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Certain paracontact metrics satisfying the critical point equation

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Abstract. The aim of this paper is to study the CPE (Critical Point Equation) on some paracontact metric manifolds. First, we prove that if a para-Sasakian metric satisfies the CPE, then it is Einstein with constant scalar curvature -2n(2n+1). Next, we prove that if (κ, μ)-paracontact metric satisfies the CPE, then it is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature -4.

1 Introduction

In [9], Kaneyuki and Williams introduce a new structure on pseudo-Riemannian geometry called paracontact structure, as a natural odd-dimensional counterpart to para-Harmitian structures, just like contact metric structures correspond to the Hermitian one. The importance of paracontact geometry comes from the theory of para-K$\breve{a}$hler manifolds. A systematic study of paracontact metric manifolds and their subclasses was started by Zamkovoy [19]. Since then, many authors have investigated paracontact geometry using various meaningful geometric conditions. We refer the reader to ([2], [5], [6], [11], [12], [14], [15], [17], [18], [20]) for some related results on paracontact geometry.

On the other hand, in [3] (for details, see Chapter 2), A. Besse studied the Hilbert-Einstein functional and proved that the critical points of this functional are the Einstein metrics. The Hilbert-Einstein functional has the following Euler-Lagrange equation

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g \lambda)g + \mathrm{Hess}_g \lambda - \lambda \,\mathrm{Ric}_g = \mathrm{Ric}_g{}^o, \tag{1}$$

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for a critical point g, where Δ_g, Ric_g, ${\mathrm{Ric}_g}^o$ and $\mathrm{Hess}\,\lambda$ are, respectively, the Laplacian, the Ricci tensor, the traceless Ricci tensor and the Hessian of the smooth function λ on M. Here $\mathcal{L}_g^*(\lambda)$ is the formal L^2-adjoint of the linearized scalar curvature operator $\mathcal{L}_g(\lambda)$. The Eq. (1) is called the Critical Point Equation (shortly, CPE). The function λ is known as the potential function. From now, we consider a metric g with a non-trivial potential function λ as a solution of the CPE and is denoted by (g,λ). Also, we can express the equation (1) in the form

$$(\mathrm{Ric}_g - \frac{r}{n-1} g)\lambda - \mathrm{Hess}_g\lambda = \frac{r}{n} g - \mathrm{Ric}_g\,. \tag{2}$$

First, we note that if λ is constant in the equation (2), then $\lambda = 0$ and g becomes Einstein. Further, tracing (2) we deduce $\Delta_g\lambda = -\frac{r}{n-1}\lambda$. From this, it follows that λ is an eigenfunction of the Laplacian. Since the Laplacian has non-positive spectrum, the scalar curvature must be positive. In [3], A. Besse first conjectured that the solution of the CPE must be Einstein (known as the CPE conjecture). To my knowledge, many authors study the CPE satisfying either some curvature conditions or some conditions on the potential functions of Riemannian manifolds. In [1], Barros and Ribeiro proved that the CPE conjecture is also true for half conformally flat. Further, in [8], Hwang proved that the CPE conjecture is valid under certain conditions on the bounds of the potential function λ. Recently, Nato [13] obtained a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein. Recently, the author considered the CPE on contact metric manifolds (see [7],[16]) and proved that the CPE conjecture is true for K-contact manifold. However, the CPE has not yet been considered on pseudo-Riemannian manifolds, for instance, paracontact metric manifolds. Hence it deserves special attention to consider the CPE on certain classes of paracontact metric manifolds. Here we characterize the solution of the CPE on certain classes of paracontact metric manifolds and prove that the CPE conjecture is true for para-Sasakian manifold.

The paper is structured as follows: in Section 2, a very brief review of paracontact geometry is given. Next, we consider the CPE on para-Sasakian manifold in Section 3, and we prove that a if a para-Sasakian metric satisfies the CPE, then it is Einstein with Einstein constant $-2n$ and has constant scalar curvature. Finally, in Section 4, we consider the CPE on (κ,μ)-paracontact manifold and prove that if (κ,μ)-paracontact metric satisfies the CPE, then it is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature.

2 Preliminaries

In this section, we shall collect some fundamental results regarding paracontact metric manifolds (for more details, see [5],[19]). A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (φ,ξ,η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η such that

$$\varphi^2 = I - \eta o\xi,\ \ \varphi(\xi) = 0,\ \ \eta o\varphi = 0,\ \ \eta(\xi) = 1, \tag{3}$$

and there exists a distribution

$$\mathcal{D} : p \in M \to \mathcal{D}_p \subset {}_pM : \mathcal{D}_p = Ker(\eta) = \{x \in T_pM : \eta(x) = 0\},$$

called paracontact distribution generated by η.

Let $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M^{2n+1}. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\,\eta(Y), \quad X, Y \in \mathfrak{X}(M), \tag{4}$$

then we say that M has an almost paracontact metric structure (φ, ξ, η, g) and g is called compatible metric. Any compatible metric g with a given almost paracontact structure is necessary for signature $(n+1, n)$. The fundamental 2-form Φ of an almost paracontact metric structure (φ, ξ, η, g) defined by

$$\Phi(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If $\Phi = d\eta$, then the manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is called a paracontact metric manifold. On a paracontact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we consider a self-adjoint operator $h = \frac{1}{2}\pounds_\xi\varphi$, where $\pounds_\xi$ is the Lie-derivative along ξ. The operator h satisfy [19]:

$$\mathrm{Tr}_g\, h = 0, \;\; h\xi = 0, \;\; h\varphi = -\varphi h.$$

On a paracontact metric manifold [19]:

$$\nabla_X \xi = -\varphi X + \varphi h X, \quad X \in \mathfrak{X}(M), \tag{5}$$

where ∇ is the operator of covariant differentiation of g. If the vector field ξ is a Killing (equivalently $h = 0$) then M is said to be a K-paracontact manifold. Moreover, on any K-paracontact manifold [19]:

$$\nabla_X \xi = -\varphi X, \tag{6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{7}$$

$$Q\xi = -2n\,\xi, \tag{8}$$

for all vector fields X, Y on M, where R is the Riemann curvature tensor of g and Q denotes the Ricci operator associated with the Ricci tensor given by $\mathrm{Ric}_g(X, Y) = g(QX, Y)$ for all vector fields X, Y on M.

Moreover, a paracontact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be para-Sasakian if the following integrality condition is satisfied $[\varphi, \varphi] = -2\, d\eta \otimes \xi$, where $[\varphi, \varphi]$ denotes the Nijenhuis tensor of φ. Equivalently, a paracontact metric manifold is said to para-Sasakian if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X,$$

for all vector fields X, Y on M, see [19]. A para-Sasakian manifold is K-paracontact [4] but the converse is true only in dimension 3, see [11].

3 On K-paracontact and para-Sasakian manifolds

In this section, we study the CPE on K-paracontact and para-Sasakian manifolds. Before entering into our main results we prove the following.

Lemma 3.1. *On a K-paracontact manifold $M^{2n+1}(\varphi,\xi,\eta,g)$, we have*

$$(i)\ (\nabla_X Q)\xi = Q\varphi X + 2n\,\varphi X,$$
$$(ii)\ (\nabla_\xi Q)X = Q\varphi X - \varphi QX, \quad X \in \mathfrak{X}(M).$$

Proof. First, taking the covariant derivative of (8) along an arbitrary vector field X on M and then using the result (6) we have the first one. Further, since ξ is Killing, we have $\pounds_\xi \operatorname{Ric}_g = 0$. This implies $(\pounds_\xi Q)X = 0$ for any vector field X on M. From which it follows

$$\begin{aligned} 0 &= \pounds_\xi(QX) - Q(\pounds_\xi X) \\ &= \nabla_\xi QX - \nabla_{QX}\xi - Q(\nabla_\xi X) + Q(\nabla_X \xi) \\ &= (\nabla_\xi Q)X - \nabla_{QX}\xi + Q(\nabla_X \xi), \end{aligned}$$

for any vector field X on M. Using (6) the last equation gives the second result. This completes the proof. □

Lemma 3.2. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a K-paracontact manifold. If (g,λ) is a non-constant solution of the CPE, then we have*

$$(Q\varphi + \varphi Q)X = -4n\,\varphi X, \quad X \in \mathfrak{X}(M). \tag{9}$$

Proof. First, we note that (1) implies $\triangle_g \lambda = -\frac{r\lambda}{2n}$. Thus, equation (1) can be exhibited as

$$\nabla_X D\lambda = (\lambda+1)QX + fX, \tag{10}$$

for all vector fields X on M; where $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$. Using this in the well-known expression of the curvature tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we obtain

$$\begin{aligned} R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + (\lambda+1)\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ + (Xf)Y - (Yf)X, \end{aligned} \tag{11}$$

for all vector fields X, Y on M. Now, substituting X by ξ in (11) and using the Lemma 3.1 we can compute

$$\begin{aligned} R(\xi,Y)D\lambda =& (\xi\lambda)QY + 2n(Y\lambda)\xi - (\lambda+1)\{\varphi QY + 2n\varphi Y\} \\ &+ (\xi f)Y - (Yf)\xi, \end{aligned} \tag{12}$$

for any vector fields Y on M. On the other hand, we obtain from (6) that

$$g(R(X,Y)\xi, Z) = g(\nabla_X \varphi)Y, Z) - g(\nabla_Y \varphi)X, Z),$$

for all vector fields X, Y on M. Applying Bianchis's first identity and the last equation we achieve $R(\xi, X)Y = (\nabla_\varphi X)Y$ for all vector fields X, Y on M. Making use of this in the scalar product of (12) with an arbitrary vector field X on M provides

$$\begin{aligned} g((\nabla_Y\varphi)X, D\lambda) =&(\lambda+1)\{g(Q\varphi Y, X) + 2ng(\varphi Y, X)\} - (\xi\lambda)g(QY, X) \\ &- (\xi f)g(X, Y) - \{2n(Y\lambda) - (Yf)\}\eta(X). \end{aligned} \tag{13}$$

Taking into account of (3), (6) and putting $X = \varphi X$, $Y = \varphi Y$ in (13) yields

$$\begin{aligned} g((\nabla_{\varphi Y}\varphi)\varphi X, D\lambda) =&(\lambda+1)\{2ng(Y, \varphi X) - g(Q\varphi Y, X\} \\ &- (\xi\lambda)g(Q\varphi Y, \varphi X) + (\xi f)\{g(X, Y) - \eta(X)\eta(Y)\}, \end{aligned} \tag{14}$$

Now, adding (13) with (14) and using (3), (8) and the well-known formula (see [19])

$$(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X\varphi)Y = 2g(X, Y)\xi - \eta(Y)(X - hX + \eta(X)\xi),$$

one can compute

$$\begin{aligned} 2\xi(\lambda - f)\, g(X, Y) =&Y\{(2n+1)\lambda - f\}\,\eta(X) \\ &+ \xi(\lambda - f)\,\eta(X)\eta(Y) - (\lambda+1)\,g(Q\varphi Y + \varphi QY, X) \\ &+ 4n(\lambda+1)\,g(\varphi X, Y) - (\xi\lambda)\,\{g(Q\varphi Y, \varphi X) - g(QY, X)\}. \end{aligned}$$

Anti-symmetrizing the foregoing equation yields

$$\begin{aligned} &Y\{(2n+1)\lambda - f\}\eta(X) - X\{(2n+1)\lambda - f\}\eta(Y) \\ &\quad + 8n(\lambda+1)\,g(\varphi X, Y) + 2(\lambda+1)\,g(Q\varphi X + \varphi QX, Y) = 0. \end{aligned} \tag{15}$$

At this point, we replace X by φX and Y by φY in (15) to achieve

$$(\lambda+1)\Big\{g(Q\varphi X, Y) + g(\varphi QX, Y) + 4n\,g(\varphi X, Y))\Big\} = 0.$$

Since λ is non-constant in the interior of M, the last equation gives us the required result. □

Thus, from the last Lemma, one can easily conclude the following result.

Theorem 3.3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a K-paracontact manifold and the Ricci operator Q commutes with paracontact structure φ. If (g, λ) is a non-constant solution of the CPE, then g is Einstein with Einstein constant $-2n$ and has constant scalar curvature $-2n(2n+1)$.*

Proof. By hypothesis, the Ricci operator Q commutes with paracontact structure φ, i.e., $Q\varphi = \varphi Q$. Applying this in (9) and then substituting X by φX, and using (3) gives that $QX = -2nX$ for any vector fields X. This shows that M is Einstein with Einstein constant $-2n$. This completes that proof. □

On para-Sasakian manifold, the Ricci operator satisfies (see [19], Lemma 3.15):

$$QX - \varphi Q\varphi X = -2n\,\eta(X)\xi, \tag{16}$$

for all vector fields X on M. Operating (16) by φ and using (3) we see that the Ricci operator Q commutes with paracontact structure φ. Hence, from Theorem 3.3, we can deduce the following result.

Theorem 3.4. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Sasakian manifold. If (g, λ) is a non-constant solution of the CPE, then g is Einstein with Einstein constant $-2n$ and has constant scalar curvature $-2n(2n+1)$.*

Remark 3.5. There are some partial answers to the CPE Conjecture. For example, Lafontaine proved that the CPE conjecture is true under conformally flat assumption with $Ker\mathcal{L}_g^*(\lambda) \neq 0$. Further, Barros and Ribeiro [1] proved that the CPE conjecture is also true for half conformally flat. Therefore, in this section we prove the CPE conjecture in a subclass paracontact metric manifolds, for instance, para-Sasakian manifolds.

4 On (κ, μ)-paracontact manifolds

In [6], Cappelletti-Montano et al introduce the notion of nullity conditions in paracontact geometry. According to them a (κ, μ)-paracontact manifold is a paracontact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{17}$$

for all vector fields X, Y on M and for some real numbers (κ, μ). The class of (κ, μ)-paracontact metric manifold contains para-Sasakian manifolds. Since then, many geometers have studied (κ, μ)-paracontact manifold and obtained various important properties of these manifolds (see, for instance, [10]). On (κ, μ)-paracontact manifold the following formulas are valid (e.g., [6]):

$$h^2 = (\kappa + 1)\varphi^2, \tag{18}$$

$$Q\xi = 2n\kappa\xi. \tag{19}$$

First of all, we recall a lemma for our main proof.

Lemma 4.1. *(See [6]) Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a (κ, μ)-paracontact manifold with $\kappa > -1$, then the Ricci operator Q of M can be expressed as*

$$\begin{aligned} QX =& [2(1-n) + n\mu]X + [2(n-1) + \mu]hX \\ &+ [2(n-1) + n(2\kappa - \mu)]\eta(X)\xi, \end{aligned} \tag{20}$$

for any vector field X on M. In this case, the scalar curvature of M is $2n(2(1-n)+\kappa+n\mu)$.

Now, we prove a lemma for later use.

Lemma 4.2. *On a (κ,μ)-paracontact manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ with $\kappa>-1$, we have*

$$(\nabla_\xi Q)X=\mu(2(n-1)+\mu)h\varphi X, \tag{21}$$

for any vector field X in M.

Proof. Replacing Y by ξ in (17) and using $\varphi^2=I-\eta\otimes\xi$, we can easily deduce that

$$R(X,\xi)\xi=\kappa\{X-\eta(X)\xi\}+\mu hX, \tag{22}$$

for any vector field X in M. Recalling a formula on paracontact metric manifold [19]:

$$(\nabla_\xi h)X=-\varphi X+\varphi h^2X-\varphi R(X,\xi)\xi, \tag{23}$$

for any vector field X in M. By virtue of (3), (18), (22) and $\varphi h=-h\varphi$, the Eq. (23) yields

$$(\nabla_\xi h)X=-\mu\varphi hX, \tag{24}$$

for any vector field X in M. Covariant derivative of (20) along ξ, we have

$$\begin{aligned}(\nabla_\xi Q)X+Q(\nabla_\xi X)=&\{2(1-n)+n\mu\}\nabla_\xi X+\{2(n-1)+\mu\}\{(\nabla_\xi h)X\\&+h(\nabla_\xi X)\}+\{2(n-1)+n(2\kappa-\mu)\}g(\nabla_\xi X,\xi)\xi,\end{aligned} \tag{25}$$

for any vector field X on M. Next, putting $X=\nabla_\xi X$ in (20), we get

$$\begin{aligned}Q(\nabla_\xi X)=&\{2(1-n)+n\mu\}\nabla_\xi X+\{2(n-1)+\mu\}h(\nabla_\xi X)\\&+\{2(n-1)+n(2\kappa-\mu)\}g(\nabla_\xi X,\xi)\xi.\end{aligned}$$

Using this and (24) in (25) we get the required result. □

Lemma 4.3. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a (κ,μ)-paracontact manifold with $\kappa>-1$. If (g,λ) is a non-constant solution of the CPE, then we have*

$$\kappa(2-\mu)=\mu(n+1). \tag{26}$$

Proof. First, differentiating (19) along an arbitrary vector field X on M and using (5) it follows that

$$(\nabla_X Q)\xi=Q(\varphi-\varphi h)X-2n\kappa\,(\varphi-\varphi h)X. \tag{27}$$

Thus, the scalar product of (11) with ξ and making use of (19), (27) gives that

$$\begin{aligned}g(R(X,Y)D\lambda,\xi)=&2n\kappa\,[(X\lambda)\eta(Y)-(Y\lambda)\eta(X)]+(\lambda+1)\,g(Q\varphi X+\varphi QX,Y)\\&-(\lambda+1)\,g(Q\varphi hX+h\varphi QX,Y)+4n\kappa\,(\lambda+1)\,g(\varphi X,Y)\\&+(Xf)\eta(Y)-(Yf)\eta(X).\end{aligned} \tag{28}$$

Replacing X by φX, Y by φY in (28) and noting that $R(\varphi X, \varphi Y)\xi = 0$ (follows from (17)), we obtain

$$(\lambda + 1)\,[Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa\,\varphi X] = 0,$$

for any vector field X on M. Since λ is non-constant in the interior M, the foregoing equation provides

$$Q\varphi X + \varphi QX + \varphi QhX + hQ\varphi X - 4n\kappa\,\varphi X = 0, \tag{29}$$

for any vector field X on M. Now, substituting X by φX in the relation (20) gives that

$$Q\varphi X = [2(1-n) + n\mu]\,\varphi X + [2(n-1) + \mu]h\varphi X.$$

On the other hand, by acting h on the previous equation and making use of (18) implies that

$$hQ\varphi X = [2(1-n) + n\mu]\,h\varphi X + (\kappa + 1)[2(n-1) + \mu]\,\varphi X.$$

Also, operating φ on (20) gives

$$\varphi QX = [2(1-n) + n\mu]\,\varphi X + [2(n-1) + \mu]\varphi hX.$$

Taking hX instead of X and using (18), the last equation reduces to

$$\varphi QhX = [2(1-n) + n\mu]\,\varphi hX + (\kappa + 1)[2(n-1) + \mu]\,\varphi X.$$

With the help of the last four equations in (29) and using (18) we obtain the required result. Hence the proof. □

Theorem 4.4. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a (κ, μ)-paracontact manifold $\kappa > -1$. If (g, λ) is a non-constant solution of the CPE, then M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature -4.*

Proof. Taking into account of (19) and (22) and substituting ξ instead of X in (28) we obtain

$$(2n+1)\kappa\{D\lambda - (\xi\lambda)\xi\} + \mu hD\lambda - (\xi f)\xi + Df = 0. \tag{30}$$

Contracting (11) over X with respect to an orthonormal basis $\{e_i\}$, we have

$$\lambda\Big\{\sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)Y, e_i) - g((\nabla_Y Q)e_i, e_i)\Big\} - r\,(Y\lambda) - 2n\,(Yf) = 0. \tag{31}$$

As the scalar curvature $r = 2n\{2(1-n)+\kappa+n\mu\}$ (from Lemma 4.1) is constant, using the formula $\operatorname{div} Q = \frac{1}{2}dr$ (follows from contraction of Bianchi's second identity) one can compute

$$\begin{aligned}\sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)Y, e_i) - g((\nabla_Y Q)e_i, e_i) &= (\operatorname{div} Q)Y - Y(\operatorname{Tr}_g Q)\\ &= \frac{1}{2}(Yr) - (Yr) = -\frac{1}{2}(Yr) = 0.\end{aligned}$$

Using this in (31) yields

$$r\,D\lambda + 2n\,Df = 0. \tag{32}$$

Combining this with (30), we get

$$2n(2n+1)\kappa\{D\lambda - (\xi\lambda)\xi\} + 2n\mu\, hD\lambda - 2n\,(\xi f)\xi - r\,D\lambda = 0. \tag{33}$$

From (10) and (19), we have

$$\nabla_\xi D\lambda = [2n\kappa(\lambda+1) + f]\xi. \tag{34}$$

Now, taking covariant derivative of (33) along ξ and using (21), (24), (34), we obtain

$$\begin{aligned}\{2n(2n+1)\kappa - r\}\{2n\kappa(\lambda+1)+f\}\xi - 2n(2n+1)\kappa\,\xi(\xi\lambda)\xi\\ + 2n\mu^2\, h\varphi D\lambda - 2n\,\xi(\xi f)\xi = 0.\end{aligned} \tag{35}$$

Next, operating (35) by φ and using (3) we have $\mu^2 hD\lambda = 0$. By the action of h and using (18), (3) gives

$$(\kappa+1)\mu^2\,(D\lambda - (\xi\lambda)\xi) = 0.$$

As $\kappa > -1$, we have either (i) $\mu = 0$, or (ii) $\mu \neq 0$.

- **Case (i):** In this case, it follows from (26) that $\kappa = 0$. Hence $R(X,Y)\xi = 0$ for any vector field X, Y on M, and therefore, M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature -4 (see [20],Theorem 3.3).

- **Case (ii):** This case yields $D\lambda = (\xi\lambda)\xi$. Differentiating this along an arbitrary vector field X together with (5) entails that $\nabla_X D\lambda = X(\xi\lambda)\xi - (\xi\lambda)(\varphi X - \varphi hX)$. Since $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$, the foregoing equation shows

  $$X(\xi\lambda)\eta(Y) - Y(\xi\lambda)\eta(X) + (\xi\lambda)\,d\eta(X,Y) = 0.$$

  Since $d\eta$ is non-zero for a paracontact metric structure, replacing X by φX and Y by φY in previous result, we get $\xi\lambda = 0$. Hence $D\lambda = 0$, and therefore, λ is constant, which is a contradiction. This completes the proof. □

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