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On square-free numbers generated from given sets of primes

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Abstract. Let x be a positive real number, and $\mathcal{P} \subset [2, \lambda(x)]$ be a set of primes, where $\lambda(x) \in \Omega(x^\varepsilon)$ is a monotone increasing function with $\varepsilon \in (0, 1)$. We examine $Q_\mathcal{P}(x)$, where $Q_\mathcal{P}(x)$ is the element count of the set containing those positive square-free integers, which are smaller than-, or equal to x, and which are only divisible by the elements of $\mathcal{P}$.

1 Introduction

In this article, we are going to investigate the number of square-free numbers that one can generate from given sets of prime numbers. More precisely, let's take a set of prime numbers $\mathcal{P}$, and denote with $Q_\mathcal{P}(x)$ the element count of the set of all those positive square-free integers, which are smaller than-, or equal to x; and which are only divisible by the elements of $\mathcal{P}$. We would like to know how $Q_\mathcal{P}(x)$ behaves asymptotically based on the structure of $\mathcal{P}$.

Following the classical notation, denote the number of *quadratfrei* integers between 1 and x as $Q(x)$. We know that $Q(x) = x/\zeta(2) + \mathcal{O}(\sqrt{x})$ for all $x \geq 1$, see for example [5, Th. 2.2]. What is required from $\mathcal{P}$ to achieve $Q_\mathcal{P}(x) \asymp x$? Indulge ourselves for a moment with an informal train of thoughts. Taking only small primes, we can select even all of them to form products which aren't greater than x. Then by the binomial theorem we have that the number of these products grow exponentially. Based on this, we can expect that there is a threshold for the number of primes in $\mathcal{P}$ somewhere between the (poly)logarithmic order, and the fractional power order, where the behaviour of $Q_\mathcal{P}(x)$ changes.

Proposition 1.1. *Let $\lambda : \mathbb{R} \to [1, +\infty)$ be a monotone increasing function which is in $\Omega(x^\varepsilon)$ with $\varepsilon \in (0, 1)$, and let $\mathcal{P}$ contain all the primes which are not greater than $\lambda(x)$.*

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Then we have

$$Q_{\mathcal{P}}(x) \asymp x\frac{\ln \lambda(x)}{\ln x}$$

as $x \to \infty$.

2 Proofs

We are going to use the fundamental lemma of the combinatorial sieve to prove our results, see [9, P I.4, Th. 4.4]. The idea is to utilise the sieve to remove those square-free integers below x, which are not divisible by a prime in $\mathcal{P}$, so we will sift with the set $\mathcal{P}' := [1, x] \cap \mathbb{P} \setminus \mathcal{P}$. Using the notation of [9, P I.4, Th. 4.4], let $\mathcal{A}$ be the set of all square-free numbers not greater than x, let $\mathcal{A}_d := \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}$, and finally let

$$P(y) := \prod_{\substack{p \le y\\ p \in \mathcal{P}'}} p.$$

We are going to give an asymptotic for

$$S(\mathcal{A}, \mathcal{P}', x) := |\{a \in \mathcal{A} : (a, P(x)) = 1\}|$$

using the mentioned sieve. To do this, on the one hand we need a multiplicative function $w \ge 0$ such that

$$|\mathcal{A}_d| = \frac{w(d)}{d}X + R_d$$

for some real number X, and all $d|P(x)$. We are going to give an asymptotic for $|\mathcal{A}_d|$ in section 2.1, and based on it an appropriate multiplicative function. On the other hand, we need to show that there exist positive constants κ and κ' such that the inequality

$$\prod_{\eta \le p \le \xi}\left(1 - \frac{w(p)}{p}\right)^{-1} < \left(\frac{\ln \xi}{\ln \eta}\right)^{\kappa}\left(1 + \frac{\kappa'}{\ln \eta}\right) \tag{1}$$

holds for $2 \le \eta \le \xi$ using our multiplicative function w. We are going to look at this in section 2.2. If these requirements hold, then uniformly in $\mathcal{A}$, X, and $u \ge 1$ we have

$$S(\mathcal{A}, \mathcal{P}', x) = X \prod_{\substack{p \le x\\ p \in \mathcal{P}'}}\left(1 - \frac{w(p)}{p}\right)(1 + \mathcal{O}(u^{-u/2})) + \mathcal{O}\Bigg(\sum_{\substack{d \le x^u\\ d|P(x)}} |R_d|\Bigg) \tag{2}$$

as $x \to +\infty$, using [9, P I.4, Th. 4.4]. We are going to use this equation to prove our results in section 2.3.

We will rely on the Dedekind psi function

$$\psi(d) = d\prod_{p|d}\left(1 + \frac{1}{p}\right) \tag{3}$$

see [2, B41], or [1, Ch. 3, Ex. 11]; and the following representation of the Euler totient function

$$\varphi(d) = d \prod_{p|d} \left(1 - \frac{1}{p}\right) \tag{4}$$

see [1, Th. 2.4]. Recall that the equality

$$\frac{\varphi(d)}{d} = \sum_{q|d} \frac{\mu(q)}{q} \tag{5}$$

holds when $d \geq 1$, see [1, Th. 2.3].

2.1 Cardinality of the sieving sets

Lemma 2.1. *For every $\delta > 0$, and $d \in Q(x^{2/3-\eta})$ with $\eta > 0$ we have*

$$|\mathcal{A}_d| = \frac{x}{\zeta(2)\psi(d)} + \mathcal{O}(x^{1/2}d^{-1/4}) + \mathcal{O}(d^{1/2+\delta}) \tag{6}$$

as $x \to +\infty$.

To prove this lemma we are going to follow the method of Prachar, see article [7]. Take note that the error term could be improved, see the article of Hooley [3]. For further developments see the article of Nunes [6] and the article of Mangerel [4].

Proof of lemma 2.1. Fix a big enough real $x > 0$, and choose an appropriate d. Then

$$|\mathcal{A}_d| = \sum_{\substack{n \leq x \\ n \equiv 0(d)}} \mu(n)^2 = \sum_{\substack{n \leq x \\ n \equiv 0(d)}} \sum_{l^2|n} \mu(l) = \sum_{\substack{l^2 m \leq x \\ l^2 m \equiv 0(d)}} \mu(l) = \sum_{l \leq X_d} + \sum_{X_d < l}$$

where $X_d := x^{1/2}d^{-1/4}$. We look at the two sums separately.

$\sum_{l \leq X_d}$ The congruence $l^2 m \equiv 0 \pmod{d}$ is always soluble for m because $(l^2, d)|0$; and the solutions can be given as $m \equiv 0 \pmod{d/(l,d)}$ because we have $(l^2, d) = (l, d)$ as d is square-free. Thus the first sum is

$$\sum_{l \leq X_d} \mu(l) \sum_{\substack{m \leq x/l^2 \\ m \equiv 0(d/(l,d))}} 1 = \sum_{l \leq X_d} \mu(l) \left(\frac{(l,d)x}{l^2 d} + \mathcal{O}(1) \right)$$

where the second item between the parentheses contributes $\mathcal{O}(X_d)$ to the result, so we are going to focus on the first item. We can split the sum along the divisors of d as

$$\frac{x}{d} \sum_{l \leq X_d} \frac{(l,d)\mu(l)}{l^2} = \frac{x}{d} \sum_{q|d} \sum_{\substack{l \leq X_d \\ (l,d)=q}} \frac{q\mu(l)}{l^2}.$$

For a given q, we can write the appropriate l as qm with $(q,m)=1$. (Observe that we only have to worry about the square-free l.) In this case $q=(qm,d)=(q,d)(m,d)$, so $(m,d)=1$, and

$$\frac{x}{d}\sum_{q|d}\sum_{\substack{m\le X_d/q\\(m,d)=1}}\frac{q\mu(qm)}{(qm)^2}=\frac{x}{d}\sum_{q|d}\frac{\mu(q)}{q}\sum_{\substack{m\le X_d/q\\(m,d)=1}}\frac{\mu(m)}{m^2}$$

as $(q,m)=1$, and μ is multiplicative. The internal sum on the right hand side can be written as

$$\sum_{\substack{m=1\\(m,d)=1}}^{\infty}\frac{\mu(m)}{m^2}+\mathcal{O}\Bigg(\sum_{m>X_d/q}\frac{1}{m^2}\Bigg)=\frac{1}{\zeta(2)}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}+\mathcal{O}(q/X_d).$$

Relying on (5) we get

$$\begin{aligned}\frac{1}{d}\sum_{q|d}\frac{\mu(q)}{q}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}&=\frac{\varphi(d)}{d^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\\&=\frac{1}{d}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}=\frac{1}{\psi(d)}\end{aligned}$$

by using (3), and (4). Concerning the sum of the remaining asymptotic term, we have

$$\frac{x}{d}\sum_{q|d}\frac{\mu(q)}{q}\mathcal{O}(q/X_d)=\frac{x}{dX_d}\sum_{q|d}c_{d,q}\mu(q)\in\mathcal{O}(X_d)$$

where $c_{d,q}\in\mathcal{O}(1)$. So we have

$$\sum_{l\le X_d}=\frac{x}{\zeta(2)\psi(d)}+\mathcal{O}(X_d).$$

$\sum_{X_d<l}$ Observe that, since $X_d<l$, we have $m<d^{1/2}$; and thus, $l\le(x/m)^{1/2}$ for a fixed m. Taking a fixed m, if the congruence $l^2m\equiv 0 \pmod d$ is soluble, then the number of solutions is $2^{\omega(d)}$, see [1, Th. 5.28], where $\omega(d)$ denotes the number of distinct prime factors of d, with the convention that $\omega(1)=0$. Once the solutions are found, we can take all the admissible values of l, for which l^2 is congruent to a solution mod d. So we have that the second sum can be bounded by some constant times

$$\sum_{m<d^{1/2}}2^{\omega(d)}\left(\left(\frac{x}{m}\right)^{1/2}d^{-1}+\mathcal{O}(1)\right)\ll 2^{\omega(d)}(x^{1/2}d^{-3/4}+d^{1/2})$$

which is in $\mathcal{O}(x^{1/2}d^{-3/4+\delta})+\mathcal{O}(d^{1/2+\delta})$ because $\omega(d)\in\mathcal{O}((\ln d)/\ln_2 d)$, see [9, Sec. 5.3].

Combining the two results, we get our statement. □

Lemma 2.2. *For $d \in Q(x)$ we have*

$$|\mathcal{A}_d| = \frac{x}{\zeta(2)\psi(d)} + \mathcal{O}\bigg(\frac{x}{d}e^{-c\sqrt{\ln x/d}}\bigg)$$

as $x \to +\infty$, where c is a positive constant.

Proof of lemma 2.2. Let $d \in Q(x)$. We can write

$$|\mathcal{A}_d| = \sum_{\substack{n\le x\\ n\equiv 0(d)}} \mu(n)^2 = \sum_{m\le x/d} \mu(md)^2.$$

The summand is zero when $(m,d) > 1$, otherwise we have $\mu(md)^2 = \mu(m)^2$ because μ is multiplicative, and d is square-free. So we have

$$\sum_{\substack{m\le x/d\\ (m,d)=1}} \mu(m)^2 = \sum_{m\le x/d} \chi_d(m)\mu(m)^2 \tag{7}$$

where χ_d is the principal character modulo d. Let

$$F_d(s;z) := \sum_{n\ge 1} \frac{\chi_d(n)\mu(n)^2 z^{\omega(n)}}{n^s}.$$

The function in the numerator of the summand is multiplicative, and when $s > 1$ we have

$$\sum_p \sum_{\nu\ge 1} \left|\frac{\chi_d(p^\nu)\mu(p^\nu)^2 z^{\omega(p^\nu)}}{p^{\nu s}}\right| = |z| \sum_p \left|\frac{\chi_d(p)}{p^s}\right| < +\infty$$

so we can apply [9, P. II.1, Th. 1.3] to get that F_d is absolutely convergent, and

$$F_d(s;z) = \prod_p \sum_{\nu\ge 0} \frac{\chi_d(p^\nu)\mu(p^\nu)^2 z^{\omega(p^\nu)}}{p^{\nu s}} = \prod_p \bigg(1 + \frac{\chi_d(p)z}{p^s}\bigg)$$

when $s > 1$. Now we are going to rely on the technique of [9, P. II.6, Sec. 6.1]. The function

$$G_d(s;z) := F_d(s;z)\zeta(s)^{-z} = \prod_p \bigg(1 + \frac{\chi_d(p)z}{p^s}\bigg)\bigg(1 - \frac{1}{p^s}\bigg)^z.$$

is expandable as a Dirichlet series $G_d(s;z) = \sum_{n\ge 1} g_{d,z}(n)/n^s$, where $g_{d,z}$ is the multiplicative function whose values on prime powers are determined by the following identities.

- When $(p,d)=1$, then $\chi_d(p)=1$; and

$$1+\sum_{\nu\geq 1} g_{d,z}(p^\nu)\xi^\nu = (1+\xi z)(1-\xi)^z$$

where $|\xi|<1$. Using the binomial theorem on the right hand side we get

$$\begin{aligned}(1+\xi z)\sum_{k=0}^{\infty}\binom{z}{k}(-\xi)^k &= (1+\xi z)\left(1-\xi z+\binom{z}{2}\xi^2-\dots\right)\\ &= 1+\xi^2\left(\binom{z}{2}-z^2\right)+\dots\end{aligned}$$

so $g_{d,z}(p)=0$. Using Cauchy's inequality, see for example [10, Sec. 2.5], we get for $|z|\leq A$ that

$$|g_{d,z}(p^\nu)|\leq M2^{\nu/2}$$

for $\nu\geq 2$, where

$$M := \sup_{\substack{|z|\leq A\\ |\xi|\leq 1/\sqrt{2}}} |(1+\xi z)(1-\xi)^z|.$$

Thus when $(p,d)=1$ we have

$$\sum_{\nu\geq 1}\frac{|g_{d,z}(p^\nu)|}{p^{\nu\sigma}} \leq M\sum_{\nu\geq 2}\frac{2^{\nu/2}}{p^{\nu\sigma}} = 2M\frac{1}{p^\sigma(p^\sigma-\sqrt{2})}.$$

- When $(p,d)>1$, then $\chi_d(p)=0$; and

$$1+\sum_{\nu\geq 1} g_{d,z}(p^\nu)\xi^\nu = (1-\xi)^z = 1-\binom{z}{1}\xi+\binom{z}{2}\xi^2-\binom{z}{3}\xi^3+\dots$$

holds. This means that

$$\sum_{\nu\geq 1}\frac{|g_{d,z}(p^\nu)|}{p^{\nu\sigma}} = \sum_{\nu\geq 1}\binom{z}{\nu}\frac{1}{p^{\nu\sigma}} = \left(1+\frac{1}{p^\sigma}\right)^z-1.$$

Now that we've given how $g_{d,z}$ behaves on prime powers, let's look at the absolute convergence of $G_d(s;z)$. We have

$$\sum_p\sum_{\nu\geq 1}\frac{|g_{d,z}(p^\nu)|}{p^{\nu\sigma}} = \sum_{(p,d)=1}\sum_{\nu\geq 1}\frac{|g_{d,z}(p^\nu)|}{p^{\nu\sigma}} + \sum_{(p,d)>1}\sum_{\nu\geq 1}\frac{|g_{d,z}(p^\nu)|}{p^{\nu\sigma}}$$

where we look at the two sums separately.

- For $\sigma > 1/2$ we have that the first sum is less than or equal to

$$2M \sum_{(p,d)=1} \frac{1}{p^\sigma(p^\sigma - \sqrt{2})} \leq 2M \sum_{p} \frac{1}{p^\sigma(p^\sigma - \sqrt{2})} \leq \frac{cM}{\sigma - 1/2}$$

  where c is an absolute constant.

- The second sum is equal to

$$\sum_{(p,d)>1} \left\{ \left(1 + \frac{1}{p^\sigma}\right)^z - 1 \right\}$$

  which is finite as d has finite number of prime divisors.

It follows that $G_d(s; z)$ is absolutely convergent for $\sigma > 1/2$, and for $\sigma \geq 3/4$, we have $G_d(s; z) \ll_{A,d} 1$. Based on [9, P. II.5, Th. 5.2], for $x \geq 3$, $N \geq 0$, $A > 0$, $0 < z \leq A$, we have

$$\sum_{n \leq x} \chi_d(n)\mu(n)^2 z^{\omega(n)} = x(\ln x)^{z-1} \left\{ \sum_{0 \leq k \leq N} \frac{\lambda_k(z)}{(\ln x)^k} + \mathcal{O}(R_N(x)) \right\}$$

where

$$\lambda_k(z) := \frac{1}{\Gamma(z-k)} \sum_{h+j=k} \frac{1}{h!j!} G_d^{(h)}(1; z)\gamma_j(z)$$

with $\gamma_j(z)$ defined as in [9, P. II.5, Th. 5.1], and $R_N(x) := R_N(x, c_1, c_2)$. The positive constants c_1, c_2, and the implicit constant in the Landau symbol depend at most on A and the convergence properties of G_d.

We are interested in the case when $z = 1$. As $\lambda_k(z) = 0$ whenever $k \geq z$, we only have to deal with $\lambda_0(1)$. We can also choose N to minimise the error term, see the discussion after [9, P. II.5, Th. 5.2], so at the end of the day we have

$$\sum_{n \leq x} \chi_d(n)\mu(n)^2 = x \left\{ \lambda_0(1) + \mathcal{O}(e^{-c\sqrt{\ln x}}) \right\} \tag{8}$$

where c is a positive constant. As $\gamma_0(1) = 1$ holds, see [9, P. II.5, Th. 5.1], we have $\lambda_0(1) = G_d(1; 1)$ which is

$$\prod_p \left(1 + \frac{\chi_d(p)}{p}\right)\left(1 - \frac{1}{p}\right) = \prod_p \left(1 - \frac{1}{p^2}\right) \prod_p \left(1 + \frac{1 - \chi_d(p)}{p}\right)^{-1} = \frac{1}{\zeta(2)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1}$$

as

$$\prod_p \left(1 + \frac{\chi_d(p)}{p}\right) = \prod_p \left(1 + \frac{1}{p}\right) \prod_p \left(1 + \frac{1 - \chi_d(p)}{p}\right)^{-1}.$$

We get the desired result from (7) using (8). □

We can set our multiplicative function as

$$w(d) := \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \tag{9}$$

and X as $x/\zeta(2)$, furthermore choose R_d based on d to have the required form for $|\mathcal{A}_d|$.

2.2 The Euler product

Lemma 2.3. *For every $y \geq 2$, we have that the inequalities*

$$\frac{4}{3}\frac{e^{-\gamma}}{\ln y}\left(1 - \frac{1}{\ln^2 y}\right) < \prod_{p \leq y}\left(1 - \frac{1}{p+1}\right) < \frac{\pi^2}{6}\frac{e^{-\gamma}}{\ln y}\left(1 + \frac{1}{\ln^2 y}\right)$$

hold, where γ is the Euler-Mascheroni constant.

Note that we could improve the exponent of the logarithmic part inside the parentheses on the right hand side, but this form will suffice for our needs.

Proof. Observe that

$$\prod_{p \leq y}\left(1 - \frac{1}{p+1}\right)\prod_{p \leq y}\left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \leq y}\left(1 - \frac{1}{p^2}\right)^{-1} \tag{10}$$

where we have the partial Euler product of the Riemann zeta function on the right hand side, see [1, Sec. 11.5]. Take note that as $\zeta(2) = \pi^2/6$, we have that the value of this product will be in $[4/3, \pi^2/6)$. Based on [8, Th. 7, Col.], we have

$$\frac{e^{-\gamma}}{\ln y}\left(1 - \frac{1}{\ln^2 y}\right) < \prod_{p \leq y}\left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\ln y}\left(1 + \frac{1}{\ln^2 y}\right) \tag{11}$$

for every $y > 1$. Combining (10) and (11) we get our statement. □

Now we are going to show that there exist positive constants κ, and κ' for our function w so that requirement (1) holds. Take note that the left hand side of requirement (1) is

$$\prod_{\eta \leq p \leq \xi}\left(1 - \frac{1}{p+1}\right)^{-1} \tag{12}$$

when we substitute w, see (9).

$\eta = 2$ When $2 \leq \xi < 3$, then the product will be $3/2$ so for example a $\kappa \geq 1$, and a $\kappa' \geq 1$ will suffice. Otherwise, when $\xi \geq 3$, then we can write

$$\prod_{p \leq \xi}\left(1 - \frac{1}{p+1}\right)^{-1} < (\ln \xi)\frac{3e^{\gamma}}{4}\left(1 - \frac{1}{\ln^2 \xi}\right)^{-1} < 8 \ln \xi$$

so a $\kappa \geq 1$, and a $\kappa' \geq 5$ will suffice in this case.

$2 < \eta < 3$ When $\eta \le \xi < 3$, then the product will be 1, for which the previous constants will be good. Otherwise, when $\xi \ge 3$, then

$$\prod_{2<p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1} < (\ln\xi)\frac{e^\gamma}{2}\left(1-\frac{1}{\ln^2\xi}\right)^{-1} < 6\ln\xi$$

so a $\kappa \ge 1$, and a $\kappa' \ge 6$ will suffice in this case.

$\eta \ge 3$ We can write product (12) as

$$\prod_{p\le\eta}\left(1-\frac{1}{p+1}\right)\prod_{p\le\xi}\left(1-\frac{1}{p+1}\right)^{-1} < \left(\frac{\ln\xi}{\ln\eta}\right)^3\frac{\pi^2}{8}\frac{(\ln^2\eta)+1}{(\ln^2\xi)-1}$$

which shows us that a $\kappa \ge 3$ will suffice, and where

$$\frac{(\ln^2\eta)+1}{(\ln^2\xi)-1} < \frac{(\ln^2\eta)+1}{(\ln^2\eta)-1} = \frac{2}{(\ln^2\eta)-1}+1$$

thus we have to guarantee that the inequality

$$\frac{\pi^2}{8}\left(\frac{2}{(\ln^2\eta)-1}+1\right) < 1+\frac{\kappa'}{\ln\eta}$$

holds, which can be done by selecting a $\kappa' \ge 14$.

Taking all the cases into consideration, it can be seen that we can select the required positive constants κ, and κ'.

2.3 The main asymptotic

Select a function λ satisfying the requirements of proposition 1.1. $\mathcal{P}$ contains all the primes which are not greater than $\lambda(x)$, so we have that the product which we have to compute in expression (2) is equal to

$$\prod_{p\le x}\left(1-\frac{1}{p+1}\right)\prod_{p\le\lambda(x)}\left(1-\frac{1}{p+1}\right)^{-1} \asymp \frac{\ln\lambda(x)}{\ln x}$$

based on lemma 2.3.

Because $|\mathcal{A}_d| = 0$ when $d > x$, we also have that $R_d = 0$ in this case. So we can choose u arbitrarily large, rendering it ineffective in the product of expression (2), while in the error term it suffice to sum just until x. By fixing a $\delta > 0$, we can write the error term as

$$\sum_{\substack{d\le x^{2/3-\eta}\\ d|P(x)}}|R_d| + \sum_{\substack{x^{2/3-\eta}<d\le x\\ d|P(x)}}|R_d| \tag{13}$$

with an $\eta > \delta$, see lemma 2.1. Then for big enough x, we can use equality (6) to write the first sum as

$$c_1 x^{1/2} \sum_{\substack{d \le x^{2/3-\eta} \\ d | P(x)}} d^{-1/4} + c_2 \sum_{\substack{d \le x^{2/3-\eta} \\ d | P(x)}} d^{1/2+\delta}$$

which is in $O(x^{1-\varepsilon(\delta,\eta)})$ with $\varepsilon(\delta,\eta) > 0$, even if we don't take the structure of $\mathcal{P}$ into consideration.

What remains is to handle the second sum in expression (13). We can use lemma 2.2 to write this sum as

$$c_3 x \sum_{\substack{x^{2/3-\eta} < d \le x \\ d | P(x)}} d^{-1} e^{-c\sqrt{\ln x/d}} = c_3 x \Bigg\{ \sum_{\substack{x^{2/3-\eta} < d \le x^{\varepsilon} \\ d | P(x)}} d^{-1} e^{-c\sqrt{\ln x/d}} + \sum_{\substack{x^{\varepsilon} < d \le x \\ d | P(x)}} d^{-1} e^{-c\sqrt{\ln x/d}} \Bigg\}. \tag{14}$$

If $\varepsilon < 2/3 - \eta$, then the first sum on the right hand side disappears. Define

$$A(y) := \sum_{\substack{n \le y \\ n | P(x)}} 1.$$

If $y < x^{\varepsilon}$, then $A(y) = 0$; otherwise when $x^{\varepsilon} \le y \le x$, then

$$\pi(y) - \pi(x^{\varepsilon}) \le A(y) \le \Phi(y, x^{\varepsilon})$$

where $\Phi(y,z)$ is the count of those $n \le y$ for which $P^-(n) > z$, with $P^-(n)$ denoting the smallest prime factor of n, see [9, P. III.6]. We have

$$\Phi(y,z) \ll \frac{y}{\ln z} \tag{15}$$

for $2 \le z \le y$, see [9, P. III.6, Th. 6.2]. Based on equation (15), and on the known approximations for the prime counting function, see [8], we can conclude that

$$A(y) \asymp \frac{y}{\ln x}$$

when $2 \le x^{\varepsilon} \le y \le x$. Now we can use Abel summation to compute the sums on the right hand side of (14). For the first sum, we get

$$\frac{A(x^{\varepsilon})}{x^{\varepsilon}} - \int_{x^{2/3-\eta}}^{x^{\varepsilon}} A(t) \frac{d}{dt} \frac{e^{-c\sqrt{\ln x/t}}}{t} \, dt \asymp \frac{1}{\ln x}$$

and for the second sum we get

$$\frac{A(x)}{x} - \frac{A(x^{\varepsilon})}{x^{\varepsilon}} e^{-c\sqrt{\ln x^{1-\varepsilon}}} - \int_{x^{\varepsilon}}^{x} A(t) \frac{d}{dt} \frac{e^{-c\sqrt{\ln x/t}}}{t} \, dt \asymp \frac{1}{\ln x}$$

for big enough x in both cases.

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References

[1] Apostol T.M.: *Introduction to Analytic Number Theory*. Springer-Verlag (1976).

[2] Guy R.K.: *Unsolved Problems in Number Theory*. Springer (2004).

[3] Hooley C.: A note on square-free numbers in arithmetic progressions. *Bulletin of the London Mathematical Society* 7 (2) (1975) 133–138.

[4] Mangerel A.P.: Squarefree Integers in Arithmetic Progressions to Smooth Moduli. *Forum of Mathematics, Sigma* 9 (1) (2021) 1–47.

[5] Montgomery H.L. and Vaughan R.C.: *Multiplicative Number Theory: I. Classical Theory*. Cambridge University Press (2006).

[6] Nunes R.M.: Squarefree numbers in arithmetic progressions. *Journal of Number Theory* 153 (1) (2015) 1–36.

[7] Prachar K.: Über die kleinste quadratfreie Zahl einer arithmetischen Reihe. *Monatshefte für Mathematik* 62 (1958) 173–176.

[8] Rosser J.B. and Schoenfeld L.: Approximate formulas for some functions of prime numbers. *Illinois Journal of Mathematics* 6 (1) (1962) 64–94.

[9] Tenenbaum G.: *Introduction to Analytic and Probabilistic Number Theory*. American Mathematical Society (2015).

[10] Titchmarsh E.C.: *The Theory of Functions*. Oxford University Press (1939).

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