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Congruences concerning Legendre polynomials modulo p^2

Aeran Kim

Abstract. In this article, we extend Z. H. Sun's congruences concerning Legendre polynomials $P_{\frac{p-1}{2}}(x)$ to $P_{\frac{p+1}{2}}(x)$ for odd prime p, which enables us to deduce some congruences resembling

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2 \pmod{p^2}.$$

1 Introduction

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}} \qquad \text{for } |t|<1.$$

It is well known [1,2,3] that

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\,(2n-2k)!}{k!\,(n-k)!\,(n-2k)!}x^{n-2k} = \frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{1}$$

where $[x]$ is the greatest integer not exceeding x and

$$(n+1)\,P_{n+1}(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x). \tag{2}$$

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Contact information:
Affiliation: Private Mathematics Academy, Republic of Korea.
Email: ae_ran_kim@hotmail.com.

Moreover,

$$P_n(x) = (-1)^n P_n(-x) \qquad \text{and} \qquad P_n(0) = \begin{cases} 0 & \text{for odd } n, \\ \dfrac{(-1)^{\frac{n}{2}}}{2^n}\dbinom{n}{\frac{n}{2}} & \text{for even } n. \end{cases} \tag{3}$$

Using the property of Legendre polynomials in Eq. (1), Z. H. Sun obtained various congruences for $P_{\frac{p-1}{2}}(x)$ modulo p^2, where p is an odd prime and x is a rational p-integer. For example, he showed that

$$\sum_{k=0}^{p-1} \frac{1}{16^k}\binom{2k}{k}^2 \left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k}\binom{2k}{k}^2 \left(x^k - \left(\frac{x}{p}\right) x^{-k}\right) \equiv 0 \pmod{p} \quad \text{for } x \not\equiv 0 \pmod{p},$$

where $\left(\frac{x}{p}\right)$ is the Jacobi symbol. Inspired by these results, we consider $P_{\frac{p+1}{2}}(x)$ and get some congruences modulo p^2 resembling $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k(2k-1)^2}\binom{2k}{k}^2$ with odd prime p. Especially using the relation $P_{\frac{p-1}{2}}(x)$, $P_{\frac{p+1}{2}}(x)$ and (2), we construct a congruence $P_{\frac{p-3}{2}}(x)$ modulo p^2.

2 Main results

Proposition 2.1. *For $n \in \mathbb{N}$ we have*

$$P_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k.$$

Proof. Refer to [1]. □

Corollary 2.2. *Let p be an odd prime and $k \in \left\{1, 2, \cdots, \frac{p+1}{2}\right\}$. Then*

$$\binom{\frac{p+1}{2}+k}{2k} \equiv \frac{(p+2k+1)(p+2k-1)}{(-16)^k(2k-1)^2}\binom{2k}{k}\left(p^2\sum_{i=1}^{k-1}\frac{1}{(2i-1)^2} - 1\right) \pmod{p^4}.$$

Proof. We can easily know that

$$\begin{aligned}
&\binom{\frac{p+1}{2}+k}{2k} \\
&= \frac{\left(\frac{p+1}{2}+k\right)\left(\frac{p+1}{2}+k-1\right)\left(\frac{p+1}{2}+k-2\right)\cdots\left(\frac{p+1}{2}-k+1\right)}{(2k)!} \\
&= \frac{(p+2k+1)\,(p+2k-1)\,(p+2k-3)\cdots(p-2k+3)}{2^{2k}\cdot(2k)!} \\
&= \frac{(p+2k+1)\,(p+2k-1)\cdot(p^2-1^2)\,(p^2-3^2)\cdots\left(p^2-(2k-3)^2\right)}{2^{2k}\cdot(2k)!} \\
&\equiv \frac{(p+2k+1)\,(p+2k-1)}{2^{2k}\cdot(2k)!}\Big[(-1)^{k-2}p^2\Big\{1^2\cdot 3^2\cdots(2k-7)^2\,(2k-5)^2 \\
&\qquad + 1^2\cdot 3^2\cdots(2k-7)^2\,(2k-3)^2+\cdots+3^2\cdots(2k-5)^2\,(2k-3)^2\Big\} \\
&\qquad + (-1)^{k-1}\cdot 1^2\cdot 3^2\cdots(2k-5)^2\,(2k-3)^2\Big] \pmod{p^4} \\
&= \frac{(p+2k+1)\,(p+2k-1)}{2^{2k}\cdot(2k)!}(-1)^k\cdot 1^2\cdot 3^2\cdots(2k-5)^2\,(2k-3)^2 \\
&\qquad \times\left(p^2\sum_{i=1}^{k-1}\frac{1}{(2i-1)^2}-1\right) \pmod{p^4}.
\end{aligned}$$

Then, since

$$\begin{aligned}
1^2\cdot 3^2\cdots(2k-5)^2\,(2k-3)^2 &= \frac{(2k-3)!^2}{2^2\cdot 4^2\cdots(2k-6)^2\,(2k-4)^2} \\
&= \frac{(2k-3)!^2}{(2^{k-2})^2\cdot(k-2)!^2} \\
&= \frac{(2k-3)!}{2^{2k-4}}\cdot\frac{k^2\,(k-1)^2}{(2k-2)\,(2k-1)\,(2k)}\cdot\frac{(2k)!}{k!^2} \\
&= \frac{(2k-3)!}{2^{2k-2}}\cdot\frac{k\,(k-1)}{2k-1}\cdot\binom{2k}{k},
\end{aligned}$$

the above identity can be written as

$$\begin{aligned}
\binom{\frac{p+1}{2}+k}{2k} &\equiv \frac{(p+2k+1)\,(p+2k-1)}{2^{2k}\cdot(2k)!}(-1)^k\cdot\frac{(2k-3)!}{2^{2k-2}}\cdot\frac{k\,(k-1)}{2k-1}\cdot\binom{2k}{k} \\
&\qquad\times\left(p^2\sum_{i=1}^{k-1}\frac{1}{(2i-1)^2}-1\right) \pmod{p^4} \\
&= \frac{(p+2k+1)\,(p+2k-1)}{(-16)^k\,(2k-1)^2}\binom{2k}{k}\left(p^2\sum_{i=1}^{k-1}\frac{1}{(2i-1)^2}-1\right) \pmod{p^4}. \qquad \square
\end{aligned}$$

Lemma 2.3. *Let p be an odd prime and let x be a variable. Then*

$$P_{\frac{p+1}{2}}(x) \equiv -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k \pmod{p^2}.$$

Proof. From Proposition 2.1 and Corollary 2.2 we observe that

$$\begin{aligned}
P_{\frac{p+1}{2}}(x) &= \sum_{k=0}^{\frac{p+1}{2}} \binom{\frac{p+1}{2}+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k \\
&= 1+\sum_{k=1}^{\frac{p+1}{2}} \binom{\frac{p+1}{2}+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k \\
&\equiv 1+\sum_{k=1}^{\frac{p+1}{2}} \frac{(p+2k+1)\,(p+2k-1)}{(-16)^k\,(2k-1)^2}\binom{2k}{k}\left(p^2\sum_{i=1}^{k-1}\frac{1}{(2i-1)^2}-1\right) \\
&\quad \times \binom{2k}{k}\left(\frac{x-1}{2}\right)^k \pmod{p^4} \\
&\equiv 1-\sum_{k=1}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k \pmod{p^2} \\
&= -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k,
\end{aligned}$$

where we use $(p+2k+1)\,(p+2k-1)=(p+2k)^2-1\equiv 4pk+4k^2-1 \pmod{p^2}$. □

Proposition 2.4. *For $n\in\mathbb{N}$ we have*

$$P_{\frac{p-1}{2}}(x) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k \pmod{p^2}.$$

Proof. See [1]. □

Lemma 2.5. *Let p be an odd prime and let x be a variable. Then*

$$\begin{aligned}
P_{\frac{p-3}{2}}(x) \equiv &-\sum_{k=0}^{\frac{p-3}{2}} \frac{4\,(2px+2p+1)\,k^2-4p\,(2x-1)\,k+2px-2p-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k \\
&-p\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2}.
\end{aligned}$$

In particular, we have $P_{\frac{p-3}{2}}(1)\equiv 1 \pmod{p^2}$.

Proof. Replacing $n = \frac{p-1}{2}$ in (2) and multiplying $2\,(p+1)$ on both sides, we obtain

$$P_{\frac{p-3}{2}}(x) \equiv (2p+1)\,P_{\frac{p+1}{2}}(x) - 2pxP_{\frac{p-1}{2}}(x) \pmod{p^2}.$$

Then, applying Lemma 2.3 and Proposition 2.4 to this identity, we have

$$\begin{aligned}
P_{\frac{p-3}{2}}(x) &\equiv (2p+1)\left[-\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k\right] \\
&\qquad - 2px\cdot\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k \\
&= (2p+1)\left[-\sum_{k=0}^{\frac{p-3}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k\right. \\
&\qquad - \frac{4p\cdot\frac{p-1}{2}+4\left(\frac{p-1}{2}\right)^2-1}{(-16)^{\frac{p-1}{2}}\left(2\cdot\frac{p-1}{2}-1\right)^2}\binom{2\cdot\frac{p-1}{2}}{\frac{p-1}{2}}^2\left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\
&\qquad \left.- \frac{4p\cdot\frac{p+1}{2}+4\left(\frac{p+1}{2}\right)^2-1}{(-16)^{\frac{p+1}{2}}\left(2\cdot\frac{p+1}{2}-1\right)^2}\binom{2\cdot\frac{p+1}{2}}{\frac{p+1}{2}}^2\left(\frac{x-1}{2}\right)^{\frac{p+1}{2}}\right] \\
&\qquad - 2px\left[\sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k\right. \\
&\qquad \left.+ \frac{1}{16^{\frac{p-1}{2}}}\binom{2\cdot\frac{p-1}{2}}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\right] \pmod{p^2}.
\end{aligned} \tag{4}$$

Here the 2nd term is

$$\begin{aligned}
&\frac{4p\cdot\frac{p-1}{2}+4\left(\frac{p-1}{2}\right)^2-1}{(-16)^{\frac{p-1}{2}}\left(2\cdot\frac{p-1}{2}-1\right)^2}\binom{2\cdot\frac{p-1}{2}}{\frac{p-1}{2}}^2\left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\
&\qquad\qquad \equiv \frac{-4p}{16^{\frac{p-1}{2}}\,(-4p+4)}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\
&\qquad\qquad \equiv \frac{-p}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2},
\end{aligned}$$

and the 3rd term is

$$\begin{aligned}
&\frac{4p\cdot\frac{p+1}{2}+4\left(\frac{p+1}{2}\right)^2-1}{(-16)^{\frac{p+1}{2}}\left(2\cdot\frac{p+1}{2}-1\right)^2}\binom{2\cdot\frac{p+1}{2}}{\frac{p+1}{2}}^2\left(\frac{x-1}{2}\right)^{\frac{p+1}{2}}\\
&\qquad=\frac{2p\,(p+1)+(p+1)^2-1}{16^{\frac{p+1}{2}}p^2}\binom{p+1}{\frac{p+1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p+1}{2}}\\
&\qquad\equiv\frac{4p}{16^{\frac{p-1}{2}}\cdot 16p^2}\cdot\left(\frac{(p+1)\,p}{\frac{p+1}{2}\cdot\frac{p+1}{2}}\right)^2\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p+1}{2}}\\
&\qquad\equiv\frac{4p}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\left(\frac{1-x}{2}\right)\\
&\qquad\equiv\frac{2p\,(1-x)}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\pmod{p^2}.
\end{aligned}$$

So, we can write Eq. (4) as

$$
\begin{aligned}
P_{\frac{p-3}{2}}(x) \equiv{}& (2p+1)\left[-\sum_{k=0}^{\frac{p-3}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k\right.\\
&\left.+\frac{p}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}-\frac{2p\,(1-x)}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\right]\\
&-2px\left[\sum_{k=0}^{\frac{p-3}{2}}\frac{1}{16^k}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k+\frac{1}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\right]\\
\equiv{}& -\sum_{k=0}^{\frac{p-3}{2}}\frac{(2p+1)\,(4pk+4k^2-1)}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k\\
&+\frac{p}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}-\frac{2p\,(1-x)}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\\
&-\sum_{k=0}^{\frac{p-3}{2}}\frac{2px}{16^k}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k-\frac{2px}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\\
={}& -\sum_{k=0}^{\frac{p-3}{2}}\frac{1}{16^k}\left[\frac{(2p+1)\,(4pk+4k^2-1)}{(2k-1)^2}+2px\right]\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k\\
&-\frac{p}{16^{\frac{p-1}{2}}}\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\\
\equiv{}& -\sum_{k=0}^{\frac{p-3}{2}}\frac{4\,(2px+2p+1)\,k^2-4p\,(2x-1)\,k+2px-2p-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k\\
&-p\binom{p-1}{\frac{p-1}{2}}^2\left(\frac{1-x}{2}\right)^{\frac{p-1}{2}}\pmod{p^2},
\end{aligned}
$$

where we utilize $16^{\frac{p-1}{2}} \equiv \left(\frac{16}{p}\right) = 1 \pmod p$ by Legendre's original definition. □

Theorem 2.6. *Let p be an odd prime and let x be a variable. Then*

$$
\sum_{k=0}^{\frac{p+1}{2}}\frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(x^k-(-1)^{\frac{p+1}{2}}\,(1-x)^k\right)\equiv 0 \pmod{p^2}.
$$

Proof. By (3) and Lemma 2.3 we note that

$$\begin{aligned}
&\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{1-x}{2}\right)^k \\
&= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x-1}{2}\right)^k \\
&\equiv -P_{\frac{p+1}{2}}(x) \\
&= -(-1)^{\frac{p+1}{2}} P_{\frac{p+1}{2}}(-x) \\
&\equiv -(-1)^{\frac{p+1}{2}}\left[-\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{-x-1}{2}\right)^k\right] \\
&= (-1)^{\frac{p+1}{2}}\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\frac{x+1}{2}\right)^k \pmod{p^2},
\end{aligned}$$

and so

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left[\left(\frac{1-x}{2}\right)^k-(-1)^{\frac{p+1}{2}}\left(\frac{x+1}{2}\right)^k\right]\equiv 0 \pmod{p^2}.$$

Now putting $x := 1-2y$ in the above identity, we conclude that

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(y^k-(-1)^{\frac{p+1}{2}}(1-y)^k\right)\equiv 0 \pmod{p^2}.$$

□

Theorem 2.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k\,(2k-1)^2}\binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4, \\ \dfrac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}}\dbinom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

Proof. When $p\equiv 1 \pmod 4$, taking $x=\frac{1}{2}$ in Theorem 2.6 we obtain

$$\begin{aligned}
0 &\equiv \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k\,(2k-1)^2}\binom{2k}{k}^2\left(\left(\frac{1}{2}\right)^k+\left(\frac{1}{2}\right)^k\right) \\
&= 2\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k\,(2k-1)^2}\binom{2k}{k}^2 \pmod{p^2}.
\end{aligned}$$

Then, since $2 \nmid p^2$, we have $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k(2k-1)^2}\binom{2k}{k}^2 \equiv 0 \pmod{p^2}$. Next, when $p \equiv 3 \pmod 4$, setting $x=0$ in Lemma 2.3 and applying Eq. (3) we get

$$\begin{aligned}\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k(2k-1)^2}\binom{2k}{k}^2 &= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-16)^k(2k-1)^2}\binom{2k}{k}^2\left(-\frac{1}{2}\right)^k \\ &\equiv -P_{\frac{p+1}{2}}(0) \\ &= -\frac{(-1)^{\frac{p+1}{4}}}{2^{\frac{p+1}{2}}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}} \\ &= \frac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2}.\end{aligned}$$

So, the proof is complete. □

Theorem 2.8. *Let p be an odd prime and $r \in \{1, 2, \cdots, \frac{p+1}{2}\}$. Then*

$$\begin{aligned}&\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k(2k-1)^2}\binom{2k}{k}^2 k(k-1)\cdots(k-r+1) \\ &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 | (p+3-2r), \\ (-1)^{\frac{p-3+2r}{4}} 2^{-\frac{p+1}{2}} \dfrac{\left(\frac{p+1}{2}+r\right)!}{\frac{p+1-2r}{4}!\,\frac{p+1+2r}{4}!} \pmod{p^2} & \text{if } 4 | (p+1-2r).\end{cases}\end{aligned}$$

Proof. First, according to Lemma 2.3 we can see that

$$\begin{aligned}&\frac{d^r}{dx^r}P_{\frac{p+1}{2}}(x)\Big|_{x=0} \\ &\equiv -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-32)^k(2k-1)^2}\binom{2k}{k}^2 \cdot \frac{d^r}{dx^r}(x-1)^k\Big|_{x=0} \\ &= -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-32)^k(2k-1)^2}\binom{2k}{k}^2 \cdot k(k-1)\cdots(k-r+1)(x-1)^{k-r}\Big|_{x=0} \\ &= -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{(-32)^k(2k-1)^2}\binom{2k}{k}^2 k(k-1)\cdots(k-r+1)(-1)^{k-r} \\ &= (-1)^{-r+1}\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{32^k(2k-1)^2}\binom{2k}{k}^2 k(k-1)\cdots(k-r+1) \pmod{p^2}.\end{aligned} \tag{5}$$

Second by (1), we have

$$\begin{aligned}
&\frac{d^r}{dx^r}P_{\frac{p+1}{2}}(x)\Big|_{x=0}\\
&=\frac{1}{2^{\frac{p+1}{2}}}\sum_{k=0}^{\left[\frac{p+1}{4}\right]}\frac{(-1)^k(p+1-2k)!}{k!\left(\frac{p+1}{2}-k\right)!\left(\frac{p+1}{2}-2k\right)!}\cdot\frac{d^r}{dx^r}x^{\frac{p+1}{2}-2k}\Big|_{x=0}\\
&=\frac{1}{2^{\frac{p+1}{2}}}\sum_{k=0}^{\left[\frac{p+1}{4}\right]}\frac{(-1)^k(p+1-2k)!}{k!\left(\frac{p+1}{2}-k\right)!\left(\frac{p+1}{2}-2k\right)!}\cdot\left(\frac{p+1}{2}-2k\right)\left(\frac{p+1}{2}-2k-1\right)\\
&\qquad\times\cdots\left(\frac{p+1}{2}-2k-r+1\right)\cdot\begin{cases}0 & \text{if } \frac{p+1}{2}-2k-r\neq 0,\\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } \frac{p+1}{2}-2k-r=0\end{cases} \qquad (6)\\
&=\frac{1}{2^{\frac{p+1}{2}}}\sum_{k=0}^{\left[\frac{p+1}{4}\right]}\frac{(-1)^k(p+1-2k)!}{k!\left(\frac{p+1}{2}-k\right)!\left(\frac{p+1}{2}-2k-r\right)!}\\
&\qquad\times\begin{cases}0 & \text{if } r\not\equiv\frac{p+1}{2}\pmod 2,\\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } r=\frac{p+1}{2}-2k\end{cases}\\
&=\begin{cases}0 & \text{if } r\not\equiv\frac{p+1}{2}\pmod 2,\\ \frac{1}{2^{\frac{p+1}{2}}}\cdot\frac{(-1)^k(p+1-2k)!}{k!\left(\frac{p+1}{2}-k\right)!} & \text{if } r=\frac{p+1}{2}-2k.\end{cases}
\end{aligned}$$

Therefore, equating (5) with (6), we deduce that

$$\begin{aligned}
&\sum_{k=0}^{\frac{p+1}{2}}\frac{4pk+4k^2-1}{32^k(2k-1)^2}\binom{2k}{k}^2k(k-1)\cdots(k-r+1)\\
&\equiv\begin{cases}0\pmod{p^2} & \text{if } r\not\equiv\frac{p+1}{2}\pmod 2,\\ \frac{(-1)^{r-1+k}(p+1-2k)!}{2^{\frac{p+1}{2}}k!\left(\frac{p+1}{2}-k\right)!}\pmod{p^2} & \text{if } r=\frac{p+1}{2}-2k\end{cases}\\
&=\begin{cases}0\pmod{p^2} & \text{if } 4|(p+3-2r),\\ (-1)^{\frac{p-3+2r}{4}}2^{-\frac{p+1}{2}}\frac{\left(\frac{p+1}{2}+r\right)!}{\frac{p+1-2r}{4}!\,\frac{p+1+2r}{4}!}\pmod{p^2} & \text{if } 4|(p+1-2r).\end{cases}
\end{aligned}$$

□

Corollary 2.9. *Let p be an odd prime. Then*

(a)

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{k\left(4pk+4k^2-1\right)}{32^k\left(2k-1\right)^2}\binom{2k}{k}^2 \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \dfrac{\frac{p+3}{2}!}{\frac{p-1}{4}!\frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

(b)

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{k^2\left(4pk+4k^2-1\right)}{32^k\left(2k-1\right)^2}\binom{2k}{k}^2 \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \dfrac{\frac{p+3}{2}!}{\frac{p-1}{4}!\frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \dfrac{\frac{p+5}{2}!}{\frac{p-3}{4}!\frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. The proof of part (a) is obvious by letting $r = 1$ in Theorem 2.8. In a similar manner, $r = 2$ in Theorem 2.8 shows that

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{\left(4pk+4k^2-1\right)}{32^k\left(2k-1\right)^2}\binom{2k}{k}^2 k(k-1) \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \dfrac{\frac{p+5}{2}!}{\frac{p-3}{4}!\frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Using this fact, the result of part (a), and $k^2 = k\left(k-1\right)+k$ we can deduce part (b). □

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