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On square-free numbers generated from given sets of primes

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Abstract. Let x be a positive real number, and $\mathcal{P} \subset [2, \lambda(x)]$ be a set of primes, where $\lambda(x) \in o(x^{1/2})$ is a monotone increasing function. We examine $Q_{\mathcal{P}}(x)$ for different sets $\mathcal{P}$, where $Q_{\mathcal{P}}(x)$ is the element count of the set containing those positive square-free integers, which are smaller than-, or equal to x, and which are only divisible by the elements of $\mathcal{P}$.

1 Introduction

In this article, we are going to investigate the number of square-free numbers that one can generate from given sets of prime numbers. More precisely, let's take a set of prime numbers $\mathcal{P}$, and denote with $Q_{\mathcal{P}}(x)$ the element count of the set of all those positive square-free integers, which are smaller than-, or equal to x; and which are only divisible by the elements of $\mathcal{P}$. We would like to know how $Q_{\mathcal{P}}(x)$ behaves asymptotically based on the structure of $\mathcal{P}$.

Following the classical notation, denote the number of *quadratfrei* integers between 1 and x as $Q(x)$. We know that $Q(x) \sim (6/\pi^2)x$ as $x \to +\infty$, see for example [5, Th. 2.2]. What is required from $\mathcal{P}$ to achieve $Q_{\mathcal{P}}(x) \asymp x$? Indulge ourselves for a moment with an informal train of thoughts. Taking only small primes, we can select even all of them to form products which aren't greater than x. Then by the binomial theorem we have that the number of these products grow exponentially. Based on this, we can expect that there is a threshold for the number of primes in $\mathcal{P}$ somewhere between the (poly)logarithmic order, and the fractional power order, where the behaviour of $Q_{\mathcal{P}}(x)$ changes.

We are going to look at two scenarios. During the first scenario, $\mathcal{P}$ will contain all the primes which are not greater than an x dependent bound $\lambda(x)$.

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Proposition 1.1. *Let $\lambda : \mathbb{R} \to [1, +\infty)$ be a monotone increasing function which is in $o(x^{1/2})$, and let $\mathcal{P}$ contain all the primes which are not greater than $\lambda(x)$. Then we have*

$$Q_{\mathcal{P}}(x) \asymp x \frac{\ln \lambda(x)}{\ln x}$$

as $x \to +\infty$.

It can be seen from this result, that taking all the primes below x^{ε} for $\varepsilon > 0$ in this scenario is enough to have $Q_{\mathcal{P}}(x) \asymp x$, but a (poly)logarithmic bound is not enough. In the second scenario, $\mathcal{P}$ will contain only those primes which are not greater than the bound $\lambda(x)$, and which fall into certain congruence classes modulo q.

Proposition 1.2. *Let $q > 0$ be an integer, and $m_1, \ldots, m_k \in \mathbb{N}$ be pairwise distinct relative primes to q. Furthermore, let $\lambda : \mathbb{R} \to [1, +\infty)$ be a monotone increasing function which is in $o(x^{1/2})$, and let $\mathcal{P}$ contain all those primes p which are not greater than $\lambda(x)$, and for which $p \equiv m_i \pmod{q}$ holds for some $i \in \{1, \ldots, k\}$. Then we have*

$$Q_{\mathcal{P}}(x) \asymp x \frac{(\ln \lambda(x))^{k/\varphi(q)}}{\ln x}$$

as $x \to +\infty$,where φ is the Euler totient function.

To prove our results, we are going to use the Dedekind psi function

$$\psi(d) = d \prod_{p \mid d} \left(1 + \frac{1}{p}\right) \tag{1}$$

see [2, B41], or [1, Ch. 3, ex. 11]; and the following representation of the Euler totient function

$$\varphi(d) = d \prod_{p \mid d} \left(1 - \frac{1}{p}\right) \tag{2}$$

see [1, Th. 2.4]. Recall that the equality

$$\frac{\varphi(d)}{d} = \sum_{q \mid d} \frac{\mu(q)}{q} \tag{3}$$

holds when $d \geq 1$, see [1, Th. 2.3].

2 Proofs

We are going to use the fundamental lemma of the combinatorial sieve to prove our results, see [8, Th. 4.4]. The idea is to utilise the sieve to remove those square-free integers below x, which are not divisible by a prime in $\mathcal{P}$. We would want to sift with the set of primes below x which are not in $\mathcal{P}$, but out of technicality, we are going to use the set $\mathcal{P}' := [1, x^{1/2}] \cap \mathbb{P} \setminus \mathcal{P}$. This set doesn't contain all the required primes, but it will remove all the unwanted composite square-free integers below x. (Next to the primes in $\mathcal{P}'$ of course.) Actually, our concern could be that it will leave the primes in $(x^{1/2}, x]$ intact, but the contribution of these numbers can be bounded by some constant times $x/\ln x$, which can be neglected in our case.

Using the notation of [8, Th. 4.4], let $\mathcal{A}$ be the set of all square-free numbers not greater than x, let $\mathcal{A}_d := \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}$, and finally let

$$P(y) := \prod_{\substack{p \le y \\ p \in \mathcal{P}'}} p.$$

We are going to give an asymptotic for

$$S(\mathcal{A}, \mathcal{P}', x^{1/2}) := |\{a \in \mathcal{A} : \gcd(a, P(x^{1/2})) = 1\}|$$

using the mentioned sieve. To do this, on the one hand we need a multiplicative function $w \ge 0$ such that

$$|\mathcal{A}_d| = \frac{w(d)}{d} X + R_d \tag{4}$$

for some real number X, and all $d | P(x^{1/2})$. We are going to give an asymptotic for $|\mathcal{A}_d|$ in section 2.1, and based on it, and appropriate multiplicative function. On the other hand, we need to show that there exist positive constants κ, and κ' such that the inequality

$$\prod_{\eta \le p \le \xi} \left(1 - \frac{w(p)}{p}\right)^{-1} < \left(\frac{\ln \xi}{\ln \eta}\right)^{\kappa} \left(1 + \frac{\kappa'}{\ln \eta}\right) \tag{5}$$

holds for $2 \le \eta \le \xi$ using our multiplicative function w. We are going to look at this in section 2.2. If these requirements hold, then uniformly in $\mathcal{A}$, X, and $u \ge 1$ we have

$$S(\mathcal{A}, \mathcal{P}', x^{1/2}) = X(1 + \mathcal{O}(u^{-u/2})) \prod_{\substack{p \le x^{1/2} \\ p \in \mathcal{P}'}} \left(1 - \frac{w(p)}{p}\right) + \mathcal{O}\Bigg(\sum_{\substack{d \le x^{u/2} \\ d | P(x^{1/2})}} R_d\Bigg) \tag{6}$$

as $x \to +\infty$, using [8, Th. 4.4]. We are going to use this equation to prove our results in section 2.3.

2.1 Cardinality of the sieving sets

First we have to show that we can write $|\mathcal{A}_d|$ as in expression (4).

Lemma 2.1. *For every $\varepsilon > 0$, and $d \in Q(x^{2/3-\eta})$ with $\eta > 0$ we have*

$$|\mathcal{A}_d| = \frac{x}{\zeta(2)\psi(d)} + \mathcal{O}(x^{1/2} d^{-1/4}) + \mathcal{O}(d^{1/2+\varepsilon})$$

as $x \to +\infty$.

We are going to follow the method of Prachar, see article [6]. Take note that the error term could be improved, see the article of Hooley [3].

Proof. Fix a big enough real $x > 0$, and choose an appropriate d. Then

$$|\mathcal{A}_d| = \sum_{\substack{n\le x\\ n\equiv 0(d)}} \mu(n)^2 = \sum_{\substack{n\le x\\ n\equiv 0(d)}} \sum_{l^2|n} \mu(l) = \sum_{\substack{l^2m\le x\\ l^2m\equiv 0(d)}} \mu(l) = \sum_{l\le X_d} + \sum_{X_d<l}$$

where $X_d := x^{1/2}d^{-1/4}$. We look at the two sums separately.

$\sum_{l\le X_d}$ The congruence $l^2m \equiv 0 \pmod d$ is always soluble for m because $(l^2, d)|0$; and the solutions can be given as $m \equiv 0 \pmod{d/(l, d)}$ because we have $(l^2, d) = (l, d)$ as d is square-free. Thus the first sum is

$$\sum_{l\le X_d} \mu(l) \sum_{\substack{m\le x/l^2\\ m\equiv 0(d/(l,d))}} 1 = \sum_{l\le X_d} \mu(l)\left(\frac{(l, d)x}{l^2d} + \mathcal{O}(1)\right)$$

where the second item between the parentheses contributes $\mathcal{O}(X_d)$ to the result, so we are going to focus on the first item. We can split the sum along the divisors of d as

$$\frac{x}{d}\sum_{l\le X_d} \frac{(l, d)\mu(l)}{l^2} = \frac{x}{d}\sum_{q|d} \sum_{\substack{l\le X_d\\ (l,d)=q}} \frac{q\mu(l)}{l^2}.$$

For a given q, we can write the appropriate l as qm with $(q, m) = 1$. (Observe that we only have to worry about the square-free l.) In this case $q = (qm, d) = (q, d)(m, d)$, so $(m, d) = 1$, and

$$\frac{x}{d}\sum_{q|d} \sum_{\substack{m\le X_d/q\\ (m,d)=1}} \frac{q\mu(qm)}{(qm)^2} = \frac{x}{d}\sum_{q|d} \frac{\mu(q)}{q} \sum_{\substack{m\le X_d/q\\ (m,d)=1}} \frac{\mu(m)}{m^2}$$

as $(q, m) = 1$, and μ is multiplicative. The internal sum on the right hand side can be written as

$$\sum_{\substack{m=1\\ (m,d)=1}}^{\infty} \frac{\mu(m)}{m^2} + \mathcal{O}\left(\sum_{m>X_d/q} \frac{1}{m^2}\right) = \frac{1}{\zeta(2)}\prod_{p|d}\left(1 - \frac{1}{p^2}\right)^{-1} + \mathcal{O}(q/X_d).$$

Relying on (3) we get

$$\begin{aligned}\frac{1}{d}\sum_{q|d} \frac{\mu(q)}{q}\prod_{p|d}\left(1 - \frac{1}{p^2}\right)^{-1} &= \frac{\varphi(d)}{d^2}\prod_{p|d}\left(1 - \frac{1}{p^2}\right)^{-1}\\ &= \frac{1}{d}\prod_{p|d}\left(1 + \frac{1}{p}\right)^{-1} = \frac{1}{\psi(d)}\end{aligned}$$

by using (1), and (2). Concerning the sum of the remaining asymptotic term, we have

$$\frac{x}{d}\sum_{q|d}\frac{\mu(q)}{q}\mathcal{O}(q/X_d)=\frac{x}{dX_d}\sum_{q|d}c_{d,q}\mu(q)\in\mathcal{O}(X_d)$$

where $c_{d,q}\in\mathcal{O}(1)$. So we have

$$\sum_{l\leq X_d}=\frac{x}{\zeta(2)\psi(d)}+\mathcal{O}(X_d).$$

$\sum_{X_d<l}$ Observe that, since $X_d<l$, we have $m<d^{1/2}$; and thus, $l\leq(x/m)^{1/2}$ for a fixed m. Taking a fixed m, if the congruence $l^2m\equiv 0 \pmod d$ is soluble, then the number of solutions is $2^{\omega(d)}$, see [1, Th. 5.28], where $\omega(d)$ denotes the number of distinct prime factors of d, with the convention that $\omega(1)=0$. Once the solutions are found, we can take all the admissible values of l, for which l^2 is congruent to a solution mod d. So we have that the second sum can be bounded by some constant times

$$\sum_{m<d^{1/2}}2^{\omega(d)}\left(\left(\frac{x}{m}\right)^{1/2}d^{-1}+\mathcal{O}(1)\right)\ll 2^{\omega(d)}(x^{1/2}d^{-3/4}+d^{1/2})$$

which is in $\mathcal{O}(x^{1/2}d^{-3/4+\varepsilon})+\mathcal{O}(d^{1/2+\varepsilon})$ because $\omega(d)\in\mathcal{O}((\ln d)/\ln_2 d)$, see [8, Sec. 5.3].

Combining the two results, we get our statement. □

Based on this proposition, we can set our multiplicative function as

$$w(d):=\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1} \tag{7}$$

and X as $x/\zeta(2)$, furthermore choose $R_d\in\mathcal{O}(x^{1/2}d^{-1/4})+\mathcal{O}(d^{1/2+\varepsilon})$, to have the required form for $|\mathcal{A}_d|$.

2.2 The Euler product

Lemma 2.2. *For every $y\geq 2$, we have that the inequalities*

$$\frac{4}{3}\frac{e^{-\gamma}}{\ln y}\left(1-\frac{1}{\ln^2 y}\right)<\prod_{p\leq y}\left(1-\frac{1}{p+1}\right)<\frac{\pi^2}{6}\frac{e^{-\gamma}}{\ln y}\left(1+\frac{1}{\ln^2 y}\right)$$

hold, where γ is the Euler-Mascheroni constant.

Note that we could improve the exponent of the logarithmic part inside the parentheses on the right hand side, but this form will suffice for our needs.

Proof. Observe that

$$\prod_{p\le y}\left(1-\frac{1}{p+1}\right)\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}=\prod_{p\le y}\left(1-\frac{1}{p^2}\right)^{-1} \tag{8}$$

where we have the partial Euler product of the Riemann zeta function on the right hand side, see [1, Sec. 11.5]. Take note that as $\zeta(2)=\pi^2/6$, we have that the value of this product will be in $[4/3,\pi^2/6)$. Based on [7, Th. 7, Col.], we have

$$\frac{e^{-\gamma}}{\ln y}\left(1-\frac{1}{\ln^2 y}\right)<\prod_{p\le y}\left(1-\frac{1}{p}\right)<\frac{e^{-\gamma}}{\ln y}\left(1+\frac{1}{\ln^2 y}\right) \tag{9}$$

for every $y>1$. Combining (8) and (9) we get our statement. □

Now we are going to show that there exist positive constants κ, and κ' for our function w so that requirement (5) holds. Take note that the left hand side of requirement (5) is

$$\prod_{\eta\le p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1} \tag{10}$$

when we substitute w, see (7).

$\eta=2$ When $2\le\xi<3$, then the product will be $3/2$ so for example a $\kappa\ge 1$, and a $\kappa'\ge 1$ will suffice. Otherwise, when $\xi\ge 3$, then we can write

$$\prod_{p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1}<(\ln\xi)\frac{3e^{\gamma}}{4}\left(1-\frac{1}{\ln^2\xi}\right)^{-1}<8\ln\xi$$

so a $\kappa\ge 1$, and a $\kappa'\ge 5$ will suffice in this case.

$2<\eta<3$ When $\eta\le\xi<3$, then the product will be 1, for which the previous constants will be good. Otherwise, when $\xi\ge 3$, then

$$\prod_{2<p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1}<(\ln\xi)\frac{e^{\gamma}}{2}\left(1-\frac{1}{\ln^2\xi}\right)^{-1}<6\ln\xi$$

so a $\kappa\ge 1$, and a $\kappa'\ge 6$ will suffice in this case.

$\eta\ge 3$ We can write product (10) as

$$\prod_{p\le\eta}\left(1-\frac{1}{p+1}\right)\prod_{p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1}<\left(\frac{\ln\xi}{\ln\eta}\right)^3\frac{\pi^2}{8}\frac{(\ln^2\eta)+1}{(\ln^2\xi)-1}$$

which shows us that a $\kappa\ge 3$ will suffice, and where

$$\frac{(\ln^2\eta)+1}{(\ln^2\xi)-1}<\frac{(\ln^2\eta)+1}{(\ln^2\eta)-1}=\frac{2}{(\ln^2\eta)-1}+1$$

thus we have to guarantee that the inequality

$$\frac{\pi^2}{8}\left(\frac{2}{(\ln^2\eta)-1}+1\right)<1+\frac{\kappa'}{\ln\eta}$$

holds, which can be done by selecting a $\kappa' \geq 14$.

Taking all the cases into consideration, it can be seen that we can select the required positive constants κ, and κ'.

2.3 The main asymptotic

We choose u as 1 in (6), and based on lemma 2.1 we get that the error term is

$$\sum_{\substack{d\leq x^{1/2}\\ d\mid P(x^{1/2})}} R_d = cx^{1/2}\sum_{\substack{d\leq x^{1/2}\\ d\mid P(x^{1/2})}} d^{-1/4}+c'\sum_{\substack{d\leq x^{1/2}\\ d\mid P(x^{1/2})}} d^{1/2+\varepsilon}$$

where the first sum is in $\mathcal{O}(x^{7/8})$, and the second sum is in $\mathcal{O}(x^{3/4})$, so the contribution of this sum can be neglected. What remains is to compute the Euler product in expression (6) for the different sets of primes. First we prove proposition 1.1.

Proof. Select a function λ satisfying the requirements of proposition 1.1. $\mathcal{P}$ contains all the primes which are not greater than $\lambda(x)$, so we have that the product which we have to compute in expression (6) is equal to

$$\prod_{p\leq x^{1/2}}\left(1-\frac{1}{p+1}\right)\prod_{p\leq\lambda(x)}\left(1-\frac{1}{p+1}\right)^{-1}\asymp\frac{\ln\lambda(x)}{\ln x}$$

based on lemma 2.2. □

To prove proposition 1.2, we need the following lemma.

Lemma 2.3. *Let q and m be integers, such that $q\geq 1$, furthermore $1\leq m\leq q$, and $(m,q)=1$. Then we have*

$$\prod_{\substack{p\leq y\\ p\equiv m(q)}}\left(1-\frac{1}{p+1}\right)\asymp\frac{1}{(\ln y)^{1/\varphi(q)}}+\mathcal{O}\left(\frac{1}{(\ln y)^{1+1/\varphi(q)}}\right)$$

as $y\to+\infty$.

Proof. Let q and m be integers satisfying the requirements given in the lemma. As in the proof of lemma 2.2, we can write

$$\prod_{\substack{p\leq y\\ p\equiv m(q)}}\left(1-\frac{1}{p+1}\right)\prod_{\substack{p\leq y\\ p\equiv m(q)}}\left(1-\frac{1}{p}\right)^{-1}=\prod_{\substack{p\leq y\\ p\equiv m(q)}}\left(1-\frac{1}{p^2}\right)^{-1} \tag{11}$$

where the product on the right hand side is a positive constant. Based on the article of Williams [9] we have

$$\prod_{\substack{p\le y\\ p\equiv m(q)}}\left(1-\frac{1}{p}\right)=\frac{C(q,m)}{(\ln y)^{1/\varphi(q)}}+\mathcal{O}\left(\frac{1}{(\ln y)^{1+1/\varphi(q)}}\right) \tag{12}$$

as $y\to+\infty$, where $C(q,m)$ is a positive constant. Combining (11) and (12) we get our statement. □

It worths mentioning the article of Languasco and Zaccagnini [4], where the authors improved the expression given for the constant by Williams, furthermore the error term for different scenarios. Now we prove proposition 1.2.

Proof. Select an appropriate modulus q, residues $m_1,\dots,m_k$, and a function λ all satisfying the requirements of proposition 1.2. $\mathcal{P}$ contains all those primes p which are not greater than $\lambda(x)$, and for which $p\equiv m_i \pmod q$ holds for some $i\in\{1,\dots,k\}$. So the product which we have to compute in (6) is equal to

$$\prod_{p\le x^{1/2}}\left(1-\frac{1}{p+1}\right)\prod_{i=1}^{k}\prod_{\substack{p\le\lambda(x)\\ p\equiv m_i(n)}}\left(1-\frac{1}{p+1}\right)^{-1}\asymp\frac{(\ln\lambda(x))^{k/\varphi(n)}}{\ln x}$$

based on lemma 2.3. □

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