On a theorem of J. Shallit concerning Fibonacci partitions

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1 Introduction

Let \( f_1 = 1, f_2 = 2 \) and \( f_i = f_{i-1} + f_{i-2} \) for \( i > 2 \) be the sequence of Fibonacci numbers. Observe that the “conventional” definition of Fibonacci numbers is different, see http://en.wikipedia.org/wiki/Fibonacci_number.

A Fibonacci partition of a positive integer \( n \) is a representation of \( n \) as an unordered sum of distinct Fibonacci numbers, which are referred to as the parts of the Fibonacci partition.

Let \( \Phi_h(n) \) be the quantity (the cardinality of the set) of Fibonacci partitions of \( n \) with \( h \) parts. J. Shallit has established the following interesting property of the function \( \Phi_h(n) \): for integers \( n > 0, d \geq 2 \) and \( i \), let \( r_{d,i}(n) \) be the quantity of all Fibonacci partitions of \( n \) with number of parts \( \equiv i \mod d \). Then, (see [3, Th. 2])

\[
|r_{3,i}(n) - r_{3,i+1}(n)| \leq 1.
\]

To prove this inequality, J. Shallit used a technique of automata theory.

Set

\[
\Phi(n; t) := \sum_{h>0} \Phi_h(n)t^h.
\]

In [4], I obtained a formula which expresses \( \Phi(n; t) \) as determinant of a tridiagonal matrix depending on \( n \). In §2 of this note, I establish Theorem 2.6 on a property of such determinants.

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In §3, I explain (Theorem 3.1) how the mentioned (see [4]) formula for $\Phi(n; t)$ together with Theorem 2.6 imply not only Shallit’s result, but also the formula

$$(r_{3,0}(n) - r_{3,1}(n)) \cdot (r_{3,0}(n) - r_{3,2}(n)) \cdot (r_{3,1}(n) - r_{3,2}(n)) = 0.$$ 

2 3–special polynomials
Let $d \geq 2$ be an integer number. For any $g(t) = \sum_{h \geq 0} a_h t^h \in \mathbb{Z}[t]$, define

$$\|g(t)\| := \sum_{h \geq 0} a_h, \quad R_i(g(t)) := \sum_{h \equiv i \mod d} a_h, \quad \text{where } i \in \{0, 1, \ldots, d-1\}.$$ 

Let $K_d[T] := \mathbb{Z}[T]/(T^d - 1)$. Define a map $R^{(d)} : \mathbb{Z}[t] \to K_d[T]$ by the formula

$$R^{(d)}(g(t)) := R_0(g(t)) + R_1(g(t))T + \cdots + R_{d-1}(g(t))T^{d-1}.$$ 

The following Lemma is subject to easy direct verification.

2.1. Lemma. The map $R^{(d)} : \mathbb{Z}[t] \to K_d[T]$ is a homomorphism of $\mathbb{Z}$–algebras.

In this Section, I consider only the case $d = 3$. For brevity, set $K := K_3[T]$ and $R := R^{(3)}$.

For any $g(t) \in \mathbb{Z}[t]$, we obviously have

$$R((1 + t + t^2) \cdot g(t)) = \|g(t)\| \cdot \varphi(T), \quad \text{where } \varphi(T) := 1 + T + T^2. \quad (1)$$

2.2. Definition. We say that $a + bT + cT^2 \in K$ is a special element if either $a = b = c$, or $|a - b| + |a - c| + |b - c| = 2$.

Formula (1) easily implies

2.3. Lemma. An element $A[T] \in K$ is special if and only if

$$A[T] \cdot (T - 1) \in M[T] := \{0, \pm (T - 1), \pm T(T - 1), \pm T^2(T - 1)\}.$$ 

2.4. Corollary. Any product of special elements is a special element.

2.5. Definition. We say that $g(t) \in \mathbb{Z}[t]$ is a 3–special polynomial if $R(g(t))$ is a special element.

In what follows, $A = (a_1, a_2, \ldots, a_m)$ is either a vector with integer non-negative coordinates if $m > 0$, or the empty set if $m = 0$. Let us define a polynomial

$$\Delta(A; t) := \Delta(a_1, \ldots, a_m; t) \in \mathbb{Z}[t]$$

by the formulas

$$\Delta(\emptyset; t) := 1, \quad \Delta(0; t) := 0, \quad \Delta(a; t) := t + t^2 + \cdots + t^a \quad \text{for } a > 0,$$
\[ \Delta(a_1, \ldots, a_m; t) := \Delta(a_1, \ldots, a_{m-1}; t) \cdot \Delta(a_m; t) - \Delta(a_1, \ldots, a_{m-2}; t) \cdot t^{a_m+1} \quad \text{if } m \geq 2. \]  

Obviously, for \( m > 0 \),

\[
\Delta(a_1, a_2, \ldots, a_m; t) = \begin{vmatrix}
\Delta(a_1; t) & t^{a_2+1} & 0 & 0 & \cdots & 0 \\
1 & \Delta(a_2; t) & t^{a_3+1} & 0 & \cdots & 0 \\
& \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \Delta(a_{m-1}; t) & t^{a_m+1} \\
0 & 0 & \cdots & 0 & 1 & \Delta(a_m; t)
\end{vmatrix}
\]

The main result of this note is

2.6. Theorem. For any \( A = (a_1, a_2, \ldots, a_m) \), the polynomial \( \Delta(A; t) \) is a 3-special one.

The proof uses the following auxiliary claim.

2.7. Lemma. Let \( \varepsilon(A) := (\varepsilon(a_1), \ldots, \varepsilon(a_m)) \), where \( \varepsilon(a) := a - 3 \lfloor \frac{a}{3} \rfloor \). Then,

\[ R(\Delta(A; t)) = R(\Delta(\varepsilon(A); t)) + k \cdot \varphi(T), \quad \text{where } k = k(A) \in \mathbb{Z}. \]

**Proof.** Let us prove by induction on \( m \). For \( m = 1 \) and \( a \geq 1 \), we have

\[ \Delta(a; t) = t \left( 1 + t^3 + \cdots + t^{3 \lfloor \frac{a}{3} \rfloor} \right) (1 + t + t^2) + t^{3 \lfloor \frac{a}{3} \rfloor} \cdot \Delta(\varepsilon(a); t). \]

Applying \( R \) to both sides of this equality we obtain

\[ R(\Delta(a; t)) = R(\Delta(\varepsilon(a); t)) + k \cdot \varphi(T), \quad \text{where } k = 1 + \left\lfloor \frac{a}{3} \right\rfloor. \tag{3} \]

For \( m \geq 2 \), let us apply \( R \) to expression (2). The induction hypothesis, Lemma 2.1, formulas (1) and (3), the obvious formula \( R(t^a) = T^{\varepsilon(a)} \), and a short computation yield the required result. \( \square \)

**Proof of Theorem 2.6.** In view of Lemma 2.7, it suffices to assume that \( a_i \in \{0, 1, 2\} \) for any \( i = 1, 2, \ldots, m \). Keeping Lemma 2.3 in mind, define

\[ S(a_1, \ldots, a_m) := R(\Delta(a_1, \ldots, a_m; t)) \cdot (T - 1) \in K. \]

The expression (2) and formula \( \varphi(T) \cdot (T - 1) = 0 \) easily imply the recurrent formula

\[
S(a_1, \ldots, a_m) = \begin{cases}
-S(a_1, \ldots, a_{m-2}) \cdot T & \text{if } a_m = 0, \\
S(a_1, \ldots, a_{m-1}) \cdot T + S(a_1, \ldots, a_{m-2}) \cdot (T + 1) & \text{if } a_m = 1, \\
-S(a_1, \ldots, a_{m-1}) - S(a_1, \ldots, a_{m-2}) & \text{if } a_m = 2.
\end{cases} \tag{4}
\]

By Lemma 2.3 it remains to show that \( S(a_1, \ldots, a_m) \in M[T] \).

Let us prove this by induction on \( m \). For \( m = 1, 2 \), the claim is directly checked. In particular, \( S(0) = 0 \) and \( S(a, 0) = -T(T - 1) \).
For $a_m = 0$, the last expressions and formula (4) imply the theorem by induction for any $m \geq 1$. Therefore, assume that $a_m = 1$ or $a_m = 2$. From expressions (4) it is not difficult to obtain the expressions

$$S(a_1, \ldots, a_{m-1}, 1) = \begin{cases} S(a_1, \ldots, a_{m-2}, 2) \cdot T^2 & \text{if } a_{m-1} = 0, \\ -S(a_1, \ldots, a_{m-2}, 2) \cdot (T + 1) & \text{if } a_{m-1} = 1, \\ S(a_1, \ldots, a_{m-3}, a_{m-2} + 2) \cdot T & \text{if } a_{m-1} = 2, \end{cases}$$

$$S(a_1, \ldots, a_{m-1}, 2) = \begin{cases} -S(a_1, \ldots, a_{m-3}, a_{m-2} + 2) \cdot T & \text{if } a_{m-1} = 0, \\ S(a_1, \ldots, a_{m-2}, 2) \cdot (T + 1) & \text{if } a_{m-1} = 1, \\ S(a_1, \ldots, a_{m-3}) & \text{if } a_{m-1} = 2. \end{cases}$$

Since

$$(T - 1)(T + 1) = -T^2(T - 1),$$

these expressions and the induction hypothesis complete the proof. \qed

### 3 An application to Fibonacci partitions

In §2 of the article [4], for any positive integer $n$, a certain sequence is uniquely defined

$$\alpha(n) = \{\alpha_1(n), \alpha_2(n), \ldots, \alpha_{k(n)}(n)\}$$

where $\alpha_{k}(n)$ is a vector with positive integer coordinates for any $k = 1, 2, \ldots, k(n)$, and it is shown ([4, Th.2.11]) that

$$\Phi(n; t) = \Delta(\alpha_1(n); t) \cdot \Delta(\alpha_2(n); t) \cdot \ldots \cdot \Delta(\alpha_{k(n)}(n); t).$$

By Theorem 2.6 the polynomial $\Delta(\alpha_{k}(n); t)$ is a 3-special one for any $k$. Thus, Lemma 2.1 and Corollary 2.4 imply

3.1. **Theorem.** For any integer $n > 0$, the polynomial $\Phi(n; t)$ is a 3-special one.

3.2. **Remark.** Using arguments similar to those in §2 (where $d = 3$ is replaced with $d = 2$) and the formula for $\Phi(n; t)$ one can easily show that $\vert r_{2,0}(n) - r_{2,1}(n) \vert \leq 1$ for any positive integer $n$. It is obvious that this inequality is equivalent to the analytic identity

$$\prod_{i=1}^{\infty} (1 - x^{f_i}) = 1 + \sum_{n=1}^{\infty} \chi(n)x^n, \quad \text{where } \vert \chi(n) \vert \leq 1. \quad (6)$$

For other proofs of this identity, see [1],[2] and [4].

In addition to that, an interesting result of Y. Zhao should be mentioned. Namely, Proposition 2 of the article [5] implies the polynomial identity

$$\prod_{a \leq i \leq b} (1 - x^{f_i}) = 1 + \sum_{n} \chi_{a,b}(n)x^n, \quad \text{where } \vert \chi_{a,b}(n) \vert \leq 1,$$

which is valid for any positive integers $a \leq b$. 
3.3. Conjecture. For positive integers $a \leq b$, let

$$M(a, b) := \{f_a, f_{a+1}, \ldots, f_b\}.$$ 

For integers $n > 0$ and $i$, let $r_{3,i}^{(a,b)}(n)$ be the quantity of Fibonacci partitions of $n$ with parts from the set $M(a, b)$ and with number of parts $\equiv i \mod 3$.

Then, $|r_{3,i}^{(a,b)}(n) - r_{3,j}^{(a,b)}(n)| \leq 1$ for any $i, j \in \{0, 1, 2\}$. Moreover,

$$\left(r_{3,0}^{(a,b)}(n) - r_{3,1}^{(a,b)}(n)\right) \cdot \left(r_{3,0}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n)\right) \cdot \left(r_{3,1}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n)\right) = 0.$$

References


