

Qualitative analysis of strictly non-Volterra quadratic dynamical systems with continuous time

Xaydar Raupovich Rasulov

Abstract. In this article, a continuous analogue of strictly non-Volterra quadratic dynamical systems with continuous time and points of equilibrium is investigated, a phase portrait of the system is constructed, numerical solutions are found, and a comparative analysis is carried out with a particular solution of the system.

1 Introduction

The main theoretical and practical studies on the study of the dynamics of a free population, various problems of physics and economics are reduced to dynamic systems. In this, a special role is played by the quadratic stochastic operators. In this regard, quadratic operators attract the attention of specialists in various fields of mathematics and its applications (see for example [2], [21]). Therefore, studies of quadratical stochastic operators remain relevant.

The concept of a quadratic stochastic operator was first formulated in the article [2]. S. Ulam [21] posed the problem of studying the behavior of the trajectories of quadratic stochastic operators. This problem is mainly solved for Volterra operators (see [5], [6], [7], [8], [14], [4]) of discrete time. In [19], a definition of strictly non-Volterra quadratic stochastic operators is introduced, which is a subclass of non-Volterra operators. But the class of non-Volterra operators has been little studied.

Depending on the problem, either continuous time can be considered (when the states of the system are of interest at each moment), or discrete (when the states of the system are of interest at separate isolated moments of time). System behavior depends on parameters,

MSC 2020: 34C05, 34C40, 34C60, 37C25

Keywords: quadratic stochastic operators, phase portrait, equilibrium positions, trajectory, qualitative analysis.

Affiliation:

Rasulov Xaydar Raupovich, Department of Mathematical analysis, Bukhara state university,
Institute of Mathematics, Bukhara branch, Uzbekistan Academy of Sciences, Bukhara,
Uzbekistan
E-mail: email@xrasulov71@mail.ru

the exact values of which are often unknown. It is fundamentally impossible to solve such problems numerically for all possible parameter values. Therefore, methods are needed that will make it possible to analyze the behavior of solutions to a dynamical system without using a computer.

As we noted above, quadratic stochastic operators of the Volterra class in discrete time have been studied quite deeply. Here are some of them [17], [9], [12], [13], [18]. Also, quadratic stochastic operators with continuous time, that is, various modifications of the predator and prey model and other models, have been studied by many authors in [1], [19] and [10].

In turn, the class of strictly non-Volterra quadratic stochastic operators with continuous time (hereinafter referred to as quadratic dynamical systems) has been studied relatively less. Since there is no general theory studying such operators. Note the works [16].

The most useful aspect of the qualitative theory of dynamical systems, be more correct, dynamical systems with continuous time, is that many important properties of solutions can be predicted in advance without having explicit solutions of the equations.

In this article, we study a qualitative analysis of a quadratic dynamical system, a continuous analogue of the non quadratic stochastic Volterra operator from [15]. The system is shown to come to a simple differential equation, solutions are found, and they are studied separately. Equilibrium points of the system are found and a phase portrait is given.

Analytical solutions of the basic system in some assumptions have been found. In contrast to the results of the study of discrete time dynamic systems, the trajectory of the solutions is a clearly expresses of the aspiration to the equilibrium point at an exponential speed. With the help of computer calculations, numerical solutions were compared with analytical solutions, graphs were drawn, and trajectories were found to overlap over time.

2 Formulation of the problem

In this article, we study a continuous analogue of one quadratic stochastic operator from [15], which in our case has the form:

$$\begin{cases} \dot{x}_0 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - x_0 = f_0(x_0, x_1, x_2), \\ \dot{x}_1 = \frac{1}{2}x_0^2 + \frac{1}{2}x_2^2 + 2x_0x_2 - x_1 = f_1(x_0, x_1, x_2), \\ \dot{x}_2 = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + 2x_0x_1 - x_2 = f_2(x_0, x_1, x_2) \end{cases} \quad (1)$$

or in vector form $\dot{x}(t) = f(x(t))$, where $x(t) = (x_0(t), x_1(t), x_2(t))$, is the state of some system at the instant of continuous time at $t \geq 0, x_0(t) \geq 0, x_1(t) \geq 0, x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$.

3 Main results

Adding all the equations of system (1) and denoting $x_0 + x_1 + x_2 = X$ we obtain an ordinary differential equation

$$X' = X^2 - X, \quad (2)$$

which is a type with shared variables.

Equation (2) has the following solution:

$$X = \frac{1}{1 - Ce^t}, \tag{3}$$

where $C = const \neq 0$. For a fixed negative C , formula (3) gives one solution located in the line $0 < X < 1$. For a fixed positive C , formula (3) defines two solutions, one of which, defined on the interval $-\infty < t < -\ln C$, is located in the half-plane $X > 1$, and the other, defined on the interval $-\ln C < t < +\infty$, is located in the half-plane $X < 0$. In addition to solutions located in these regions indicated above, equation (2) has two more solutions $X \equiv 0$ and $X \equiv 1$, which are formally obtained from (3) with $C = 0$ and $C = \infty$.

This means that the phase portrait of equation (2) consists of five phase curves: two singular points 0 and 1, two rays $(-\infty, 0)$ and $(1, +\infty)$, and an interval $(0, 1)$. Here, the point $X = 0$ is stable, and the point $X = 1$ is an unstable equilibrium.

Let's take a look at each solution separately. Let us show that only one solution $X = 1$ corresponds to the considered system (1).

So, $X = 0$ i.e., $x_0 + x_1 + x_2 = 0$. From here, we find $x_0 = -x_1 - x_2$ and put on the second and third equations of system (2)

$$\begin{cases} \dot{x}_0 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - x_0, \\ \dot{x}_1 = \frac{1}{2}x_1^2 - x_2^2 - x_1x_2 - x_1, \\ \dot{x}_2 = -x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - x_2. \end{cases} \tag{4}$$

After some transformations of the second and third equations of system (4), we obtain an ordinary Abel differential equation of the second kind with respect to x_1 , which does not admit a solution in quadratures [11],[20], [22]. Hence we can say that system (4) in the general case has no solution in the analytical form.

Definition 3.1. Equilibrium positions of system (1) are such points $x^*(t)$ of the phase space, which $f(x^*(t)) = 0$. Obviously, $x^*(t)$ is a solution of system (1), since $\dot{x}^* = 0$.

System (4) has 4 equilibrium points: $N_1(0, 0, 0)$, $N_2(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3})$, $N_3(-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3})$ and $N_4(-\frac{2}{3}, \frac{4}{3}, -\frac{2}{3})$, of which only N_1 is located on the border of the first octant. If $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$, for all $t \geq 0$, then the solution $X = 0$ cannot describe the original problem.

Although, the solution of the system located in the line $0 < X < 1$

$$X = \frac{1}{1 - Ce^t}$$

for a fixed negative C satisfies the condition for $t \geq 0$, $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$, but does not satisfy the condition $x_0(t) + x_1(t) + x_2(t) = 1$. And the solution of the system located on the lines $X < 0$ and $X > 1$ does not satisfy any condition which, for $t \geq 0$, $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$.

We can conclude that only one solution $X \equiv 1$ of equation (2) describes the considered problem (1).

From the condition $x_0(t) + x_1(t) + x_2(t) = 1$ we find $x_2 = 1 - x_0 - x_1$ and assuming the first and second equations of system (1) we get:

$$\begin{cases} \dot{x}_0 = \frac{1}{2}x_0^2 - x_1^2 - x_0x_1 - 2x_0 + x_1 + \frac{1}{2}, \\ \dot{x}_1 = -x_0^2 + \frac{1}{2}x_1^2 - x_0x_1 + x_0 - 2x_1 + \frac{1}{2}, \\ \dot{x}_2 = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + 2x_0x_1 - x_2. \end{cases} \quad (5)$$

By substitution, the first two equations of the system (5)

$$\begin{cases} 3x_0 + 3x_1 = v_0, \\ x_0 - x_1 = v_1, \end{cases}$$

are reduced to the system of equations:

$$\begin{cases} v_0' = \frac{3}{4}v_1^2 - \frac{1}{4}v_0^2 - v_0 + 3, \\ v_1' = \frac{1}{2}v_0v_1 - 3v_1. \end{cases}$$

Finding v_0 and, accordingly, v_0' from the second equation of this system and substituting into the first equation and setting $v_1' = Y$, $v_1'' = YY'$, we obtain the Abel equation of the second type:

$$v_1YY' = \frac{1}{2}Y^2 - 4v_1Y + \frac{3}{8}v_1^4 - 6v_1^2,$$

which does not admit a solution by quadratures [11],[20], [12]. Hence we can say that system (1) in the general case has no solution in the analytical form. Below, we show that system (1), under some assumptions about $x_0(t)$, $x_1(t)$, $x_2(t)$ admits solutions in quadratures.

We find the equilibrium positions of system (1), that is, consider the equation $f(x) = 0$.

$$\begin{cases} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - x_0 = 0, \\ \frac{1}{2}x_0^2 + \frac{1}{2}x_2^2 + 2x_0x_2 - x_1 = 0, \\ \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + 2x_0x_1 - x_2 = 0. \end{cases}$$

This system has 8 points of equilibrium position: $M_1(0, 0, 0)$, $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $M_3(-1, -1, 3)$, $M_4(-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3})$, $M_5(-1, 3, -1)$, $M_6(3, -1, -1)$, $M_7(-\frac{2}{3}, \frac{4}{3}, -\frac{2}{3})$ and $M_8(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3})$, where c is an arbitrary constant.

Considering the conditions of the problem $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$, then we obtain a single point of equilibrium position $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, corresponding to system (1).

Let us investigate the stability of the solution at the point $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of system (1). We calculate at the point M_2 the Jacobian:

$$A = \begin{pmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \frac{\partial f_0}{\partial x_2} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

where $\det|A| = 4 \neq 0$. Linearized system (1) has the following form:

$$\begin{cases} x'_0 = -x_0 + x_1 + x_2 - \frac{1}{3}, \\ x'_1 = x_0 - x_1 + x_2 - \frac{1}{3}, \\ x'_2 = x_0 + x_1 - x_2 - \frac{1}{3}. \end{cases} \quad (6)$$

Assuming $u = x_0 - \frac{1}{3}$, $v = x_1 - \frac{1}{3}$, $w = x_2 - \frac{1}{3}$, we get

$$A = \begin{cases} u' = -u + v + w, \\ v' = u - v + w, \\ w' = u + v - w. \end{cases} \quad (7)$$

Here, the corresponding matrix is $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, its determinant is $\det A = 4 \neq 0$, and its eigenvalues are $\lambda_{1,2} = -2$, $\lambda_3 = 1$. This means that the point of equilibrium position M_2 is an unstable rest point.

Let n_+ , n_0 , n_- be the number of eigenvalues of A (taking into account their multiplicity) with positive, equal to zero and negative real parts, respectively (see [3]).

Definition 3.2. The equilibrium position of the dynamical system (7) is called hyperbolic if $n_0 = 0$, that is, there are no eigenvalues located on the imaginary axis. A hyperbolic equilibrium is called a hyperbolic saddle if $n_+n_- \neq 0$. Hence, on the other hand, the equilibrium position M_2 is a hyperbolic saddle. The general solution of system (6) has the form:

$$\begin{cases} x_0 = \frac{1}{3} - (C_0 + C_1)e^{-2t} + C_2e^t, \\ x_1 = \frac{1}{3} + C_0e^{-2t} + C_2e^t, \\ x_2 = \frac{1}{3} + C_1e^{-2t} + C_2e^t, \end{cases} \quad (8)$$

where $C_0, C_1, C_2 = \text{const.}$

If $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$, then we have to put $C_2 = 0$ and the solution will look like:

$$\begin{cases} x_0 = \frac{1}{3} - (C_0 + C_1)e^{-2t}, \\ x_1 = \frac{1}{3} + C_0e^{-2t}, \\ x_2 = \frac{1}{3} + C_1e^{-2t}. \end{cases} \quad (9)$$

It follows that, as $t \rightarrow +\infty$ the solution of system (5) $(x_0(t), x_1(t), x_2(t))$ tends to the point $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ exponentially fast.

There are two constants in (9) that cannot uniquely determine the solution to the Cauchy problem for system (6). If we put $x_1(t) = x_2(t)$, then the Cauchy problem for system (6) has a unique solution.

In studying the Cauchy problem for equation (1), proceeding from the above, we assume $x_1(t) = x_2(t)$. Substituting $x_0 = 1 - x_1 - x_2$ into the second and third equations of system (1), we obtain:

$$\begin{cases} \dot{x}_0 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - x_0, \\ \dot{x}_1 = \frac{1}{2}x_1^2 - x_2^2 - x_1x_2 - 2x_1 + x_2 + \frac{1}{2}, \\ \dot{x}_2 = -x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 + x_1 - 2x_2 + \frac{1}{2}. \end{cases} \quad (10)$$

Assume $x_1 = x_2 = v$. Then the second and third equations take the form:

$$v' = -\frac{3}{2}v^2 - v + \frac{1}{2}. \quad (11)$$

The phase portrait of equation (11) consists of five phase curves: two singular points $v = -1$ and $v = \frac{1}{3}$, two rays $(-\infty, -1)$ and $(\frac{1}{3}, +\infty)$ and an interval $(-1, \frac{1}{3})$. Here, the point $v = -1$ is stable, and the point $v = \frac{1}{3}$ is an unstable equilibrium.

We solve equation (11) and put the solution in the first equation of system (10). Considering $x_0(t) \geq 0$, $x_1(t) \geq 0$, $x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$, we find the general solution of the system (10). The solution of system (10) in the line $0 < x_1(t), x_2(t) < \frac{1}{3}$ has the following form:

$$\begin{cases} x_0 = \frac{1}{3} + \frac{8Ce^{-2t}}{3(3+Ce^{-2t})} + C_2e^{-t}, \\ x_1 = \frac{1-Ce^{-2t}}{3+Ce^{-2t}}, \\ x_2 = \frac{1-Ce^{-2t}}{3+Ce^{-2t}}, \end{cases} \quad (12)$$

and in the line $\frac{1}{3} < x_1(t), x_2(t) < 1$

$$\begin{cases} x_0 = \frac{1}{3} - \frac{8Ce^{-2t}}{3(3-Ce^{-2t})} + C_3e^{-t}, \\ x_1 = \frac{1+Ce^{-2t}}{3-Ce^{-2t}}, \\ x_2 = \frac{1+Ce^{-2t}}{3-Ce^{-2t}}, \end{cases} \quad (13)$$

where $C, C_2, C_3 = const$.

From (12) and (13) we obtain, respectively, that $x_0(t) + x_1(t) + x_2(t) = 1 + C_2e^{-t}$ and $x_0(t) + x_1(t) + x_2(t) = 1 + C_3e^{-t}$. Hence it follows that the sum $x_0(t) + x_1(t) + x_2(t)$ tends to unity exponentially fast as $t \rightarrow +\infty$, and not separately (see [15]). Now, consider the case when $x_0(t) = x_1(t) = x_2(t) = X$. Substituting this in (2), taking into account the invariance of the equations of the system, we obtain the ordinary differential equation

$$X' = 3X^2 - X, \quad (14)$$

which is of the shared variable type too. As above, it can be shown that equation (14) has the following solution:

$$X = \frac{1}{3 - Ce^t}, \quad (15)$$

where $C = const \neq 0$. For a fixed negative C , formula (15) gives one solution located in the line $0 < X < \frac{1}{3}$. For a fixed positive C , formula (15) defines two solutions, one of which,

defined on the interval $-\infty < t < \ln \frac{3}{C}$, is located in the half-plane $X > \frac{1}{3}$, and the other, defined on the interval $\ln \frac{3}{C} < t < +\infty$, is located in the half-plane $X < 0$. In addition to solutions located in these regions indicated above, equation (14) has two more solutions $X \equiv 0$ and $X \equiv \frac{1}{3}$, which are formally obtained from (15) with $C = 0$ and $C = \infty$.

The phase portrait of equation (14) consists of five phase curves: two singular points 0 and $\frac{1}{3}$, two rays $(-\infty, 0)$ and $(\frac{1}{3}, +\infty)$ and an interval $(0, \frac{1}{3})$. Here, the point $X = 0$ is stable, and the point $X = \frac{1}{3}$ is an unstable equilibrium.

Taking into account, $x_0(t) \geq 0, x_1(t) \geq 0, x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$, as above, we obtain that in this case $x_0(t) = x_1(t) = x_2(t) = \frac{1}{3}$, i.e. the considered dynamical system is at rest.

As mentioned above, we failed to find an analytical solution to the Cauchy problem for the system (1) in the general case. In this regard, with the help of the MathCAD program, solutions of the Cauchy problem for system (1) were found and the phase portrait of the trajectory (with an accuracy of 0.001) was compiled in fig. 1,2. The phase portrait of system (1) is as follows:

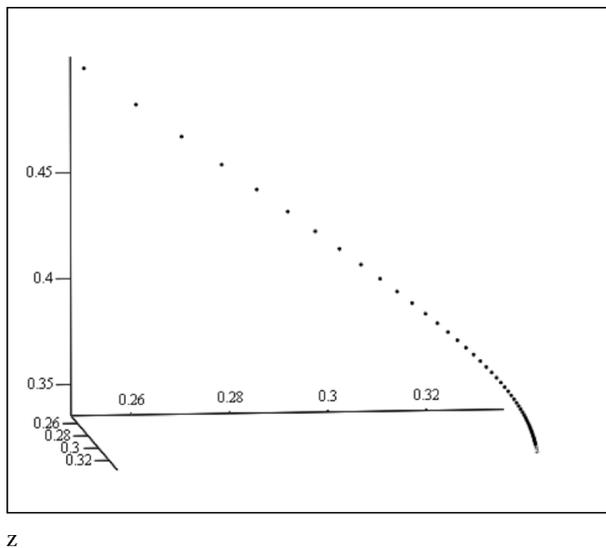
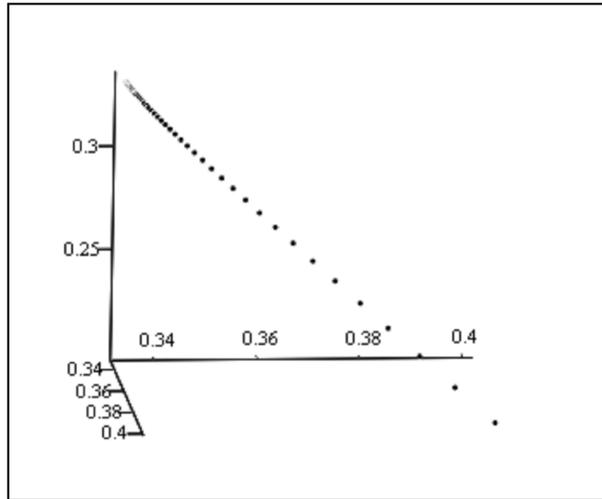


Figure 1: Initial conditions $x_0(0) = 0.5, x_1(0) = 0.25, x_2(0) = 0.25, Z = (x_2, x_1, x_0)$ (in the line $0 < x_1(t), x_2(t) < \frac{1}{3}, x_1(t) = x_2(t)$).



Z

Figure 2: Initial conditions $x_0(0) = 0.2$, $x_1(0) = 0.4$, $x_2(0) = 0.4$, $Z = (x_2, x_1, x_0)$ (in the line $\frac{1}{3} < x_1(t), x_2(t) < 1$, $x_1(t) = x_2(t)$).

Also, for cases where $x_1(t) = x_2(t)$ and $0 < x_1(t), x_2(t) < \frac{1}{3}$ and $\frac{1}{3} < x_1(t), x_2(t) < 1$ compared solutions of system (1), calculated using the MathCAD program with (12) and (13), respectively (fig. 3-6).

To compare the numerical and analytical solutions (12) and (13) on the graph, we denote by $x_0(t), x_1(t), x_2(t)$ numerical solutions, and by $y(t), u(t)$ and $w(t)$ analytic solutions from (12) and (13), respectively.

As a result of the research, it was found that the difference between the numerical solutions (1) and (12), (13) does not exceed 0.001, and for $t \geq 4$, the numerical and analytical solutions almost coincide and together tend to the rest point $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

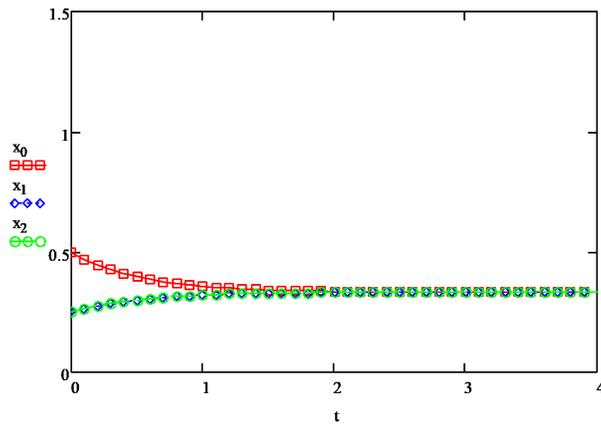


Figure 3: The graph of the numerical solution of system (1) with the initial conditions $x_0(0) = 0.5$, $x_1(0) = 0.25$, $x_2(0) = 0.25$ (in the line $0 < x_1(t)$, $x_2(t) < \frac{1}{3}$, $x_1(t) = x_2(t)$),

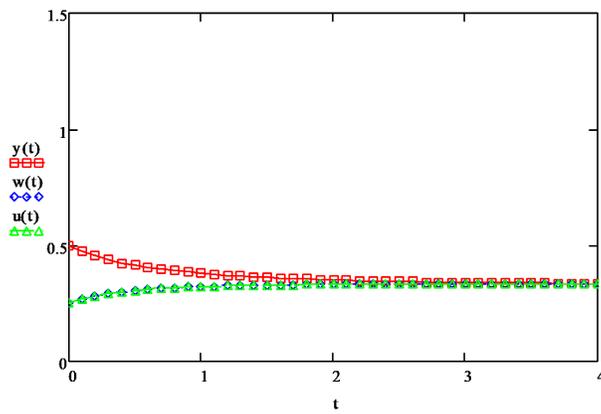


Figure 4: Graph of function (12), solution of system (10) with initial values $y(0) = 0.5$, $u(0) = 0.25$, $w(0) = 0.25$.

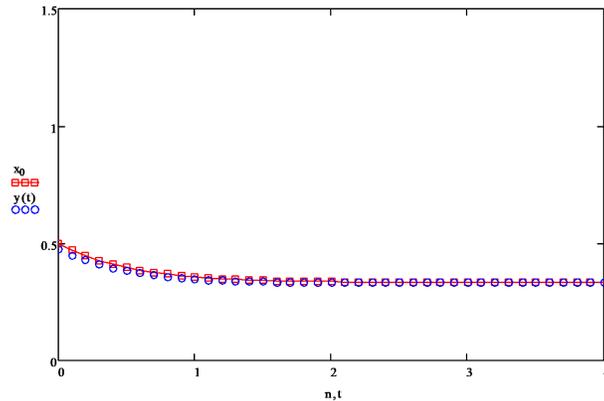


Figure 5: Graph of comparison of the numerical solution of system (1) $x_0(t)$ and analytical solutions (12) $y(t)$ with the same initial values.

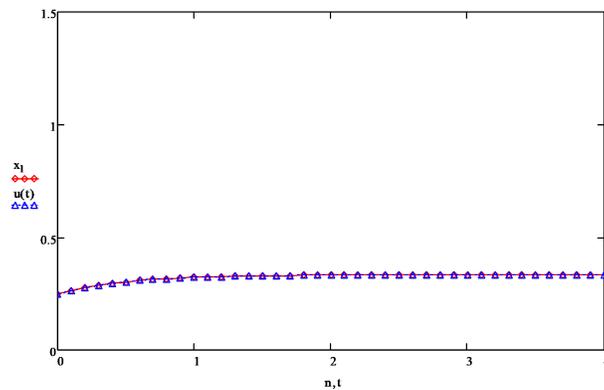


Figure 6: Comparison graph of the numerical solution of system (1) $x_1(t)$ and the analytical solution (12) $w(t)$.

A similar picture takes place when comparing the solution of system (1) with analytical solutions (13) in the line $\frac{1}{3} < x_1(t), x_2(t) < 1, x_1(t) = x_2(t)$, with the same initial conditions.

Thereby, the following theorem has been proved.

Theorem. System (1) with the conditions $t \geq 0, x_0(t) \geq 0, x_1(t) \geq 0, x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$ has a unique fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is a hyperbolic saddle. In addition, the solution (12) and (13) of system (10) exponentially tends to the solution of the system (1).

The results of this paper show that the dynamics of an analogue of strictly non-Volterra operators with continuous time is much richer than the dynamics of non-Volterra operators with discrete time (see [15]). This is also seen from the phase portrait of system (1), which

consists of five curves. It was found that $x_0(t), x_1(t), x_2(t)$ tends to the equilibrium point $M_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ according to formulas (12) and (13), and the sum $x_0(t) + x_1(t) + x_2(t)$ tends to unity exponentially fast as $t \rightarrow +\infty$.

In addition, in the absence of conditions

$$x_0(t) \geq 0, \quad x_1(t) \geq 0, \quad x_2(t) \geq 0 \quad \text{and} \quad x_0(t) + x_1(t) + x_2(t) = 1,$$

the trajectory defined by system (1) in overall, has several curves.

Each strictly non-Volterra quadratic operator is an interesting example in the theory of multidimensional nonlinear dynamical systems with various trajectory behavior.

A comparative analysis of the results obtained in [15] and in the present paper shows that the equilibrium positions of system (1) coincide with the fixed point of the operator [15] only when $x_0(t) \geq 0, x_1(t) \geq 0, x_2(t) \geq 0$ and $x_0(t) + x_1(t) + x_2(t) = 1$ and both of these trajectories tend to the equilibrium position exponentially fast.

In addition, we can say that considering an analog of a quadratic operator (studied in [15]) with continuous time gives some advantage. Since it has been established that on the basis of computer calculations it can be said that for $t \geq 4$ solutions of system (1) obtained using MathCAD (more than 100 solutions with different initial values are calculated and compared) coincide with the solutions obtained analytically (12) and (13).

The author thanks prof. U.A. Rozikov for useful discussions.

References

- [1] Angelina M. L., Melissa A.P., Thomas S., Nakul Ch.: Mathematical modeling of mosquito dispersal in a heterogeneous environment. *Mathematical Biosciences* 241 (2013) 198–216.
- [2] Bernshtein S. N.: Solution of one mathematical problem related to the theory of inheritance. *Scientific Notes of the Research Department of Ukraine. Department of Mathematics [in russian]* (1) (1924) 83–115.
- [3] Bratus A. S., Novozhilov A. S. and Platonov A. P.: *Dynamical systems and models of biology*. Moscow Fizmatlit [in russian] (2009).
- [4] Ganikhodzhaev N. N. and Zanin D. V.: On a necessary condition for the ergodicity of quadratic operators defined on the two-dimensional simplex. *Russian Math. Surveys* 59 (3) (2004) 571–572.
- [5] Ganikhodzhaev R. N.: Quadratic stochastic operators, Lyapunov functions, and tournaments. *Russian Acad. Sci. Math. sb. [in russian]* 183 (8) (1992) 119–140.
- [6] Ganikhodzhaev R. N. and Sarimsakov A. T.: Mathematical model of the coalition of biological systems. *Dokl. AN UzSSR [in russian]* (3) (1992) 14–17.
- [7] Ganikhodzhaev R. N.: A family of quadratic stochastic operators, operating in S_2 . *Dokl. AN UzSSR [in russian]* (1) (1989) 3–5.
- [8] Ganikhodzhaev R. N. and Eshmamatova D. B.: Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories. *Vladikavkaz. matem. zhurn., [in russian]* 8 (2) (2006) 12–28.

- [9] Ganikhodzhaev R. N., Mukhamedov F. M. and Rozikov U. A.: Quadratic stochastic operators and processes: results and open problems. *Inf. Dim. Anal. Quant. Prob. Rel. Fields* 14 (2) (2011) 279–335.
- [10] Jicai Huang, Shigui Ruan and Jing Song: Bifurcations in a predator-prey system of Leslie type with generalized Holling type III functional response. *J. Differential Equations* (257) (2014) 1721–1752.
- [11] Kamke E.: *Handbook of ordinary differential equations. Moscow, publishing house.* Moscow, Nauka [in russian] (1976).
- [12] Kesten H.: Quadratic transformations: a model for population growth. *Advances in Appl. Probability* (2) (1970) 1–82.
- [13] Lyubich Yu. I.: *Mathematical Structures in Population Genetics (Mathematics, 22)*. Springer-Verlag, Berlin (1992).
- [14] Mukhamedov F. M.: Infinite-dimensional quadratic Volterra operators. *Russian Math. Surveys* 55 (6) (2000) 1161–1162.
- [15] Mukhitdinov R. T.: On a strictly non-Volterra quadratic operator. Abstracts of the international conference Operator algebras and quantum probability theory, University, Tashkent [in russian] (2005). .
- [16] Rasulov Kh. R.: On a continuous time F - quadratic dynamical system. *Uzbek mathematical journal* (4) (2018) 126–130.
- [17] Rozikov U. A. and Zhamilov U. U.: The dynamics of strictly non-Volterra quadratic stochastic operators on the 2-simplex. *Russian Acad. Sci. Math. Sb.* 200 (9) (2009) 1339–1351.
- [18] Rozikov U. A. and Zhamilov U. U.: F-quadratic stochastic operators. *Math. Notes* 83 (4) (2008) 554–559.
- [19] Tianran Zhang, Wendi Wang: Hopf bifurcation and bistability of a nutrient-phytoplankton-zooplankton model. *Applied Mathematical Modeling* 36 (12) (2012) 6225–6235.
- [20] Tokmachev M. S.: Abel differential equations of the second kind. *Bulletin of Novgorod State University [in russian]* (22) (2002) 19–23.
- [21] Ulam S. M.: *A collection of mathematical problems.* New York-London, Interscience Publ. (1960).
- [22] Zaitsev V. F. and Polyanin A. D.: *Handbook on ordinary differential equations.* Moscow, Fizmatlit house [in russian] (2001).

Received: December 13, 2020

Accepted for publication: March 04, 2021

Communicated by: Utkir Rozikov