$ of degree at most $n-1$ and $b_n \in \Lambda$. Hence by induction

$$\begin{aligned}\overline{a(X)}b'(X)a(X) &= \overline{a(X)}(b''(X) + b_nX^n)a(X) \\ &= \overline{a(X)}b''(X)(a(X)) + (\overline{a(X)}b_n a(X))X^n \in \Lambda[X].\end{aligned}\quad \square$$

To define Bak's unitary group or the general quadratic group, we fix a (non-zero) central element $\lambda \in R$ with $\lambda\bar{\lambda} = 1$, and then consider the form

$$\psi_n = \begin{pmatrix} 0 & I_n \\ \lambda I_n & 0 \end{pmatrix}.$$

Bak's Unitary or General Quadratic Groups GQ:

$$\text{GQ}(2n, R, \lambda) = \{\sigma \in \text{GL}(2n, R) \mid \bar{\sigma}\psi_n\sigma = \psi_n\}.$$

Elementary Quadratic Matrices:

Let $\rho : \{1, 2, \dots, n\} \rightarrow \{n+1, n+2, \dots, 2n\}$ be defined by $\rho(i) = n+i$. Let e_{ij} be the matrix with 1 in the ij -th position and 0's elsewhere. For $a \in R$, and $1 \leq i, j \leq n$, we define

$$qe_{ij}(a) = I_{2n} + ae_{ij} - \bar{a}e_{\rho(j)\rho(i)} \text{ for } i \neq j,$$

$$qr_{ij}(a) = \begin{cases} I_{2n} + ae_{i\rho(j)} - \lambda\bar{a}e_{j\rho(i)} & \text{for } i \neq j \\ I_{2n} + ae_{\rho(i)j} & \text{for } i = j, \end{cases}$$

$$ql_{ij}(a) = \begin{cases} I_{2n} + ae_{\rho(i)j} - \bar{\lambda}\bar{a}e_{\rho(j)i} & \text{for } i \neq j \\ I_{2n} + ae_{\rho(i)j} & \text{for } i = j. \end{cases}$$

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda\bar{a}$, and hence it forces that $a \in \Lambda_{max}(R)$. One checks that these above matrices belong to $GQ(2n, R, \Lambda)$; cf. [1].)

n -th Elementary Quadratic Group $EQ(2n, R, \Lambda)$: The subgroup generated by $q\varepsilon_{ij}(a)$, $qr_{ij}(a)$ and $ql_{ij}(a)$, for $a \in R$ and $1 \leq i, j \leq n$. For uniformity we denote the elementary generators of $EQ(2n, R, \Lambda)$ by $\eta_{ij}(\ast)$.

It is clear that the stabilization map takes generators of $EQ(2n, R, \Lambda)$ to the generators of $EQ(2(n+1), R, \Lambda)$.

Remark 2.4. Throughout this article we shall assume that all ideals of R are involution-invariant, i.e., if J is an ideal, then $\bar{J} = J$. We also assume that $2n \geq 6$.

Definition 2.5. The relative general quadratic subgroup of $GQ(2n, R, \Lambda)$ with respect to the ideal J is defined by the set $\{\alpha \in GQ(2n, R, \Lambda) \mid \alpha \equiv I_{2n} \pmod{J}\}$ and it is denoted by $GQ(2n, R, \Lambda, J)$.

Definition 2.6. Let (R, Λ) be a form ring and $J \subset R$ be an ideal. The subgroup of $GQ(2n, R, \Lambda)$ generated by the matrices of the form $\eta_{ij}(x)\eta_{ji}(a)\eta_{ij}(x)^{-1}$, where $x \in R$ and $a \in J$, is called the relative elementary subgroup and is denoted by $EQ(2n, R, \Lambda, J)$.

Notation 2.7. A row $(a_1, a_2, \dots, a_n) \in R^n$ is said to be unimodular if there exists a vector $(b_1, b_2, \dots, b_n) \in R^n$ such that $\sum_{i=1}^n a_i b_i = 1$. The set of all unimodular rows of length n is denoted by $Um_n(R)$. We denote the set of all unimodular rows of length n which are congruent to $e_1 = (1, 0, \dots, 0)$ modulo the ideal J by $Um_n(R, J)$. For an ideal $J \subset R$ the extended ideal $J \otimes_R R[X]$ of $R[X]$ is denoted by $J[X]$. We will mostly use localizations with respect to two types of multiplicatively closed subsets of R , viz. $S = \{1, s, s^2, \dots\}$, where $s \in R$ is a non-nilpotent non-zero divisor; and $S = R \setminus \mathfrak{m}$ for some $\mathfrak{m} \in \text{Max}(R)$. By $J_s[X]$ and $J_{\mathfrak{m}}[X]$ we shall mean the extension of $J[X]$ in $R_s[X]$ and $R_{\mathfrak{m}}[X]$ respectively.

Definition 2.8. Let (R, Λ) be a form ring and $J \subset R$ be an ideal of R . The excision ring of R with respect to the ideal J is denoted by $R \oplus J$ and is defined by the set $\{(r, i) \mid r \in R, i \in J\}$ with the addition defined by $(r, i) + (s, j) = (r + s, i + j)$, and the multiplication defined by $(r, i)(s, j) = (rs, rj + is + ij)$. We can extend the involution $-$ to the ring $R \oplus J$ by setting $(r, i) = (\bar{r}, \bar{i})$. The element $(\lambda, 0) \in R \oplus J$ satisfies $(\lambda, 0)(\bar{\lambda}, 0) = (1, 0)$, where $(1, 0)$ is the identity element of $R \oplus J$. We can observe that the additive subgroup $(\Lambda \oplus J) \cap \Lambda_{max}(R \oplus J)$ satisfies the properties of $(\lambda, 0)$ -parameter on $R \oplus J$. We fix the notation $\Gamma \oplus J$ for the subgroup $(\Lambda \oplus J) \cap \Lambda_{max}(R \oplus J)$. Hence we get the form ring $(R \oplus J, \Gamma \oplus J)$.

Recall that there is a natural map $f : R \oplus J \rightarrow R$ given by $f(r, i) = r + i$. This map induces a canonical homomorphism on $GQ(2n, R \oplus J, \Gamma \oplus J)$. We shall use the same notation f to denote this map.

The following two key lemmas are proved in [10] for the traditional classical groups. Proofs are similar for the general quadratic groups.

Lemma 2.9. For $\alpha \in \text{EQ}(2n, R, \Lambda, J)$, there exists a matrix $\tilde{\alpha} \in \text{EQ}(2n, R \oplus J, \Gamma \oplus J)$ such that $f(\tilde{\alpha}) = \alpha$.

Lemma 2.10. For $\alpha \in \text{GQ}(2n, R, \Lambda, J)$, there exists a matrix $\tilde{\alpha} \in \text{GQ}(2n, R \oplus J, \Gamma \oplus J)$ such that $f(\tilde{\alpha}) = \alpha$.

3 Relative L-G Principle for the Transvection Subgroups

It is known that any module finite ring (i.e., finite over its center) R can be written as a direct limit of its finitely generated subrings. Also, $G(R, \lambda) = \varinjlim G(R_i, \lambda_i)$, where the limit is taken over all finitely generated subring of R . Hence, we can assume that $C(R)$ is Noetherian. For the rest of this section, we shall consider R to be a module finite ring with identity.

The local-global principle for the transvection subgroups (absolute cases) of the full automorphism groups was established in [4] for the traditional classical groups. Then, in [8] it was generalized for the general quadratic groups. In this section we deduce the relative L-G principle for the transvection subgroups.

The following results are proved in [10] for traditional classical groups of free modules, and the steps of the proof for the general quadratic groups are identical. Therefore, we state these results without proof.

For any column vector $v \in (R^{2n})^t$ we consider the row vector $\tilde{v} = \bar{v}^t \psi_n$.

Definition 3.1. We define a map $M : (R^{2n})^t \times (R^{2n})^t \rightarrow M(2n, R)$ and the inner product \langle , \rangle as follows:

$$M(v, w) = v \cdot \tilde{w} - \bar{\lambda} \tilde{w} v, \quad \langle v, w \rangle = \tilde{v} \cdot w$$

Lemma 3.2. Let (R, Λ) be a form ring and $v \in \text{EQ}(2n, R, \Lambda, J)e_1$. Let $w \in J^{2n}$ be a column vector such that $\langle v, w \rangle = 0$. Then $I_{2n} + M(v, w) \in \text{EQ}(2n, R, \Lambda, J)$.

Theorem 3.3 (Relative L-G principle). Let R be a ring and $J \subset R$ be an ideal of R . Let $\alpha(X) \in \text{GQ}(2n, R[X], \Lambda[X], J[X])$, with $\alpha(0) = I_{2n}$ be such that for every maximal ideal $\mathfrak{m} \in \text{Max}(C(R))$, we have $\alpha_{\mathfrak{m}}(X) \in \text{EQ}(2n, R_{\mathfrak{m}}[X], \Lambda_{\mathfrak{m}}[X], J_{\mathfrak{m}}[X])$. In that case, $\alpha(X) \in \text{EQ}(2n, R[X], \Lambda[X], J[X])$.

We recall some definitions and fix notations.

Definition 3.4. Let (R, Λ) be a form ring and P be a right R -module. A map $f : P \times P \rightarrow R$ is said to be a sesquilinear form if $f(pa, qb) = \bar{a}f(p, q)b$ for all $p, q \in P$ and $a, b \in R$. A map $q : P \rightarrow R/\Lambda$ is said to be a quadratic form if $q(p) = f(p, p) + \Lambda$, where f is a sesquilinear form on P . With respect to a sesquilinear form on P , we can define an associated λ -Hermitian form $h : P \times P \rightarrow R$ by $h(p, q) = f(p, q) + \lambda f(q, p)$. The triplet (P, h, q) is called a quadratic module.

Definition 3.5. Let (P, q, h) be a quadratic module and $\text{GL}(P)$ be the full automorphism group of P . The quadratic module P is said to be non-singular if P is a projective R -module and the associated λ -Hermitian form is non-singular. For a non-singular P , the general quadratic group of P is defined as follows:

$$\text{GQ}(P, \Lambda, q, h) = \{\alpha \in \text{GL}(P) \mid h(\alpha u, \alpha v) = h(u, v), q(\alpha u) = q(u)\}$$

We will denote by $\text{GQ}(P, \Lambda, J)$ the set $\{\alpha \in \text{GQ}(P, \Lambda) \mid \alpha \equiv \text{Id} \pmod{JP}\}$.

Definition 3.6. Let (P, h, q) be a quadratic module over (R, Λ) and $J \subset R$ be an ideal of R . Let $u, v \in P$ and $a \in R$ be such that $f(u, u) \in \Lambda$, $h(u, v) = 0$ and $f(v, v) = a \pmod{\Lambda}$. Then the transvection map $\sigma = \sigma_{u,v,a} : P \rightarrow P$ is defined by

$$\sigma(x) = x + uh(v, x) - v\bar{\lambda}h(u, x) - u\bar{\lambda}ah(u, x).$$

The set of all transvections of P will be denoted as $\text{T}(P, \Lambda)$. A map $\sigma \in \text{T}(P, \Lambda)$ is said to be a transvection relative to J if either u or v belongs to the submodule JP . The set of all transvections relative to the ideal J will be denoted by $\text{T}(P, \Lambda, J)$.

Definition 3.7. Let (P, h, q) be a quadratic module over a form ring (R, Λ) and $J \subset R$ be an ideal of R . Let Q be the quadratic module $P \perp \mathbb{H}(R)$, where $\mathbb{H}(R)$ denotes the hyperbolic form $R \perp R^*$. Then the transvections which are of the form

$$q = (p, a, b) \mapsto (p - aq, a, b + h(p, q)),$$

or,

$$q = (p, a, b) \mapsto (p - bq, a + h(p, q), b),$$

where $a \in R$, $b \in R^*$, $p, q \in P$, are called elementary transvections. The set of all elementary transvections is denoted by $\text{ET}(Q, \Lambda)$. An elementary transvection is said to be elementary transvection relative to J if $q \in JQ$. The subgroup of $\text{ET}(Q, \Lambda)$ generated by elementary transvections relative to J is denoted by $\text{ET}(JQ)$. And we denote $\text{ET}(Q, \Lambda, JQ)$ by the normal closure of $\text{ET}(JQ)$ in $\text{ET}(Q, \Lambda)$. We also use the notation $\text{ET}(Q, \Lambda, J)$ for the group $\text{ET}(Q, \Lambda, JQ)$.

Notation 3.8. Let P be a finitely generated quadratic R module of rank $2n$ with a fixed form $\langle \cdot, \cdot \rangle$. Denote by Q the module $P \perp \mathbb{H}(R)$ and by $Q[X]$ the module $(P \perp \mathbb{H}(R))[X]$. We assume that the rank of the quadratic module is $2n \geq 6$.

We shall also assume the following two hypotheses:

(H1) For every maximal ideal \mathfrak{m} of R , the quadratic module $Q_{\mathfrak{m}}$ is isomorphic to $R_{\mathfrak{m}}^{2n+2}$ for the standard bilinear form $\mathbb{H}(R_{\mathfrak{m}}^{n+1})$.

(H2) For every non-nilpotent $s \in R$, if the projective module M_s is a free R_s -module, then the quadratic module M_s is isomorphic to R_s^{2n+2} for the standard bilinear form $\mathbb{H}(R_s^{n+1})$.

Lemma 3.9 ([7, Lemma 5.7], [10, Lemma 3.6]). *The group*

$$\mathrm{GQ}(2n, R[X], \Lambda[X], (X)) \cap \mathrm{EQ}(2n, R[X], \Lambda[X], J[X])$$

is generated by the elements of the form

$$\varepsilon \eta_{ij}(Xh(X)) \varepsilon^{-1},$$

where $\varepsilon \in \mathrm{EQ}(2n, R, \Lambda)$, $h(X) \in J[X]$ and $\eta_{ij}(\ast)$ are the elementary generators of the group $\mathrm{EQ}(2n, R[X], \Lambda[X])$.

Lemma 3.10 ([10, Corollary 3.8]). *If $\eta = \eta_1 \eta_2 \dots \eta_r$, where each η_j is an elementary generator, and $h(Y) \in J[Y]$, then there are elements $h_t(X, Y) \in J[X, Y]$ such that*

$$\eta \eta_{pq}(X^{2^r} h(Y)) \eta^{-1} = \prod_{t=1}^k \eta_{p_t q_t}(X^m h(X, Y)).$$

We now recall some standard results:

Lemma 3.11. *Let R be a ring and K a finitely presented left (right) R -module, and let L be any left (right) R -module. Then we have a natural isomorphism:*

$$f : \mathrm{Hom}_R(K, L)[X] \rightarrow \mathrm{Hom}_{R[X]}(K[X], L[X]).$$

Lemma 3.12. *Let S be a multiplicatively close subset of a ring R . Let K be a finitely presented R -module and L be any R -module. Then we have a natural isomorphism*

$$g : S^{-1}(\mathrm{Hom}_R(K, L)) \rightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}K, S^{-1}L).$$

The following lemma is used frequently (sometimes in a subtle way) in the proof of the main results.

Lemma 3.13 (cf. [19, Lemma 5.1]). *Let A be Noetherian ring and $0 \neq s \in A$. Then there exists a natural number k such that the homomorphism*

$$\mathrm{G}(A, s^k A, s^k \lambda) \rightarrow \mathrm{G}(A_s, \lambda_s)$$

induced by the localization homomorphism $A \rightarrow A_s$ is injective.

Now we prove the relative L-G principle by using Lemma 2.9.

Proposition 3.14 (Relative Dilation Principle). *Let R be an almost commutative ring (i.e., an associative ring which is finite over its center $C(R)$) and $J \subset R$ be an ideal. Let P and Q be as in 3.8. Let s be a non-nilpotent element of R such that P_s is a free module. Let $\sigma(X) \in \mathrm{GQ}(Q[X], \Lambda[X], J[X])$ with $\sigma(0) = Id$ and suppose additionally that $\sigma_s(X) \in \mathrm{EQ}(2n+2, R_s[X], \Lambda_s[X], J_s[X])$. Then there exists $\hat{\sigma}(X) \in \mathrm{ET}(Q[X], \Lambda[X], J[X])$ and $l > 0$ such that $\hat{\sigma}(X)$ localizes to $\sigma(bX)$ for some $b \in (s^l)$ and $\hat{\sigma}(0) = Id$.*

Proof. Since elementary transvections can always be lifted, then we may assume that R is reduced. We will show that there exists $l > 0$ such that

$$\sigma(bX) \in \text{ET}(Q[X], \Lambda[X], J[X])$$

for all $b \in (s^k)$ and for all $k \geq l$.

As $\sigma(0) = \text{Id}$, by Lemma 3.9, we can write $\sigma_s(X) = \prod_k \gamma_k \eta_{i_k j_k}(X \lambda_k(X)) \gamma_k^{-1}$, where $\gamma_k \in \text{EQ}(2n + 2, R_s, \Lambda_s)$, and $\lambda_k(X) \in J_s[X]$. Hence, by the proof of Lemma 2.9, there exists $\tilde{\sigma}_{(s,0)}(X) \in \text{EQ}(2n + 2, (R_s \oplus J_s)[X], (\Gamma_s \oplus J_s)[X])$ such that

$$\tilde{\sigma}_{(s,0)}(X) = \prod_k \tilde{\gamma}_k \widetilde{\eta_{i_k j_k}}(0, X \lambda_k(X)) \tilde{\gamma}_k^{-1},$$

where $\phi_s(\tilde{\gamma}_k) = \gamma_k, (0, X \lambda_k(X)) \in (R \oplus J)_{(s,0)}[X]$, the $\widetilde{\eta_{i_k j_k}}(*)$ are the elementary generators of $\text{EQ}(2n + 2, (R_s \oplus J_s)[X], (\Gamma_s \oplus J_s)[X])$ and $\phi : R \oplus J \rightarrow R$ is defined by $\phi((a, i)) = a + i$ and $\Gamma_s \oplus J_s = (\Gamma \oplus J)_s$ is the localization of $\Gamma \oplus J$ with respect to the set $\{1, s, s^2, \dots\}$. Hence, for $d > 0$, we have that

$$\tilde{\sigma}_{(s,0)}(XT^{2d}) = \prod_k \tilde{\gamma}_k \widetilde{\eta_{i_k j_k}}((0, XT^{2d} \lambda_k(XT^{2d})) \tilde{\gamma}_k^{-1},$$

for some $\gamma_k \in \text{EQ}(2n + 2, R_s, \Lambda_s)$. Using Lemma 3.10 and standard commutator formulas (see [1, Lemma 3.16, pg. 43]), we get that $\tilde{\sigma}_{(s,0)}(XT^{2d}) = \prod_t \widetilde{\eta_{p_t q_t}}(T \mu_t(X))$, for some $\mu_t(X) \in (R \oplus J)_{(s,0)}[X]$ with $p_t = 1$ or $q_t = 1$.

Since P_s is a free R_s -module, then we have

$$(P \oplus J)_{(s,0)}[X, T] \cong (R \oplus J)_{(s,0)}^{2n}[X, T] \cong (P \oplus J)_{(s,0)}[X, T]^*.$$

Thus using the isomorphism, polynomials in $(P \oplus J)_{(s,0)}[X, T]$ can be regarded as linear forms.

First we consider the case: $p_t = 1$. Let $p_1^*, p_2^*, \dots, p_n^*, p_{-1}^*, \dots, p_{-n}^*$ be the standard basis of $(P \oplus J)_{(s,0)}$. Let $s^m p_i^* \in P \oplus J$ for some $m > 0$ and $i = \pm 1, \pm 2, \dots, \pm n$. Let $e_{\pm i}^*$ be the standard basis of $(R \oplus J)^{2n}$. Then for $q_t = \pm i$, consider the element $T \mu_t(X) e_{\pm i}^*$ as an element in $(P \oplus J)_{(s,0)}[X, T]^*$. As $(P \oplus J)_{(s,0)}$ is free, by Lemma 3.11, we can say $T \mu_t(X) e_{\pm i}^*$ is a polynomial in T . Again, by Lemma 3.12, there exists $k_1 > 0$ such that k_1 is the maximum power of $(s, 0)$ occurring in the denominator of $\mu_t(X) e_{\pm i}^*$. Choose $l_1 \geq \max(k, m)$.

Now consider the case $q_t = 1$. Then, for $p_t = \pm j$, $T \mu_t(X) e_{\pm j}^* \in (P \oplus J)_{(s,0)}[X, T]$. By a similar argument we can consider $T \mu_t(X) e_{\pm j}^*$ as a polynomial in T and hence there exists $k_2 > 0$ such that k_2 is maximum power of $(s, 0)$ occurred in $\mu_t e_{\pm j}^*$. Now choose $l_2 \geq \max(k_2, m)$. For $l \geq \max(l_1, l_2)$, under the transformation $T \mapsto (s, 0)^l T$, $\tilde{\sigma}_{(s,0)}((b, 0)XT^{2d})$ is defined over $(Q \oplus J)[X, T]$, i.e., there exists some element

$$\tilde{\sigma}(X, T) \in \text{ET}((Q \oplus J)[X, T])$$

such that $\tilde{\sigma}_{(s,0)}(X, T) = \tilde{\sigma}_{(s,0)}((b, 0)XT^{2d})$. Replacing $T = (1, 0)$ and using Lemma 3.13, one gets $\tilde{\sigma}((b, 0)X) \in \text{ET}(Q[X] \oplus J[X], (\Gamma \oplus J)[X], 0 \oplus J[X])$. Hence, the result follows by applying ϕ . □

Consequence: **Relative L-G principle for the transvection subgroups:**

Theorem 3.15. *Let R be an almost commutative ring and $J \subset R$ be an ideal. Let P and Q be as in 3.8. Let $\sigma(X) \in \text{GQ}(Q[X], \Lambda[X], J[X])$ with $\sigma(0) = \text{Id}$.*

If $\sigma_{\mathfrak{m}}(X) \in \text{EQ}(2n + 2, R_{\mathfrak{m}}[X], \Lambda_{\mathfrak{m}}[X], J_{\mathfrak{m}}[X])$ for all $\mathfrak{m} \in \text{Max}(C(R))$, then we have $\sigma(X) \in \text{ET}(Q[X], \Lambda[X], J[X])$.

Proof. Follows by arguing as in the proof of [8, Lemma 3.10], and using Proposition 3.14. \square

4 Relative Stability for Quadratic K_1

The aim of this section is to establish the K_1 -stability of the relative transvection groups as an application of Theorem 3.15. For the absolute case we refer to [5] and [8].

Definition 4.1. Let R be an associative ring with identity and $J \subset R$ an ideal. Consider the ring $D = \{(a, b) \in R \times R : a - b \in J\}$ with addition and multiplication defined component wise. We call it the double ring relative to the ideal J . For a form ring (R, Λ) , one extends the involution of $- : R \rightarrow R$ to the ring D , defining $- : D \rightarrow D$ by $(a, b) = (\bar{a}, \bar{b})$. We fix the element (λ, λ) and define $\Lambda' = \{(a, b) \in \Lambda \times \Lambda \mid a - b \in J\}$. Then one can show that (D, Λ') is a form ring.

Definition 4.2. Let (R, Λ, λ) and (S, Λ', λ') be two form rings. A ring homomorphism $f : R \rightarrow S$ is said to be a morphism of form rings if $f(\bar{r}) = \overline{f(r)}$, $f(\Lambda) \subseteq \Lambda'$ and $f(\lambda) = \lambda'$.

Lemma 4.3. *Let (R, Λ) be a form ring and $J \subset R$ be a two sided ideal. Then the form rings (D, Λ') and $(R \oplus J, \Lambda \oplus J)$ are isomorphic.*

Proof. Consider the homomorphisms $f : D \rightarrow R \oplus J$ defined by $f(a, b) = (a, b - a)$ and $g : R \oplus J \rightarrow D$ defined by $g(a, i) = (a, a + i)$. It can be checked that both f and g are form homomorphisms. They are inverses of each other. \square

Lemma 4.4 ([20, Proposition 3.1]). *Let A be a commutative Noetherian ring of (Krull) dimension d and $J \subset A$ be an ideal. Then the ring D is also a commutative ring of dimension d .*

Proof. Clearly D_A is a commutative Noetherian ring. We first prove that dimension of $A \oplus J$ is d , and the rest follows from Lemma 4.3. Clearly A can be identified with the sub-ring $\{(r, 0) \mid r \in A\}$ of $A \oplus J$. The element $(0, i)$ is integral over A since we have that $(0, i)^2 - (i, 0)(0, i) = (0, 0)$. Hence every element of $A \oplus J$ is integral over R , and therefore $\dim(A) = \dim(A \oplus J)$. \square

Recall K_1 -stability result of Bak–Petrov–Tang (cf. [5]) for general quadratic groups in the absolute case.

Theorem 4.5 ([5]). *Let R be an almost commutative with $\dim(C(R)) = d$. Consider the form ring (R, Λ) . Then the stabilization map*

$$\frac{\text{GQ}(2n, R, \Lambda)}{\text{EQ}(2n, R, \Lambda)} \rightarrow \frac{\text{GQ}(2n+2, R, \Lambda)}{\text{EQ}(2n+2, R, \Lambda)}$$

is an isomorphism for $2n \geq \max(6, 2d+4)$.

We prove the above result in the relative case.

Theorem 4.6. *Let R be a form ring which is finitely generated over its center $C(R)$, with $\dim(C(R)) = d$ and let $J \subset R$ be a two-sided ideal of R . Then the stabilization map*

$$\frac{\text{GQ}(2n, R, \Lambda, J)}{\text{EQ}(2n, R, \Lambda, J)} \rightarrow \frac{\text{GQ}(2n+2, R, \Lambda, J)}{\text{EQ}(2n+2, R, \Lambda, J)}$$

is an isomorphism for $2n \geq \max(6, 2d+4)$.

Proof. Consider the stabilization map $\phi : \text{KQ}_{1,2n} \rightarrow \text{KQ}_{1,2n+2}$, where

$$\text{KQ}_{1,2n}(R, \Lambda, J) = \frac{\text{GQ}(2n, R, \Lambda, J)}{\text{EQ}(2n, R, \Lambda, J)}.$$

By [17, 5.3.22], we have following sequences are exact

$$1 \longrightarrow \text{GQ}(2n, R, \Lambda, J) \xrightarrow{i} \text{GQ}(2n, D, \Lambda') \xrightarrow{p_2} \text{GQ}(2n, R, \Lambda) \longrightarrow 1,$$

$$1 \longrightarrow \text{EQ}(2n, R, \Lambda, J) \xrightarrow{i} \text{EQ}(2n, D, \Lambda') \xrightarrow{p_2} \text{EQ}(2n, R, \Lambda) \longrightarrow 1,$$

where the map i is induced from the map $M_{2n}(J) \rightarrow M_{2n}(D)$ given by $i(\alpha) = (\alpha, I_{2n})$ and the map p_2 is induced from the projection map $p_2 : D \rightarrow R$ given by $p_2((a, b)) = b$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{EQ}(2n, R, \Lambda, J) & \longrightarrow & \text{EQ}(2n, D, \Lambda') & \longrightarrow & \text{EQ}(2n, R, \Lambda) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{GQ}(2n, R, \Lambda, J) & \longrightarrow & \text{GQ}(2n, D, \Lambda') & \longrightarrow & \text{GQ}(2n, R, \Lambda) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{KQ}_{1,2n}(R, \Lambda, J) & \longrightarrow & \text{KQ}_{1,2n}(D, \Lambda') & \longrightarrow & \text{KQ}_{1,2n}(R, \Lambda) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

Taking limits over n in the lower row, we get an exact sequence

$$1 \longrightarrow \mathrm{KQ}_1(R, \Lambda, J) \longrightarrow \mathrm{KQ}_1(D, \Lambda') \longrightarrow \mathrm{KQ}_1(R, \Lambda) \longrightarrow 1,$$

and for any n , we get a homomorphism of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{KQ}_{1,2n}(R, \Lambda, J) & \longrightarrow & \mathrm{KQ}_{1,2n}(D, \Lambda') & \longrightarrow & \mathrm{KQ}_{1,2n+2}(R, \Lambda) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{KQ}_1(R, \Lambda, J) & \longrightarrow & \mathrm{KQ}_1(D, \Lambda') & \longrightarrow & \mathrm{KQ}_1(R, \Lambda) \longrightarrow 1. \end{array}$$

For $2n \geq 2d + 4$ the right most map is an isomorphism by Theorem 4.5. Since R is finitely generated over $C(R)$, then it can be checked that D is finitely generated over $C(D)$. Since $C(D)$ is a double ring of $C(R)$ relative to the ideal $C(R) \cap J$, by Lemma 4.4, we get $\dim(C(D)) = d$. Hence for $2n \geq 2d + 4$, the middle homomorphism is also an isomorphism by Theorem 4.5. Hence it follows that the left most map is an isomorphism. \square

Now we prove the K_1 -stability of relative transvection groups as an application of relative local-global principle of transvection groups. For this we need to recall the following result of Vaserstein.

Lemma 4.7 ([8, Lemma 4.2]). *Let (R, Λ) be a form ring finitely generated over $C(R)$ with Krull dimension of $C(R) = d$ and $J \subset R$ be an two-sided ideal of R . Let P, Q be as in Notation 3.8. Let the rank of Q be $2n \geq \mathrm{Max}(6, 2d + 2)$. Then the group of elementary transvections $\mathrm{ET}(Q \perp \mathbb{H}(R), J)$ acts transitively on the set $\mathrm{Um}(Q \perp \mathbb{H}(R), J)$ of unimodular elements which are congruent to $(0, \dots, 0, 1, 0)$ modulo J .*

Theorem 4.8. *Let (R, Λ) be a form ring finitely generated over $(C(R))$ with Krull dimension d and let $J \subset R$ be a two-sided ideal. Let P, Q be as defined in Notation 3.8 and let the rank of Q be $2n \geq \max(6, 2d + 4)$. Then the stabilization map*

$$i_{2n} : \mathrm{KQ}_{1,2n}(Q, J) \rightarrow \mathrm{KQ}_{1,2n+2}(Q \perp \mathbb{H}(R), J)$$

is an isomorphism.

Proof. The surjectivity part follows from Lemma 4.7. In order to prove the injectivity part, let $\alpha \in \mathrm{GQ}(Q, J)$ be such that $\tilde{\alpha} = \alpha \perp \mathrm{Id}$ lies in $\mathrm{ET}(M \perp \mathbb{H}(R), J)$. Let $\varphi(X)$ be the isotropy between $\tilde{\alpha}$ and Id . Now by similar argument as given in [8, Theorem 4.4], we can get an isotropy $\tilde{\varphi}(X) \in \mathrm{GQ}(Q[X], J[X])$ between α and Id . Now, by localizing $\tilde{\varphi}$ at a maximal ideal $\mathfrak{m} \in \mathrm{Max}(C(R))$, $\tilde{\varphi}_{\mathfrak{m}}(X)$ becomes (relative) stably elementary. By Theorem 4.6 it follows that $\tilde{\varphi}_{\mathfrak{m}}(X)$ is actually (relative) elementary. Therefore, by the relative L-G principle (Theorem 3.15) one gets $\tilde{\varphi}(X) \in \mathrm{ET}(Q[X], J[X])$. Hence $\alpha = \tilde{\varphi}(1) \in \mathrm{ET}(Q, J)$. \square

5 Structure of Unstable Quadratic K_1 Groups; Relative Case

We devote this section to discuss the study of nilpotent property of (relative) unstable K_1 -groups. Throughout this section we assume R is a commutative ring with identity, i.e., we are considering the trivial involution and $n \geq 3$. By $SQ(2n, R, \lambda)$ we shall denote the subgroup of $GQ(2n, R, \lambda)$ with matrices of determinant 1, and analogously for the relative cases.

Let G be a group. Define Z^i inductively by $Z^0 = G$, $Z^1 = [G, G]$ and $Z^i = [G, G^{i-1}]$. The group G is said to be nilpotent if $Z^r = \{e\}$ for some $r > 0$.

Definition 5.1. A group G is called nilpotent-by-abelian if it has a normal subgroup H such that H is nilpotent and G/H is abelian.

In [2], A.Bak proved that the unstable $K_1(R)$ group of $GL_n(R)$ is nilpotent-by-abelian for $n \geq 3$, and hence $K_1(R)$ is solvable. Later it was generalized by R. Hazrat for the general quadratic groups over module finite rings (cf. [18]). In [19] Hazrat–Vavilov revisited this problem for ordinary classical Chevalley groups (that is, types A, C, and D) and finally extends it further to the exceptional Chevalley groups (that is, types E, F, and G). A simpler and shorter proof is given in [4] for the linear, symplectic and orthogonal groups (absolute cases). In [7], the first author proved this result for the general Hermitian groups and the same proof also works for the general quadratic groups. The relative cases are proved in [4] for the traditional classical groups. In this article we prove the result for the relative general quadratic groups, and consequently we get the result for the module cases as an application of (relative) L-G principle for the transvection subgroups.

Theorem 5.2. *The quotient group $SQ(2n, R, \Lambda, J)/EQ(2n, R, \Lambda, J)$ is nilpotent for $n \geq 3$. The class of nilpotency is at most $\max(1, d + 2 - n)$ where $d = \dim R$.*

Proof. Note that for $n \geq d + 2$, the quotient group

$$G = SQ(2n, R, \Lambda, J)/EQ(2n, R, \Lambda, J)$$

is abelian and hence nilpotent. So we consider the case $n < d + 2$. Let us fix a natural number n . We prove the theorem by induction on $d = \dim R$. Let $m = d + 2 - n$ and $\alpha = [\beta, \gamma]$ for some $\beta \in G$ and $\gamma \in Z^{m-1}$. Clearly the result is true for $d = 0$. Let $\tilde{\beta}$ be the pre-image of β and $\tilde{\gamma}$ be the pre-image of γ under the map

$$SQ(2n, R, \Lambda, J) \rightarrow G.$$

Consider the double ring (D, Λ') as defined in 4.1. By [17, 5.3.22], we have the following exact sequences:

$$1 \longrightarrow SQ(2n, R, J, \Lambda) \xrightarrow{i} SQ(2n, D, \Lambda') \xrightarrow{p_2} SQ(2n, R, \Lambda) \longrightarrow 1$$

$$1 \longrightarrow EQ(2n, R, J, \Lambda) \xrightarrow{i} EQ(2n, D, \Lambda') \xrightarrow{p_2} EQ(2n, R, \Lambda) \longrightarrow 1$$

where the map i is induced from the map $M_{2n}(J) \rightarrow M_{2n}(D)$ given by $i(\alpha) = (\alpha, I_{2n})$ and p_2 is induced from the projection map $p_2 : D \rightarrow R$ given by $p_2((a, b)) = b$. Now, in the group $H = \text{SQ}(2n, D, \Lambda')/\text{EQ}(2n, D, \Lambda')$ we have $i(\tilde{\gamma}) \in H^m$. Since $\dim(R) = \dim(D)$ (by Lemma 4.4), by the result of the absolute cases, we get $[i(\tilde{\beta}), i(\tilde{\gamma})] \in \text{EQ}(2n, D, \Lambda')$. It can be checked that $i([\tilde{\beta}, \tilde{\gamma}]) = [i(\tilde{\beta}), i(\tilde{\gamma})]$ and the image of this element under p_2 is identity. Hence by the above diagram one gets $[\tilde{\beta}, \tilde{\gamma}] \in \text{EQ}(2n, R, J)$, consequently $[\beta, \gamma]$ is trivial in G . Next, consider the pre-images β_1 of $\tilde{\beta}$ and γ_1 of $\tilde{\gamma}$ under the map $\text{SQ}(2n, R \oplus J, \Gamma \oplus J) \rightarrow \text{SQ}(2n, R, \Lambda, J)$. Now by the result of absolute cases we get $[\beta_1, \gamma_1] \in \text{EQ}(2n, R \oplus J, \Gamma \oplus J)$. It can also be checked that the same commutator $[\beta_1, \gamma_1]$ is in $\text{EQ}(2n, R \oplus J, \Gamma \oplus J, 0 \oplus J)$. By projecting $R \oplus J$ onto R , one gets $[\tilde{\beta}, \tilde{\gamma}] \in \text{EQ}(2n, R, J, \Lambda)$, and hence $\alpha = \{1\}$ in G . \square

Corollary 5.3. *Let (R, Λ) be a commutative form ring and $J \subset R$ be an ideal of R . Then the quotient group $\text{GQ}(2n, R, \Lambda, J)/\text{ET}(2n, R, \Lambda, J)$ is nilpotent-by-abelian for $n \geq 3$.*

Corollary 5.4. *Let (R, Λ) be a commutative form ring and $J \subset R$ be an ideal of R . Consider the notation as in 3.8. Let $d = \dim(R)$ and $t =$ the local rank of Q . The quotient group $\text{T}(Q, J)/\text{ET}(Q, J)$ is nilpotent of class at most $\max(1, d + 3 - t/2)$.*

Proof. The proof is same as [4, Theorem 4.1]. \square

6 Bass' Nil Group $\text{NKQ}_1(\mathbf{R})$

The group $\text{NK}_1(R) = \ker(K_1(R[X]) \rightarrow K_1(R))$; $X = 0$ is called the Bass' nil-group of R . This is the subgroup consisting of elements $[\alpha(X)] \in K_1(R[X])$ such that $[\alpha(0)] = [I]$, and hence $K_1(R[X]) \cong \text{NK}_1(R) \oplus K_1(R)$. The aim of the next sections is to study some properties of Bass nil-groups NK_1 for the general quadratic groups or Bak's unitary groups.

In this section we recall some basic definitions and properties of the representatives of $\text{NKQ}_1(R)$. We represent any element of $M_{2n}(R)$ as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } a, b, c, d \in M_n(R).$$

For a matrix as above, we call $\begin{pmatrix} a & b \end{pmatrix}$ the upper half of α . Let (R, λ, Λ) be a form ring. By setting $\bar{\Lambda} = \{\bar{a} \mid a \in \Lambda\}$ we get another form ring $(R, \bar{\lambda}, \bar{\Lambda})$. We can extend the involution of R to $M_n(R)$ by setting $(a_{ij})^* = (\bar{a}_{ji})$. For details, see [21], [22].

Definition 6.1. Let (R, λ, Λ) be a form ring. A matrix $\alpha = (a_{ij}) \in M_n(R)$ is said to be Λ -Hermitian if $\alpha = -\lambda\alpha^*$ and all the diagonal entries of α are contained in Λ . A matrix $\beta \in M_n(R)$ is said to be $\bar{\Lambda}$ -Hermitian if $\beta = -\bar{\lambda}\beta^*$ and all the diagonal entries of β are contained in $\bar{\Lambda}$.

Remark 6.2. A matrix $\alpha \in M_n(R)$ is Λ -Hermitian if and only if α^* is $\bar{\Lambda}$ -Hermitian.

Lemma 6.3 ([21, Example 2]). *Let $\beta \in \text{GL}_n(R)$ be a Λ -Hermitian matrix. Then the matrix $\alpha^*\beta\alpha$ is Λ -Hermitian for every $\alpha \in \text{GL}_n(R)$.*

Definition 6.4. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{M}_{2n}(R)$ be a matrix. Then α is said to be a Λ -quadratic matrix if one of the following equivalent conditions holds:

1. $\alpha \in \text{GQ}(2n, R, \Lambda)$ and the diagonal entries of the matrices a^*c, b^*d are in Λ ,
2. $a^*d + \lambda c^*b = \text{I}_n$ and the matrices a^*c, b^*d are Λ -Hermitian,
3. $\alpha \in \text{GQ}(2n, R, \Lambda)$ and the diagonal entries of the matrices ab^*, cd^* are in Λ ,
4. $ad^* + \lambda bc^* = \text{I}_n$ and the matrices ab^*, cd^* are Λ -Hermitian.

Lemma 6.5. *Let $\alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \text{M}_{2n}(R)$. Then $\alpha \in \text{GQ}^\lambda(2n, R, \Lambda)$ if and only if we have $a \in \text{GL}_n(R)$ and $d = (a^*)^{-1}$.*

Proof. Let $\alpha \in \text{GQ}^\lambda(2n, R, \Lambda)$. In view of condition 2. of Definition 6.4, we have, $a^*d = \text{I}_n$. Hence a is invertible and $d = (a^*)^{-1}$. The converse holds by condition 2. of Definition 6.4. \square

Definition 6.6. Let $\alpha \in \text{GL}_n(R)$ be a matrix. A matrix of the form $\begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^*)^{-1} \end{pmatrix}$ is denoted by $\mathbb{H}(\alpha)$ and is said to be hyperbolic.

Remark 6.7. In a similar way we can show that a matrix of the form $T_{12}(\beta) := \begin{pmatrix} \text{I}_n & \beta \\ 0 & \text{I}_n \end{pmatrix}$ is a Λ -quadratic matrix if and only if β is $\bar{\Lambda}$ -Hermitian. Similarly, a matrix of the form $T_{21}(\gamma) := \begin{pmatrix} \text{I}_n & 0 \\ \gamma & \text{I}_n \end{pmatrix}$ is a Λ -quadratic matrix if and only if γ is Λ -Hermitian.

Likewise, in the quadratic case we can define the notion of Λ -elementary quadratic groups in the following way:

Definition 6.8. The Λ -elementary quadratic group is denoted by $\text{EQ}^\lambda(2n, R, \Lambda)$ and defined to be the group generated by the $2n \times 2n$ matrices of the following forms:

- $\mathbb{H}(\alpha)$, here $\alpha \in \text{E}_n(R)$,
- $T_{12}(\beta)$, where β is $\bar{\Lambda}$ -Hermitian, and
- $T_{21}(\gamma)$, where γ is Λ -Hermitian.

Lemma 6.9. *Let $A = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \text{M}_{2n}(R)$. Then $A \in \text{GQ}^\lambda(2n, R, \Lambda)$ if and only if we have $\alpha \in \text{GL}_n(R)$, $\delta = (\alpha^*)^{-1}$ and $\alpha^{-1}\beta$ is $\bar{\Lambda}$ -Hermitian. In this case*

$$A \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}.$$

Proof. Let $A \in \text{GQ}^\lambda(2n, R, \Lambda)$. Then, by condition 4. of Definition 6.4, we have that $\alpha\delta^* = I_n$ and $\alpha\beta^*$ is Λ -Hermitian. Hence α is invertible and $\delta = (\alpha^*)^{-1}$. For $\alpha^{-1}\beta$, we get

$$(\alpha^{-1}\beta)^* = \beta^*(\alpha^{-1})^* = \alpha^{-1}(\alpha\beta^*)(\alpha^{-1})^*,$$

which is Λ -Hermitian by Lemma 6.3. Hence $\alpha^{-1}\beta$ is $\bar{\Lambda}$ -Hermitian. Conversely, condition on A fulfills condition 4. of Definition 6.4. Hence A is Λ -quadratic. Now, since $\alpha^{-1}\beta$ is $\bar{\Lambda}$ -Hermitian, $T_{12}(-\alpha^{-1}\beta) \in \text{EQ}^\lambda(2n, R, \Lambda)$, and $AT_{12}(\alpha^{-1}\beta) = \mathbb{H}(\alpha)$. Thus $A \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}$. \square

Arguing similarly one gets the following:

Lemma 6.10. *Let $B = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in M_{2n}(R)$. Then $B \in \text{GQ}^\lambda(2n, R, \Lambda)$ if and only if $\alpha \in \text{GL}_n(R)$, $\delta = (\alpha^*)^{-1}$ and γ is Λ -Hermitian. In this case*

$$B \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}.$$

Lemma 6.11. *Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GQ}^\lambda(2n, R, \Lambda)$. If $a \in \text{GL}_n(R)$ then*

$$\alpha \equiv \mathbb{H}(a) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}.$$

Proof. By same argument as given in Lemma 6.9, we have $a^{-1}b$ is $\bar{\Lambda}$ -Hermitian. Hence $T_{12}(-a^{-1}b) \in \text{EQ}^\lambda(2n, R, \Lambda)$, and consequently $\alpha T_{12}(-a^{-1}b) = \begin{pmatrix} a & 0 \\ c & d' \end{pmatrix} \in \text{GQ}^\lambda(2n, R, \Lambda)$ for some $d' \in \text{GL}_n(R)$. Hence by Lemma 6.10, we get

$$\alpha T_{12}(-a^{-1}b) \equiv H(a) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}.$$

Hence $\alpha \equiv H(a) \pmod{\text{EQ}^\lambda(2n, R, \Lambda)}$. \square

Definition 6.12. Consider the embedding:

$$\text{GQ}^\lambda(2n, R, \Lambda) \rightarrow \text{GQ}^\lambda(2n+2, R, \Lambda),$$

We use the following notation:

$$\text{GQ}^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} \text{GQ}^\lambda(2n, R, \Lambda),$$

$$\text{EQ}^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} \text{EQ}^\lambda(2n, R, \Lambda).$$

In view of quadratic analog of Whitehead Lemma, we have that the group $\text{EQ}^\lambda(R, \Lambda)$ coincides with the commutator of $\text{GQ}^\lambda(R, \Lambda)$. Therefore, the group

$$\text{KQ}_1^\lambda(R, \Lambda) := \frac{\text{GQ}^\lambda(R, \Lambda)}{\text{EQ}^\lambda(R, \Lambda)}$$

is well-defined. The class of a matrix $\alpha \in \text{GQ}^\lambda(R, \Lambda)$ in the group $\text{KQ}_1^\lambda(R, \Lambda)$ is denoted by $[\alpha]$. In this way we obtain a K_1 -functor KQ_1^λ acting from the category of form rings to the category of abelian groups.

Definition 6.13. In the quadratic case, the kernel of the group homomorphism

$$\mathrm{KQ}_1^\lambda(R[X], \Lambda[X]) \rightarrow \mathrm{KQ}_1^\lambda(R, \Lambda)$$

induced from the form ring homomorphism $(R[X], \Lambda[X]) \rightarrow (R, \Lambda); X \mapsto 0$ is denoted by $\mathrm{NKQ}_1^\lambda(R, \Lambda)$.

Remark 6.14. Since the class of Λ -quadratic groups is a subclass of the class of quadratic groups, the local-global principle holds for Λ -quadratic groups. We use this fact throughout the next section.

7 Absence of torsion in $\mathrm{NKQ}_1^\lambda(R, \lambda)$

In [23], J. Stienstra showed that $\mathrm{NK}_1(R)$ is a $W(R)$ -module, where $W(R)$ is the ring of big Witt vectors (cf. [12] and [25]). Consequently, in ([24], §3), C. Weibel observed that if k is a unit in a commutative local ring R , then $\mathrm{SK}_1(R[X])$ has no k -torsion. Note that if R is a commutative local ring, then $\mathrm{SK}_1(R[X])$ coincides with $\mathrm{NK}_1(R)$; indeed, if R is a local ring, then $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$ for all $n > 0$. Therefore, we may replace $\alpha(X)$ by $\alpha(X)\alpha(0)^{-1}$ and assume that $[\alpha(0)] = [\mathrm{I}]$. In [6], the first author extended Weibel's result for arbitrary associative rings. In this section we prove the analog result for λ -unitary Bass nil-groups, *viz.* $\mathrm{NK}_1\mathrm{GQ}^\lambda(R, \lambda)$, where (R, λ) is the form ring as introduced by A. Bak in [1]. The main ingredient for our proof is an analog of Higman linearization (for a subclass of Bak's unitary group) due to V. Kopeiko; cf. [21]. For the general linear groups, Higman linearization (cf. [6]) allows us to show that $\mathrm{NK}_1(R)$ has a unipotent representative. The same result is not true in general for the unitary nil-groups. Kopeiko's results in [21], [22] give a complete description of the elements of $\mathrm{NK}_1\mathrm{GQ}^\lambda(R, \lambda)$ which have (unitary) unipotent representatives.

Definition 7.1. For an associative unital ring R we consider the truncated polynomial ring

$$R_t = \frac{R[X]}{(X^{t+1})}.$$

Lemma 7.2 (cf. [6], Lemma 4.1). *Let $P(X) \in R[X]$ be any polynomial. Then the following identity holds in the ring R_t :*

$$(1 + X^r P(X)) = (1 + X^r P(0))(1 + X^{r+1} Q(X)),$$

where $r > 0$ and $Q(X) \in R[X]$, with $\deg(Q(X)) < t - r$.

Proof. Let us write $P(X) = a_0 + a_1 X + \cdots + a_t X^t$. Then we can express the polynomial $P(X)$ as $P(X) = P(0) + X P'(X)$ for some $P'(X) \in R[X]$. Now, in R_t

$$\begin{aligned} (1 + X^r P(X))(1 + X^r P(0))^{-1} &= (1 + X^r P(0) + X^{r+1} P'(X))(1 + X^r P(0))^{-1} \\ &= 1 + X^{r+1} P'(X)(1 - X^r P(0) + X^{2r} (P(0))^2 - \cdots) \\ &= 1 + X^{r+1} Q(X), \end{aligned}$$

where $Q(X) \in R[X]$ with $\deg(Q(X)) < t - r$. Hence the lemma follows. \square

Remark 7.3. Iterating the above process we can write, for any polynomial $P(X) \in R[X]$,

$$(1 + XP(X)) = \prod_{i=1}^t (1 + a_i X^i)$$

in R_t , for some $a_i \in R$. By ascending induction, it follows that the a_i 's are uniquely determined.

Lemma 7.4. *Let R be an associative ring, and $k \in \mathbb{Z}$ such that $kR = R$. Let $P(X)$ be a polynomial in $R[X]$. Assume $P(0)$ lies in the center of R . Then, if*

$$(1 + X^r P(X))^{k^r} = 1$$

for some $r \geq 0$, we have that

$$(1 + X^r P(X)) = (1 + X^{r+1} Q(X))$$

in the ring R_t , for some $Q(X) \in R[X]$ with $\deg(Q(X)) < t - r$.

The following result is due to V. Kopeiko, cf. [21]. This is an analog of Higman linearization for this special case.

Proposition 7.5. *Let (R, λ) be a form ring. Then, every element of the group $\text{NKQ}_1^\lambda(R, \Lambda)$ has a representative of the form*

$$[a; b, c]_n = \begin{pmatrix} \mathbf{I}_r - aX & bX \\ -cX^n & \mathbf{I}_r + a^*X + \cdots + (a^*)^n X^n \end{pmatrix} \in \text{GQ}^\lambda(2r, R[X], \Lambda[X])$$

for some positive integers r and n , where $a, b, c \in M_r(R)$ satisfy the following conditions:

1. the matrices b and ab are Hermitian and also $ab = ba^*$,
2. the matrices c and ca are Hermitian and also $ca = a^*c$,
3. $bc = a^{n+1}$ and $cb = (a^*)^{n+1}$.

Corollary 7.6. *Let R be an associative ring. Let $[\alpha] \in \text{NKQ}_1^\lambda(R, \Lambda)$ have the representation $[a; b, c]_n$ for some $a, b, c \in M_n(R)$ according to Proposition 7.5. Then, if it is true that $(\mathbf{I}_r - aX) \in \text{GL}_r(R)$, we have $[\alpha] = [\mathbb{H}(\mathbf{I}_r - aX)]$ in $\text{NKQ}_1^\lambda(R, \Lambda)$.*

Proof. By Lemma 6.11, we have $[a; b, c]_n \equiv \mathbb{H}(\mathbf{I}_r - aX) \pmod{\text{EQ}^\lambda(2r, R[X], \Lambda[X])}$. Hence $[\alpha] = [\mathbb{H}(\mathbf{I}_r - aX)]$ in $\text{NKQ}_1^\lambda(R, \Lambda)$. \square

Theorem 7.7. *Let (R, Λ) be a form ring, where R is an associative ring with 1 and k is an invertible integer, i.e., $kR = R$. Let*

$$[\alpha(X)] = \left[\begin{pmatrix} A(X) & B(X) \\ C(X) & D(X) \end{pmatrix} \right] \in \text{NKQ}_1^\lambda(R, \Lambda)$$

with $A(X) \in \text{GL}_r(R[X])$ for some $r \in \mathbb{N}$. Then $[\alpha(X)]$ has no k -torsion.

Proof. By Theorem 7.5, $[\alpha(X)] = [[a; b, c]_n]$ for some $a, b, c \in M_s(R)$ and for some natural numbers n and s . Note that in Step 1 of the Proposition 7.5, the invertibility of the top left corner of the matrix α is not changed during the linearization process. The same is true for the remaining steps of the Proposition 7.5, and so, since the top left corner matrix is $A(X) \in \mathrm{GL}_r(R[X])$, we get $(I_s - aX) \in \mathrm{GL}_s(R[X])$. Using Corollary 7.6, one gets $[\alpha(X)] = [\mathbb{H}(I_s - aX)]$. Now let $[\alpha]$ have k -torsion, which implies that also $[\mathbb{H}(I_r - aX)]$ has k -torsion. Since $(I_r - aX)$ is invertible, it follows that a is nilpotent. Let t be such that $a^{t+1} = 0$. Since $[(I_r - aX)]^k = [I]$ in $\mathrm{NKQ}_1^\lambda(R[X], \Lambda[X])$, by arguing as in [7], we get $[I_r - aX] = [I]$ in $\mathrm{NKQ}_1^\lambda(R[X], \Lambda[X])$. This completes the proof. \square

8 Graded Analog

We recall the well-known “Swan–Weibel homotopy trick”, which is the main ingredient to handle the graded case. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded ring. An element $a \in R$ is denoted by $a = a_0 + a_1 + a_2 + \cdots$, where $a_i \in R_i$ for each i , and all but finitely many a_i 's are zero. Let $R_+ = R_1 \oplus R_2 \oplus \cdots$. A graded structure of R induces a graded structure on $M_n(R)$ (the ring of $n \times n$ matrices with entries in R).

Definition 8.1. Let $a \in R_0$ be a fixed element. We fix an element $b = b_0 + b_1 + \cdots$ in R and define a ring homomorphism $\epsilon : R \rightarrow R[X]$ as follows:

$$\epsilon(b) = \epsilon(b_0 + b_1 + \cdots) = b_0 + b_1X + b_2X^2 + \cdots + b_iX^i + \cdots .$$

Then we evaluate the polynomial $\epsilon(b)(X)$ at $X = a$ and denote the image by $b^+(a)$ i.e., $b^+(a) = \epsilon(b)(a)$. Note that $(b^+(x))^+(y) = b^+(xy)$. Observe, $b_0 = b^+(0)$. We shall use this fact frequently.

The above ring homomorphism ϵ induces a group homomorphism from $\mathrm{GL}(n, R)$ to $\mathrm{GL}(n, R[X])$ of rank n for every $n \geq 1$, i.e., for $\alpha \in \mathrm{GL}(n, R)$ we get a map

$$\epsilon : \mathrm{GL}(n, R) \rightarrow \mathrm{GL}(n, R[X]), \text{ defined by}$$

$$\alpha = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \oplus \cdots \mapsto \alpha_0 \oplus \alpha_1X \oplus \alpha_2X^2 \cdots ,$$

where $\alpha_i \in M(n, R_i)$. As above, for $a \in R_0$, we define $\alpha^+(a)$ as $\alpha^+(a) = \epsilon(\alpha)(a)$.

The graded dilation lemma and graded local-global principle are proved in [11] for linear, symplectic and orthogonal groups. Arguing in similar manner one gets:

Theorem 8.2. (Graded Local-Global Principle) *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be an almost commutative graded ring with identity 1. Let $\alpha \in \mathrm{GQ}(2n, R, \Lambda)$ be such that $\alpha \equiv I_{2n} \pmod{R_+}$. If $\alpha_{\mathfrak{m}} \in \mathrm{EQ}(2n, R_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$, for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(C(R_0))$, then $\alpha \in \mathrm{EQ}(2n, R, \Lambda)$.*

Moreover, the L-G principle for the elementary subgroups and their normality property are equivalent.

Theorem 8.3. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$ be an almost commutative graded ring with identity 1. Then the following assertions are equivalent for $2n \geq 6$:*

1. $\text{EQ}(2n, R, \Lambda)$ is a normal subgroup of $\text{GQ}(2n, R, \Lambda)$.
2. If $\alpha \in \text{GQ}(2n, R, \Lambda)$ with $\alpha^+(0) = I_{2n}$, and $\alpha_{\mathfrak{m}} \in \text{EQ}(2n, R_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m} \in \text{Max}(C(R_0))$, then $\alpha \in \text{EQ}(2n, R, \Lambda)$.

As an application of Theorem 8.2 and the Theorem 7.7, we obtain the following:

Theorem 8.4. *Let $R = R_0 \oplus R_1 \oplus \dots$ be an almost commutative graded ring with 1. Let $N = N_0 + N_1 + \dots + N_r \in M_r(R)$ be a nilpotent matrix, and let I denote the identity matrix. Let $k \in \mathbb{Z}$ be a unit in R_0 . If $[(I + N)]^k = [I]$ in $\text{NKQ}_1^\lambda(R, \Lambda)$, then $[I + N] = [I + N_0]$.*

Proof. Consider the ring homomorphism $f : R \rightarrow R[X]$ defined by

$$f(a_0 + a_1 + \dots) = a_0 + a_1X + \dots$$

Then

$$\begin{aligned} [(I + N)^k] = [I] &\Rightarrow f([(I + N)^k]) = [f(I + N)]^k = [I] \\ &\Rightarrow [(I + N_0 + N_1X + \dots + N_rX^r)]^k = [I]. \end{aligned}$$

Let \mathfrak{m} be a maximal ideal in $C(R_0)$. By Theorem 7.7, we have

$$[(I + N_0 + N_1X + \dots + N_rX^r)] = [I]$$

in $\text{NKQ}_1^\lambda(R_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$. Hence by using the local-global principle we conclude

$$[(I + N)] = [I + N_0]$$

in $\text{NK}_1\text{GQ}^\lambda(R, \Lambda)$, as required. \square

Acknowledgment

We thank the unknown referee for useful suggestions to improve the manuscript. We also thank V. Kopeiko and Sergey Sinchuk for useful discussions on Bass's nil group. Research by the first author was supported by SERB-MATRICES grant for the financial year 2020–2021. Research by the second author was supported by IISER (Pune) post-doctoral research grant, 2020–2021.

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Received: July 28, 2022

Accepted for publication: February 9, 2023

Communicated by: Roozbeh Hazrat