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On a theorem of J. Shallit concerning Fibonacci partitions

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Abstract. In this note, I prove a claim on determinants of some special tridiagonal matrices. Together with my result about Fibonacci partitions (`https://arxiv.org/pdf/math/0307150.pdf`), this claim allows one to prove one (slightly strengthened) Shallit's result (`https://arxiv.org/pdf/2007.14930.pdf`) about such partitions.

1 Introduction

Let $f_1 = 1, f_2 = 2$ and $f_i = f_{i-1} + f_{i-2}$ for $i > 2$ be the sequence of Fibonacci numbers. Observe that the "conventional" definition of Fibonacci numbers is different, see `http://en.wikipedia.org/wiki/Fibonacci_number`.

A *Fibonacci partition* of a positive integer n is a representation of n as an unordered sum of distinct Fibonacci numbers, which are referred to as the *parts* of the Fibonacci partition.

Let $\Phi_h(n)$ be the quantity (the cardinality of the set) of Fibonacci partitions of n with h parts. J. Shallit has established the following interesting property of the function $\Phi_h(n)$: for integers $n > 0$, $d \geqslant 2$ and i, let $r_{d,i}(n)$ be the quantity of all Fibonacci partitions of n with number of parts $\equiv i \bmod d$. Then, (see [3, Th. 2])

$$|r_{3,i}(n) - r_{3,i+1}(n)| \leqslant 1.$$

To prove this inequality, J. Shallit used a technique of automata theory.

Set

$$\Phi(n;t) := \sum_{h>0} \Phi_h(n)t^h.$$

In [4], I obtained a formula which expresses $\Phi(n;t)$ as determinant of a tridiagonal matrix depending on n. In §2 of this note, I establish Theorem 2.6 on a property of such determinants.

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In §3, I explain (Theorem 3.1) how the mentioned (see [4]) formula for $\Phi(n;t)$ together with Theorem 2.6 imply not only Shallit's result, but also the formula

$$\bigl(r_{3,0}(n)-r_{3,1}(n)\bigr)\cdot\bigl(r_{3,0}(n)-r_{3,2}(n)\bigr)\cdot\bigl(r_{3,1}(n)-r_{3,2}(n)\bigr)=0.$$

2 3–special polynomials

Let $d\geqslant 2$ be an integer number. For any $g(t)=\sum_{h\geqslant 0}a_h t^h\in\mathbb{Z}[t]$, define

$$\|g(t)\|:=\sum_{h\geqslant 0}a_h,\quad R_i(g(t)):=\sum_{h\equiv i \bmod d}a_h,\ \text{where}\ \ i\in\{0,1,\dots,d-1\}.$$

Let $K_d[T]:=\mathbb{Z}[T]/(T^d-1)$. Define a map $R^{(d)}:\mathbb{Z}[t]\to K_d[T]$ by the formula

$$R^{(d)}(g(t)):=R_0(g(t))+R_1(g(t))T+\dots+R_{d-1}(g(t))T^{d-1}.$$

The following Lemma is subject to easy direct verification.

2.1. Lemma. *The map $R^{(d)}:\mathbb{Z}[t]\to K_d[T]$ is a homomorphism of $\mathbb{Z}$–algebras.*

In this Section, I consider only the case $d=3$. For brevity, set $K:=K_3[T]$ and $R:=R^{(3)}$.

For any $g(t)\in\mathbb{Z}[t]$, we obviously have

$$R\bigl((1+t+t^2)\cdot g(t)\bigr)=\|g(t)\|\cdot\varphi(T),\quad \text{where}\ \varphi(T):=1+T+T^2. \tag{1}$$

2.2. Definition. We say that $a+b\,T+c\,T^2\in K$ is a *special element* if either $a=b=c$, or $|a-b|+|a-c|+|b-c|=2$.

Formula (1) easily implies

2.3. Lemma. *An element $A[T]\in K$ is special if and only if*

$$A[T]\cdot(T-1)\in M[T]:=\left\{0,\pm\,(T-1),\pm\,T(T-1),\pm\,T^2(T-1)\right\}.$$

2.4. Corollary. *Any product of special elements is a special element.*

2.5. Definition. We say that $g(t)\in\mathbb{Z}[t]$ is a 3–*special polynomial* if $R(g(t))$ is a special element.

In what follows, $A=(a_1,a_2,\dots,a_m)$ is either a vector with integer non-negative coordinates if $m>0$, or the empty set if $m=0$. Let us define a polynomial

$$\Delta(A;t):=\Delta(a_1,\dots,a_m;t)\in\mathbb{Z}[t]$$

by the formulas

$$\Delta(\emptyset;t):=1,\qquad \Delta(0;t):=0,\qquad \Delta(a;t):=t+t^2+\dots+t^a\quad\text{for}\ a>0,$$

$$\Delta(a_1,\dots,a_m;t) := \Delta(a_1,\dots,a_{m-1};t)\cdot\Delta(a_m;t) - \Delta(a_1,\dots,a_{m-2};t)\cdot t^{a_m+1} \quad \text{if } m \geqslant 2. \quad (2)$$

Obviously, for $m > 0$,

$$\Delta\left(a_1,a_2,\dots,a_m;t\right) = \begin{vmatrix} \Delta(a_1;t) & t^{a_2+1} & 0 & 0 & \dots & 0 \\ 1 & \Delta(a_2;t) & t^{a_3+1} & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & \Delta(a_{m-1};t) & t^{a_m+1} \\ 0 & 0 & \dots & 0 & 1 & \Delta(a_m;t) \end{vmatrix}.$$

The main result of this note is

2.6. Theorem. *For any $A = (a_1,a_2,\dots,a_m)$, the polynomial $\Delta(A;t)$ is a 3–special one.*

The proof uses the following auxiliary claim.

2.7. Lemma. *Let $\varepsilon(A) := (\varepsilon(a_1),\dots,\varepsilon(a_m))$, where $\varepsilon(a) := a - 3\left\lfloor \frac{a}{3}\right\rfloor$. Then,*

$$R\big(\Delta(A;t)\big) = R\left(\Delta\big(\varepsilon(A);t\big)\right) + k\cdot\varphi(T), \qquad \textit{where } \ k = k(A)\in\mathbb{Z}.$$

Proof. Let us prove by induction on m. For $m = 1$ and $a \geqslant 1$, we have

$$\Delta(a;t) = t\left(1 + t^3 + \dots + t^{3\left\lfloor \frac{a}{3}\right\rfloor}\right)(1+t+t^2) + t^{3\left\lfloor \frac{a}{3}\right\rfloor}\cdot\Delta\big(\varepsilon(a);t\big).$$

Applying R to both sides of this equality we obtain

$$R\big(\Delta(a;t)\big) = R\Big(\Delta\big(\varepsilon(a);t\big)\Big) + k\cdot\varphi(T), \qquad \text{where } \ k = 1 + \left\lfloor \frac{a}{3}\right\rfloor. \quad (3)$$

For $m \geqslant 2$, let us apply R to expression (2). The induction hypothesis, Lemma 2.1, formulas (1) and (3), the obvious formula $R(t^a) = T^{\varepsilon(a)}$, and a short computation yield the required result. □

Proof of Theorem 2.6. In view of Lemma 2.7, it suffices to assume that $a_i \in \{0,1,2\}$ for any $i = 1,2,\dots,m$. Keeping Lemma 2.3 in mind, define

$$S(a_1,\dots,a_m) := R\big(\Delta(a_1,\dots,a_m;t)\big)\cdot(T-1)\in K.$$

The expression (2) and formula $\varphi(T)\cdot(T-1) = 0$ easily imply the recurrent formula

$$S(a_1,\dots,a_m) = \begin{cases} -S(a_1,\dots,a_{m-2})\cdot T & \text{if } a_m = 0, \\ S(a_1,\dots,a_{m-1})\cdot T + S(a_1,\dots,a_{m-2})\cdot(T+1) & \text{if } a_m = 1, \\ -S(a_1,\dots,a_{m-1}) - S(a_1,\dots,a_{m-2}) & \text{if } a_m = 2. \end{cases} \quad (4)$$

By Lemma 2.3 it remains to show that $S(a_1,\dots,a_m)\in M[T]$.

Let us prove this by induction on m. For $m = 1,2$, the claim is directly checked. In particular, $S(0) = 0$ and $S(a,0) = -T(T-1)$.

For $a_m = 0$, the last expressions and formula (4) imply the theorem by induction for any $m \geqslant 1$. Therefore, assume that $a_m = 1$ or $a_m = 2$. From expressions (4) it is not difficult to obtain the expressions

$$S(a_1, \dots, a_{m-1}, 1) = \begin{cases} S(a_1, \dots, a_{m-2}, 2) \cdot T^2 & \text{if } a_{m-1} = 0, \\ -S(a_1, \dots, a_{m-2}, 2) \cdot (T+1) & \text{if } a_{m-1} = 1, \\ S(a_1, \dots, a_{m-3}, a_{m-2}+2) \cdot T & \text{if } a_{m-1} = 2, \end{cases}$$

$$S(a_1, \dots, a_{m-1}, 2) = \begin{cases} -S(a_1, \dots, a_{m-3}, a_{m-2}+2) \cdot T & \text{if } a_{m-1} = 0, \\ S(a_1, \dots, a_{m-2}, 2) \cdot (T+1) & \text{if } a_{m-1} = 1, \\ S(a_1, \dots, a_{m-3}) & \text{if } a_{m-1} = 2. \end{cases}$$

Since

$$(T-1)(T+1) = -T^2(T-1),$$

these expressions and the induction hypothesis complete the proof. □

3 An application to Fibonacci partitions

In §2 of the article [4], for any positive integer n, a certain sequence is uniquely defined

$$\alpha(n) = \big\{\alpha_1(n), \alpha_2(n), \dots, \alpha_{k(n)}(n)\big\} \tag{5}$$

where $\alpha_k(n)$ is a vector with positive integer coordinates for any $k = 1, 2, \dots, k(n)$, and it is shown ([4, Th.2.11]) that

$$\Phi(n;t) = \Delta\big(\alpha_1(n);t\big) \cdot \Delta\big(\alpha_2(n);t\big) \cdot \ldots \cdot \Delta\big(\alpha_{k(n)}(n);t\big).$$

By Theorem 2.6 the polynomial $\Delta\big(\alpha_k(n);t\big)$ is a 3–special one for any k. Thus, Lemma 2.1 and Corollary 2.4 imply

3.1. Theorem. *For any integer $n > 0$, the polynomial $\Phi(n;t)$ is a 3–special one.*

3.2. Remark. Using arguments similar to those in §2 (where $d = 3$ is replaced with $d = 2$) and the formula for $\Phi(n;t)$ one can easily show that $|r_{2,0}(n) - r_{2,1}(n)| \leqslant 1$ for any positive integer n. It is obvious that this inequality is equivalent to the analytic identity

$$\prod_{i=1}^{\infty}(1 - x^{f_i}) = 1 + \sum_{n=1}^{\infty} \chi(n) x^n, \qquad \text{where} \quad |\chi(n)| \leqslant 1. \tag{6}$$

For other proofs of this identity, see [1],[2] and [4].

In addition to that, an interesting result of Y. Zhao should be mentioned. Namely, Proposition 2 of the article [5] implies the polynomial identity

$$\prod_{a \leqslant i \leqslant b} (1 - x^{f_i}) = 1 + \sum_n \chi_{a,b}(n) x^n, \qquad \text{where} \quad |\chi_{a,b}(n)| \leqslant 1,$$

which is valid for any positive integers $a \leqslant b$.

3.3. Conjecture. *For positive integers $a \leqslant b$, let*

$$M(a,b) := \{f_a, f_{a+1}, \dots, f_b\}.$$

For integers $n > 0$ and i, let $r_{3,i}^{(a,b)}(n)$ be the quantity of Fibonacci partitions of n with parts from the set $M(a,b)$ and with number of parts $\equiv i \bmod 3$.

Then, $\left|r_{3,i}^{(a,b)}(n) - r_{3,j}^{(a,b)}(n)\right| \leqslant 1$ for any $i, j \in \{0,1,2\}$. Moreover,

$$\left(r_{3,0}^{(a,b)}(n) - r_{3,1}^{(a,b)}(n)\right) \cdot \left(r_{3,0}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n)\right) \cdot \left(r_{3,1}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n)\right) = 0.$$

References

[1] F. Ardila. The coefficients of a Fibonacci power series. *Fibonacci Quart.* **42**(3) (2004), 202–204.

[2] N. Robbins. Fibonacci Partitions. *Fibonacci Quart.* **34**(4) (1996), 306–313.

[3] Jeffrey Shallit. Robbins and Ardila meet Berstel. *Information Processing Letters*, **167** (2021) 106081.

[4] F. V. Weinstein. Notes on Fibonacci Partitions. *Experimental Math.* **25**(4) (2016) 482–499.

[5] Y. Zhao. The coefficients of a truncated Fibonacci series. *Fibonacci Quart.* **46/47** (2008/2009), 53–55.