Communications in Mathematics **32** (2024), no. 1, 129–147
DOI: https://doi.org/10.46298/cm.10822

A numerical technique for solving variable order time fractional differential-integro equations

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Abstract. In this manuscripts, we consider the coupled differential-integral equations including the variable-order Caputo fractional operator. To solve numerically these type of equations, we apply the shifted Jacobi–Gauss collocation scheme. Using this numerical method a system of algebraic equations is constructed. We solve this system with a recursive method in the nonlinear case and we solve it in linear case with algebraic formulas. Finally, for the high performance of the suggested method three Examples are illustrated.

1 Introduction

A coupled differential-integral equation including the Caputo fractional operator is introduced as follows:

$$\underbrace{{}^{C}\mathrm{D}_{x}^{\varrho_1(x)}\Big[p_1(x)\Big] + p_1'(x) + \int_0^x p_1(\upsilon)d\upsilon = \theta_1(x)}_{\textbf{the initial condition for this types of equation is } p_1(0)=p_1'(0)=0}, \tag{1}$$

$$\underbrace{{}^{C}\mathrm{D}_{x}^{\varrho_2(x)}\Big[p_2(x)\Big] + p_2'(x) + \int_0^x p_2(\upsilon)d\upsilon = \theta_2(x)}_{\textbf{the initial condition for this types of equation is } p_2(0)=p_2'(0)=0,}, \tag{2}$$

where ${}^{C}\mathrm{D}_{x}^{\varrho_1(x)}$, ${}^{C}\mathrm{D}_{x}^{\varrho_1(x)}$ are the Caputo fractional operators of variable orders $0 < \varrho_i(x) \le 1$, given by:

$${}^{C}\mathrm{D}_{x}^{\varrho_i(x)}p(x) = \begin{cases} p'(x), & \varrho_i(x) = 1, \\ \frac{1}{\Gamma(1-\varrho_i(x))}\int_0^x (x-\upsilon)^{-\varrho_i(x)} p'(\upsilon), & 0 < \varrho_i(x) \le 1. \end{cases} \tag{3}$$

MSC 2020: 26A33,35R11.
Keywords: Coupled differential-integral equation, Caputo fractional operator, Shifted fractional Jacobi collocation method, Variable-order.
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In the coupled differential-integral equations (1), (2), $p_1(x), p_2(x)$ and $\theta_1(x), \theta_2(x)$ are indistinct and definite respectively. The main reason for choosing these kind of differential-integral equations is because of their role and applications in various fields as computational and mathematical sciences [9], engineering sciences which are modelled using these types of equations, physics and mathematical models in chemistry.

Several types of integral equations (IEs) are considered very important in various functional analysis topics because of their essential role in engineering, physics, economics, and natural sciences. Many real-life applications can be well described using IEs. For instance, IEs play an important role in kinetic molecular, radiative, neutron transport, traffic, and queuing theories [3]. Fractional calculus constitutes a powerful mathematical tool in terms of analysis when it is applicable to the investigation of arbitrary-order integrals and derivatives via the generalisations of integer-order differentiation and n-fold integration. Some interesting research works concerning the relation between symmetry and generalised fractional calculus properties have been conducted [10]. Recently, the study of the integrals and derivatives of fractional order was limited to a purely mathematical context, but in recent decades, they have been applied in various natural sciences and technology fields such as theoretical physics, fluid mechanics, biology, image processing, and entropy theory which reveal a great potential of these fractional integrals and derivatives in modelling scientific phenomena [26]. Both of the theoretical and practical studies of the fractional differential equations' element have received a global research interest. The main purpose of this paper is to obtain the solution of the coupled differential-integral equations are introduced in (1) and (2) by using a numerical method that this numerical method is called a shifted Jacobi–Gauss collocation algorithm. This numerical method contains a shifted Jacobi polynomials. Obtaining a solution by using different numerical methods was studied by some authors in the mathematical field. In this paper a few numerical methods are mentioned, for Example, in [11] was studied a numerical method based on Chebyshev cardinal wavelets, the new algorithm base on the discretisation method to obtain solution of the fractional functional integral equations of variable-order in [14] was discussed, the explicit and implicit Euler methods [26], the Legendre wavelets method [4], the numerical method base on Laplace and Sumudu transform methods [9], the Adams-Bashforth-Moulton scheme [5], the optimisation method [23].

So, this article is divided into four sections as follows. In Section 2, Lemma and definitions of Jacobi polynomials, shifted Jacobi polynomials and their properties are introduced. In Section 3, approximation function and the numerical method to obtaining solution of the coupled differential-integral equation are described. In order to accuracy of the presented method four examples in Section 4 are described.

2 Some main lemma and definitions about the Jacobi polynomials and shifted Jacobi polynomials

This section deals with some definitions about the Jacobi polynomials and shifted Jacobi polynomials and Lemma, which will be applied in next section.

Definition 2.1. Let $\mu, \nu > -1, x \in [-1,1]$. Then the Jacobi polynomial of degree n is defined by [7]:

$$\mathfrak{P}_{n+1}^{\mu,\nu}(x) = (a_n^{\mu,\nu} x - b_n^{\mu,\nu})\mathfrak{P}_n^{\mu,\nu}(x) - c_n^{\mu,\nu}\mathfrak{P}_{n-1}^{\mu,\nu}(x), \tag{4}$$

where for $n = 0, 1$ the functions $\mathfrak{P}_0^{\mu,\nu}(x)$ and $\mathfrak{P}_1^{\mu,\nu}(x)$ are defined by:

$$\mathfrak{P}_0^{\mu,\nu}(x) = 1,\ \mathfrak{P}_1^{\mu,\nu}(x) = \frac{(\mu+\nu+2)x + \mu - \nu}{2}, \tag{5}$$

and the coefficients $a_n^{\mu,\nu}, b_n^{\mu,\nu}, c_n^{\mu,\nu}$ are given by:

$$\begin{aligned} a_n^{\mu,\nu} &= \frac{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}{(2n+2)(n+\mu+\nu+1)}, \\ b_n^{\mu,\nu} &= \frac{(\nu^2-\mu^2)(2n+\mu+\nu+1)}{(2n+2)(n+\mu+\nu+1)(2n+\mu+\nu)}, \\ c_n^{\mu,\nu} &= \frac{(n+\mu)(n+\nu)(2n+\mu+\nu+2)}{(n+1)(n+\mu+\nu+1)(2n+\mu+\nu)}. \end{aligned} \tag{6}$$

Also, the Jacobi polynomial has a finite series as follows:

$$\mathfrak{P}_n^{\mu,\nu}(x) = \frac{\Gamma(\mu+n+1)}{n!\Gamma(\mu+\nu+n+1)} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{\Gamma(\mu+\nu+k+n+1)}{\Gamma(\mu+k+1)} \Big(\frac{x-1}{2}\Big)^k, \tag{7}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. Let $\mu, \nu > -1$. Then the shifted Jacobi polynomial of degree n is defined by [7]:

$$\begin{aligned} \mathbb{P}_n^{\mu,\nu}(x) &= \mathfrak{P}_n^{\mu,\nu}(\underbrace{2x-1}_{2x-1\in[-1,1]}) \qquad (8) \\ &= \frac{\Gamma(\mu+n+1)}{n!\Gamma(\mu+\nu+n+1)} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{\Gamma(\mu+\nu+k+n+1)}{\Gamma(\mu+k+1)} \Big(x-1\Big)^k \\ &= \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)} x^k. \qquad (9) \end{aligned}$$

Also, the shifted Jacobi polynomial respect to the weight function $\omega^{\mu,\nu}(x) = (1-x)^{\mu} x^{\nu}$ is orthogonal, that the orthogonality condition is given as:

$$\langle \mathbb{P}_n^{\mu,\nu}(x), \mathbb{P}_m^{\mu,\nu}(x) \rangle_{\omega^{\mu,\nu}(x)} = \int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x)\mathbb{P}_m^{\mu,\nu}(x)dx = \begin{cases} 0, & n \neq m, \\ \frac{\Gamma(n+\mu+\nu+1)}{n!\Gamma(2n+\mu+\nu+1)}, & n = m, \end{cases} \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Lemma 2.3. *Let $0 < \varrho_i \leq 1, i = 1,2$. Then for any $\zeta \geq -1$ the Caputo fractional operator of variable order $\varrho_i, i = 1,2$ is given by [24]:*

$$
{}^{C}\mathrm{D}_x^{\varrho_i(x)} x^{\zeta} = \begin{cases} 0, & \zeta = constant, \\ \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\varrho_i(x)+1)} x^{\zeta-\varrho_i(x)}, & \zeta \geq -1, \\ \frac{d^m}{dx^m}(x^{\zeta}), & \varrho_i(x) = m, m \in \mathbb{N}. \end{cases} \tag{11}
$$

Then, applying the relation (11) *on* (8)*, for $i = 1,2$ we obtain:*

$$
\begin{aligned}
&{}^{C}\mathrm{D}_x^{\varrho_i(x)} \mathbb{P}_n^{\mu,\nu}(x) \\
&= \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)} \frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)} x^{k-\varrho_i(x)} \\
&= \sum_{k=0}^{n} \Omega_{k,n}^{\mu,\nu} x^{k-\varrho_i(x)}, \qquad (12) \\
&\underbrace{{}^{Caputo}\mathrm{D}_x^{\varrho_i(x)} \mathbb{P}_n^{\mu,\nu}(x)}_{for\, \varrho_i(x)=1,\, m=1} \\
&= \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \sum_{k=1}^{n} (-1)^{n-k} \frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)} x^{k-1} \\
&= \sum_{k=1}^{n} \Psi_{k,n}^{\mu,\nu} x^{k-1}, \qquad (13)
\end{aligned}
$$

where

$$
\Omega_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \times (-1)^{n-k} \frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)} \times \frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}
$$

and

$$
\Psi_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \times (-1)^{n-k} \frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}.
$$

3 The function approximation and proposed method algorithm

Due to the orthogonality condition is presented in (10) about the shifted Jacobi polynomial, any $p_1(x), p_2(x) \in L^2[0,1]$ can be approximated in terms of the shifted Jacobi polynomial as follows:

$$
p_1(x) = \sum_{k=0}^{\infty} a_k \mathbb{P}_k^{\mu,\nu}(x), \tag{14}
$$

$$
p_2(x) = \sum_{k=0}^{\infty} b_k \mathbb{P}_k^{\mu,\nu}(x), \tag{15}
$$

considering the first n sentence of the relations (14), (15), we have:

$$p_1(x) \simeq p_{1,n}(x) = \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x), \tag{16}$$

$$p_2(x) \simeq p_{2,n}(x) = \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(x), \tag{17}$$

where the coefficients a_j, b_j are calculated as:

$$a_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_1(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x)p_1(x)dx}{\| \mathbb{P}_n^{\mu,\nu}(x) \|^2}, \tag{18}$$

$$b_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_2(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x)p_2(x)dx}{\| \mathbb{P}_n^{\mu,\nu}(x) \|^2}. \tag{19}$$

Theorem 3.1. *Let $p_{1,n}$ be as an approximation of the function $p_1(x)$, that it satisfy in the Equation (1) and the following conditions for $\lambda_1, \lambda_2 \in (0,1)$ hold:*

$$\| p_{1,n}(x) - p_{1,n-1}(x) \| \leq \lambda_1 \| p_{1,n-1}(x) - p_{1,n-2}(x) \| \tag{20}$$

$$\| p'_{1,n}(x) - p'_{1,n-1}(x) \| \leq \lambda_2 \| p'_{1,n-1}(x) - p'_{1,n-2}(x) \| . \tag{21}$$

Then the sequence $p_{1,n}$ to $p_1(x)$ converges, that $p_1(x)$ is the exact solution of Equation (1).

Proof. Suppose $\mathbf{R}_n(x)$ be a residual function, which is defined as:

$$\begin{aligned} \mathbf{R}_n(x) &= {}^C\mathrm{D}_x^{\varrho_1(x)}\Big[p_{1,n-1}(x)\Big] + p'_{1,n-1}(x) + \int_0^x p_{1,n-1}(\upsilon)d\upsilon - \theta_1(x), \\ \mathbf{R}_{n-1}(x) &= {}^C\mathrm{D}_x^{\varrho_1(x)}\Big[p_{1,n-2}(x)\Big] + p'_{1,n-2}(x) + \int_0^x p_{1,n-2}(\upsilon)d\upsilon - \theta_1(x). \end{aligned} \tag{22}$$

Then, we have:

$$
\begin{aligned}
\| \mathbf{R}_n(x) &- \mathbf{R}_{n-1}(x) \| \\
&= \| {}^{C}\mathrm{D}_x^{\varrho_1(x)}\Big[p_{1,n-1}(x) - p_{1,n-2}(x)\Big] + \Big(p'_{1,n-1}(x) - p'_{1,n-2}(x)\Big) \\
&\quad + \int_0^x \Big(p_{1,n-1}(\upsilon) - p_{1,n-2}(\upsilon)d\upsilon\Big) \| \\
&= \| \frac{1}{\Gamma(1-\varrho_1(x))} \int_0^x (x-\upsilon)^{-\varrho_1(x)}\Big[p'_{1,n-1}(\upsilon) - p'_{1,n-2}(\upsilon)\Big]d\upsilon \\
&\quad + \Big(p'_{1,n-1}(x) - p'_{1,n-2}(x)\Big) \\
&\quad + \int_0^x \Big(p_{1,n-1}(\upsilon) - p_{1,n-2}(\upsilon)d\upsilon\Big) \| \\
&\leq \frac{1}{\Gamma(1-\varrho_1(x))} \int_0^x (x-\upsilon)^{-\varrho_1(x)} \| p'_{1,n-1}(\upsilon) - p'_{1,n-2}(\upsilon) \| d\upsilon \\
&\quad + \| p'_{1,n-1}(x) - p'_{1,n-2}(x) \| \\
&\quad + \int_0^x \| p_{1,n-1}(\upsilon) - p_{1,n-2}(\upsilon) \| d\upsilon \\
&= \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,n-1}(\upsilon) - p'_{1,n-2}(\upsilon) \| d\upsilon + \| p'_{1,n-1}(x) - p'_{1,n-2}(x) \| \\
&\quad + \int_0^x \| p_{1,n-1}(\upsilon) - p_{1,n-2}(\upsilon) \| d\upsilon. \qquad (23)
\end{aligned}
$$

Using (20), (21), we get:

$$
\begin{aligned}
\| \mathbf{R}_n(x) &- \mathbf{R}_{n-1}(x) \| \\
&\leq \frac{\lambda_2 x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,n-2}(\upsilon) - p'_{1,n-3}(\upsilon) \| d\upsilon + \lambda_2 \| p'_{1,n-2}(x) - p'_{1,n-3}(x) \| \\
&\quad + \lambda_1 \int_0^x \| p_{1,n-2}(\upsilon) - p_{1,n-3}(\upsilon) \| d\upsilon \\
&\ \vdots \\
&\leq \frac{\lambda_2^{n-2} x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,1}(\upsilon) - p'_{1,0}(\upsilon) \| d\upsilon + \lambda_2^{n-2} \| p'_{1,1}(x) - p'_{1,0}(x) \| \\
&\quad + \lambda_1^{n-2} \int_0^x \| p_{1,1}(\upsilon) - p_{1,0}(\upsilon) \| d\upsilon \\
&\leq \frac{\lambda_2^{n-2} x^{2-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \max_{x\in[0,1]} \| p'_{1,1}(x) - p'_{1,0}(x) \| + \lambda_2^{n-2} \max_{x\in[0,1]} \| p'_{1,1}(x) - p'_{1,0}(x) \| \\
&\quad + \lambda_1^{n-2} \max_{x\in[0,1]} \| p_{1,1}(x) - p_{1,0}(x) \| . \qquad (24)
\end{aligned}
$$

So, $\| \mathbf{R}_n(x) - \mathbf{R}_{n-1}(x) \| \to 0$ when $n \to \infty$, then $\mathbf{R}_n(x)$ a Cauchy sequence in $L^2[0,1]$ and

$L^2[0,1]$ is complete, therefore the sequence $\mathbf{R}_n(x)$ is convergent that is

$$\lim_{n\to\infty} \mathbf{R}_n(x) = \mathbf{R}(x), \tag{25}$$

where

$$\begin{aligned} \mathbf{R}_n(x) &= {}^C\mathrm{D}_x^{\varrho_1(x)}\Big[p_{1,n-1}(x)\Big] + p'_{1,n-1}(x) + \int_0^x p_{1,n-1}(\upsilon)d\upsilon - \theta_1(x), \\ \mathbf{R}(x) &= {}^C\mathrm{D}_x^{\varrho_1(x)}\Big[p_1(x)\Big] + p'_1(x) + \int_0^x p_1(\upsilon)d\upsilon - \theta_1(x). \end{aligned} \tag{26}$$

The proof is complete. □

Like the Theorem 3.1 can be considered a similar process for the Equation (2).

Theorem 3.2 ([17]). *Suppose $f(x)$ be belongs to the space $\mathbf{H}^\sigma(0,1), \sigma \geq 0$ that $\mathbf{H}^\sigma(0,1)$ is defined by:*

$$\mathbf{H}^\sigma(0,1) = \Big\{ f \in L^2(0,1) : \| f \|_{\mathbf{H}^\sigma(0,1)} = \Big(\sum_{k=0}^{\sigma} \| f^{(k)} \|^2_{L^2(0,1)} \Big)^{\frac{1}{2}} < \infty \Big\}, \tag{27}$$

and $P_{k,M}$ as the best approximation of f is considered. Then the following inequality is hold:

$$\| f - P_{k,M} \|_{L^2(0,1)} \leq cM^{-\sigma}2^{-k\sigma} \| f^{(\sigma)} \|^2_{L^2(0,1)}, \tag{28}$$

$$\| f - P_{k,M} \|_{\mathbf{H}^q(0,1)} \leq cM^{2q-\frac{1}{2}-\sigma}2^{k(q-\sigma)} \| f^{(\sigma)} \|^2_{L^2(0,1)}, \; q \geq 1. \tag{29}$$

Theorem 3.3. *Suppose $p_1(x)$ is belongs to the space $\mathbf{H}^\sigma(0,1), \sigma \geq 0$ and p_N is the best approximation of $p_1(x)$. Then we have:*

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, \tag{30}$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, \; q \geq 1, \tag{31}$$

here $\mathbb{E} = p_1(x) - p_N(x)$ is considered.

Proof. Using (1), we obtain

$$\begin{aligned}
\| \mathbb{E} \|_{L^2(0,1)} &= \| {}^C\mathrm{D}_x^{\varrho_1(x)}\Big[p_1(x)-p_N(x)\Big] + \Big(p_1'(x)-p_N'(x)\Big) \\
&\quad + \int_0^x \Big(p_1(\upsilon)-p_N(\upsilon)d\upsilon\Big) \|_{L^2(0,1)} \\
&= \| \frac{1}{\Gamma(1-\varrho_1(x))}\int_0^x (x-\upsilon)^{-\varrho_1(x)}\Big[p_1'(\upsilon)-p_N'(\upsilon)\Big]d\upsilon \\
&\quad + \Big(p_1'(x)-p_N'(x)\Big) + \int_0^x \Big(p_1(\upsilon)-p_N(\upsilon)d\upsilon\Big) \|_{L^2(0,1)} \\
&\leq \frac{1}{\Gamma(1-\varrho_1(x))} \| \int_0^x (x-\upsilon)^{-\varrho_1(x)}\Big[p_1'(\upsilon)-p_N'(\upsilon)\Big]d\upsilon \|_{L^2(0,1)} \\
&\quad + \| p_1'(x)-p_N'(x) \|_{L^2(0,1)} + \| \int_0^x \Big(p_1(\upsilon)-p_N(\upsilon)d\upsilon\Big) \|_{L^2(0,1)} . \qquad (32)
\end{aligned}$$

We use $\| f*g \|_{L^2(0,1)} \leq \| f \|_1 \| g \|_{L^2(0,1)}$ for Eq.(32), we have:

$$\begin{aligned}
\| \mathbb{E} \|_{L^2(0,1)} &\leq \frac{1}{\Gamma(1-\varrho_1(x))} \| \int_0^x (x-\upsilon)^{-\varrho_1(x)}\Big[p_1'(\upsilon)-p_N'(\upsilon)\Big]d\upsilon \|_{L^2(0,1)} \\
&\quad + \| p_1'(x)-p_N'(x) \|_{L^2(0,1)} + \| \int_0^x \Big(p_1(\upsilon)-p_N(\upsilon)d\upsilon\Big) \|_{L^2(0,1)} \\
&\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \| p_1'(x)-p_N'(x) \|_{L^2(0,1)} \\
&\quad + \| p_1'(x)-p_N'(x) \|_{L^2(0,1)} + \| \int_0^x \Big(p_1(\upsilon)-p_N(\upsilon)d\upsilon\Big) \|_{L^2(0,1)} \\
&\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \| p_1(x)-p_N(x) \|_{L^2(0,1)} \\
&\quad + \| p_1(x)-p_N(x) \|_{L^2(0,1)} + x \| p_1(x)-p_N(x) \|_{L^2(0,1)} \\
&\leq \frac{\| p_1(x)-p_N(x) \|_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2 \| p_1(x)-p_N(x) \|_{L^2(0,1)}, \qquad (33)
\end{aligned}$$

so, we employ Eqs.(28) and (29) on the relation (33), we get:

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, \qquad (34)$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)},\ q \geq 1. \qquad (35)$$

The desired result is achieved. □

Theorem 3.4. *Suppose $p_2(x)$ is belongs to the space $\mathbf{H}^\sigma(0,1), \sigma \geq 0$ and $p_{N'}$ is the best* *approximation of* $p_1(x)$*. Then we have:*

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN'^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_2(x))} + 2cN'^{-\sigma} \| p_2^{(\sigma)} \|^2_{L^2(0,1)}, \tag{36}$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN'^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_2(x))} + 2cN'^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|^2_{L^2(0,1)}, \; q \geq 1. \tag{37}$$

Proof. The proof of this theorem is similar to the proof of the Theorem 3.3. □

3.1 The shifted Jacobi-Gauss collocation algorithm

This subsection describes the proposed method for solving Eqs.(1), (2). By substituting the relations (16) and (17) in Eqs.(1) and (2), we obtain:

$$\underbrace{{}^{C}\mathrm{D}_x^{\varrho_1(x)}\Big[\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x)\Big] + \frac{d}{dx}\Big(\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x)\Big) + \int_0^x \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(\upsilon) d\upsilon = \theta_1(x),}_{\textbf{the initial condition for this types of equation is } \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(0)=\sum_{k=0}^{n} a_k \mathbb{P}_k'^{\mu,\nu}(0)=0,}$$

$$\sum_{k=0}^{n} a_k {}^{C}\mathrm{D}_x^{\varrho_1(x)} \mathbb{P}_k^{\mu,\nu}(x) + \sum_{k=0}^{n} a_k \mathbb{P}_k'^{\mu,\nu}(x) + \sum_{k=0}^{n} a_k \int_0^x \mathbb{P}_k^{\mu,\nu}(\upsilon) d\upsilon = \theta_1(x), \tag{38}$$

$$\underbrace{{}^{C}\mathrm{D}_x^{\varrho_2(x)}\Big[\sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(x)\Big] + \sum_{k=0}^{n} b_k \mathbb{P}_k'^{\mu,\nu}(x) + \int_0^x \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(\upsilon) d\upsilon = \theta_2(x),}_{\textbf{the initial condition for this types of equation is } \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(0)=\sum_{k=0}^{n} b_k \mathbb{P}_k'^{\mu,\nu}(0)=0,}$$

$$\sum_{k=0}^{n} b_k {}^{C}\mathrm{D}_x^{\varrho_1(x)} \mathbb{P}_k^{\mu,\nu}(x) + \sum_{k=0}^{n} b_k \mathbb{P}_k'^{\mu,\nu}(x) + \sum_{k=0}^{n} b_k \int_0^x \mathbb{P}_k^{\mu,\nu}(\upsilon) d\upsilon = \theta_2(x), \tag{39}$$

by applying Eqs.(12),(13) on the equations (38) and (39), we have:

$$\sum_{k=0}^{n}\sum_{m=0}^{k} a_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_1(x)} + \sum_{k=0}^{n}\sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^{n}\sum_{m=0}^{k} a_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_1(x), \tag{40}$$

$$\sum_{k=0}^{n}\sum_{m=0}^{k} b_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_2(x)} + \sum_{k=0}^{n}\sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^{n}\sum_{m=0}^{k} b_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_2(x), \tag{41}$$

where $\Upsilon_{m,k}^{\mu,\nu} = (-1)^{k-m}\frac{\Gamma(\nu+k+1)\Gamma(\mu+\nu+m+k+1)}{(m+1)!(k-m)!\Gamma(\nu+m+1)}$. Now we calculate the equations (40) and (41) in points $x_{\mu,\nu,i} = \frac{x_{\mu,\nu,i}+1}{2}$ which is the nodes of the standard Jacobi-Gauss interpolation in the interval $[-1,1]$ and their introduced in [7]. Then for $i=1,2,\ldots,n$ we get:

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{m=0}^{k} a_k\Omega_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-\varrho_1(x)} &+ \sum_{k=0}^{n}\sum_{m=1}^{k+1} a_k\Psi_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-1}+\sum_{k=0}^{n}\sum_{m=0}^{k} a_k\Upsilon_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m+1}=\theta_1(x_{\mu,\nu,i}),\\
\sum_{k=0}^{n} a_k\mathbb{P}_k^{\mu,\nu}(0) &= \sum_{k=0}^{n} a_k\mathbb{P}_k^{\prime\mu,\nu}(0)=0.
\end{aligned}\tag{42}$$

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{m=0}^{k} b_k\Omega_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-\varrho_2(x)} &+ \sum_{k=0}^{n}\sum_{m=1}^{k+1} b_k\Psi_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-1}+\sum_{k=0}^{n}\sum_{m=0}^{k} b_k\Upsilon_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m+1}=\theta_2(x_{\mu,\nu,i}),\\
\sum_{k=0}^{n} b_k\mathbb{P}_k^{\mu,\nu}(0) &= \sum_{k=0}^{n} b_k\mathbb{P}_k^{\prime\mu,\nu}(0)=0.
\end{aligned}\tag{43}$$

By solving these equations together with the initial conditions, the uncertain coefficients a_k, b_k are obtained. When the functions $\theta_1(x), \theta_2(x)$ are nonlinear, in this case the equations (42) and (43) changes to the following equations:

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{m=0}^{k} a_k\Omega_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-\varrho_1(x)} &+ \sum_{k=0}^{n}\sum_{m=1}^{k+1} a_k\Psi_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-1}+\sum_{k=0}^{n}\sum_{m=0}^{k} a_k\Upsilon_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m+1}\\
&= \theta_1(x_{\mu,\nu,i},p_1(x_{\mu,\nu,i}),p_1'(x_{\mu,\nu,i})),\\
\sum_{k=0}^{n} a_k\mathbb{P}_k^{\mu,\nu}(0) &= \sum_{k=0}^{n} a_k\mathbb{P}_k^{\prime\mu,\nu}(0)=0.
\end{aligned}\tag{44}$$

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{m=0}^{k} b_k\Omega_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-\varrho_2(x)} &+ \sum_{k=0}^{n}\sum_{m=1}^{k+1} b_k\Psi_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m-1}+\sum_{k=0}^{n}\sum_{m=0}^{k} b_k\Upsilon_{m,k}^{\mu,\nu}x_{\mu,\nu,i}^{m+1}\\
&= \theta_2(x_{\mu,\nu,i},p_2(x_{\mu,\nu,i}),p_2'(x_{\mu,\nu,i})),\\
\sum_{k=0}^{n} b_k\mathbb{P}_k^{\mu,\nu}(0) &= \sum_{k=0}^{n} b_k\mathbb{P}_k^{\prime\mu,\nu}(0)=0.
\end{aligned}\tag{45}$$

By applying a recursive method on the relations (44) and (45) can be obtained the uncertain coefficients a_k, b_k.

4 Numerical Examples

This section examines four examples by using the proposed method to show its performance and accuracy.

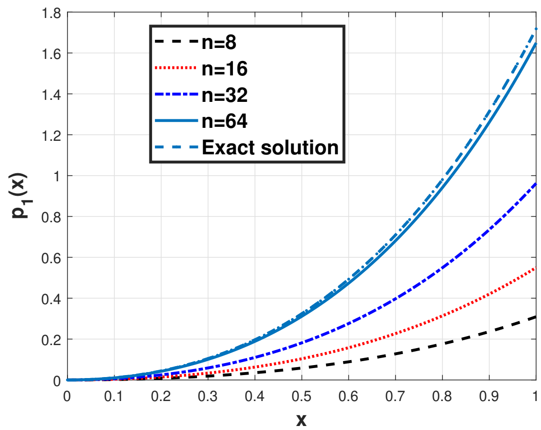

Figure 1: Numerical and exact solutions of the Example 1 with $\varrho_1(x) = 1 - 0.001x$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

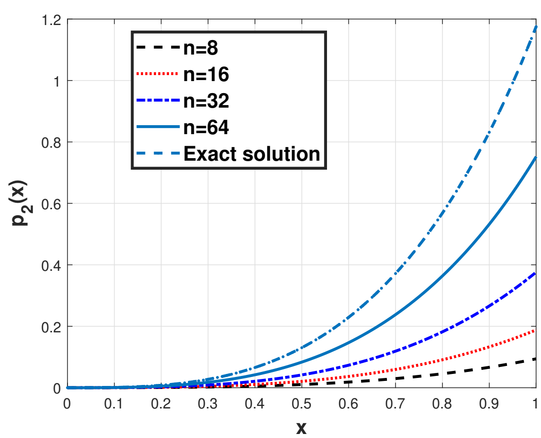

Figure 2: Numerical and exact solutions of the Example 1 with $\varrho_2(x) = 1 - \sin^2(x)$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$0.96485e-15$	$0.99077e-16$	$1.01719e-17$	$1.04410e-18$
0.2	$0.86589e-15$	$0.88993e-16$	$0.91443e-17$	$0.93940e-18$
0.3	$0.77425e-15$	$0.79650e-16$	$0.81918e-17$	$0.84231e-18$
0.4	$0.68954e-15$	$0.71009e-16$	$0.73105e-17$	$0.75244e-18$
0.5	$0.61139e-15$	$0.63033e-16$	$0.64967e-17$	$0.66940e-18$
0.6	$0.53947e-15$	$0.55688e-16$	$0.57467e-17$	$0.59284e-18$
0.7	$0.47343e-15$	$0.48940e-16$	$0.50573e-17$	$0.52242e-18$
0.8	$0.41297e-15$	$0.42757e-16$	$0.44252e-17$	$0.45780e-18$
0.9	$0.35778e-15$	$0.37110e-16$	$0.38473e-17$	$0.39869e-18$

Table 1: The absolute error for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\varrho_1(x) = 1 - 0.001x$ for Example 1.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$1.02256e-15$	$1.12266e-17$	$1.22921e-19$	$1.34242e-21$
0.2	$1.46252e-17$	$1.58973e-19$	$1.72428e-21$	$1.86639e-23$
0.3	$0.68232e-17$	$0.75878e-19$	$0.84084e-21$	$0.92869e-23$
0.4	$0.42820e-17$	$0.48438e-19$	$0.54532e-21$	$0.61123e-23$
0.5	$0.24707e-17$	$0.28622e-19$	$0.32932e-21$	$0.37658e-23$
0.6	$0.12620e-17$	$0.15145e-19$	$0.17989e-21$	$0.21170e-23$
0.7	$0.05314e-17$	$0.06759e-19$	$0.08446e-21$	$0.10393e-23$
0.8	$0.01572e-17$	$0.02239e-19$	$0.03073e-21$	$0.04091e-23$
0.9	$0.00196e-17$	$0.00383e-19$	$0.00663e-21$	$0.01053e-23$

Table 2: The absolute error for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\varrho_2(x) = 1 - \sin^2(x)$ for Example 1.

4.1 Example 1.

We consider the following coupled differential-integral equation of order $\varrho_1(x) = 1 - 0.001x$ and $\varrho_2(x) = 1 - \sin^2(x)$:

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_1(x)}\Big[p_1(x)\Big] &+ p_1'(x) + \int_0^x p_2(\upsilon)d\upsilon = \theta_1(x), \\ p_1(0) &= p_1'(0) = 0, \end{aligned} \tag{46}$$

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_2(x)}\Big[p_2(x)\Big] &+ p_2'(x) + \int_0^x p_1(\upsilon)d\upsilon = \theta_2(x), \\ p_2(0) &= p_2'(0) = 0, \end{aligned} \tag{47}$$

where

$$\theta_1(x) = x^2\cosh(x) - 2t\sinh(x) + 2\cosh(x) + 2(e^x + 2xe^x - 1)$$

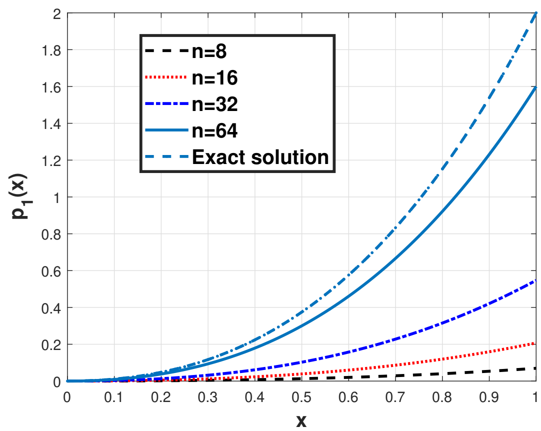

Figure 3: Numerical and exact solutions of the Example 2 with $\varrho_1(x) = \sin(\frac{x}{3})$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

and

$$\theta_2(x) = 4x\cosh(x) + 2x^2\cosh(x) + e^x(x-1).$$

The analytical solutions for these questions are $p_1(x) = xe^x - x, p_2(x) = x^2\sinh(x)$. This example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 1 and 2. In the Tables 1 and 2, the absolute error between the approximate and analytical solutions are displayed.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$0.20113e-15$	$0.20639e-16$	$0.21173e-17$	$0.21717e-18$
0.2	$0.18095e-15$	$0.18586e-16$	$0.19087e-17$	$0.19595e-18$
0.3	$0.16210e-15$	$0.16669e-16$	$0.17136e-17$	$0.17611e-18$
0.4	$0.14456e-15$	$0.14883e-16$	$0.15317e-17$	$0.15760e-18$
0.5	$0.12827e-15$	$0.13223e-16$	$0.13626e-17$	$0.14037e-18$
0.6	$0.11319e-15$	$0.11685e-16$	$0.12058e-17$	$0.12439e-18$
0.7	$0.09928e-15$	$0.10265e-16$	$0.10609e-17$	$0.10960e-18$
0.8	$0.08649e-15$	$0.08959e-16$	$0.09275e-17$	$0.09598e-18$
0.9	$0.07479e-15$	$0.07762e-16$	$0.08051e-17$	$0.08347e-18$

Table 3: The absolute error for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\varrho_1(x) = \sin(\frac{x}{3})$ for Example 2.

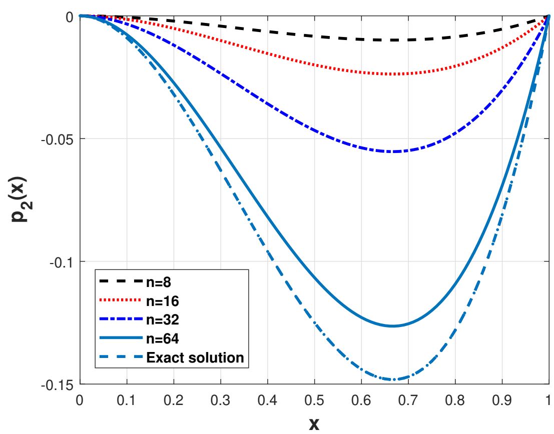

Figure 4: Numerical and exact solutions of the Example 2 with $\varrho_2(x) = \frac{x}{3}$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

4.2 Example 2.

Consider the following coupled differential-integral equation with $\varrho_1(x) = \sin(\frac{x}{3})$ and $\varrho_2(x) = \frac{x}{3}$:

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_1(x)}\Big[p_1(x)\Big] &+ p_1'(x) + \int_0^x p_1(\upsilon)d\upsilon = 4x(x+1) + 2x^2 + \frac{x^4}{4} + \frac{x^3}{3}, \\ p_1(0) &= p_1'(0) = 0, \end{aligned} \tag{48}$$

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_2(x)}\Big[p_2(x)\Big] &+ p_2'(x) + \int_0^x p_2(\upsilon)d\upsilon = 4x(x-1) + 2x^x + \frac{x^4}{4} - \frac{x^3}{3}, \\ p_2(0) &= p_2'(0) = 0, \end{aligned} \tag{49}$$

the exact solutions of Eqs.(48),(49) are $p_1(x) = x^2(x+1), p_2(x) = x^2(x-1)$. This example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 3 and 4. In the Tables 3 and 4, the absolute error between the approximate and analytical solutions are displayed.

4.3 Example 3.

Next, we describe the nonlinear coupled differential-integral equation

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_1(x)}\Big[p_1(x)\Big] &+ p_1'(x) + \int_0^x p_1(\upsilon)d\upsilon = 2x^2 + \frac{x^4}{12} + \frac{x^9}{27} - p_1^3(x), \\ p_1(0) &= p_1'(0) = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} {}^{C}\mathrm{D}_x^{\varrho_2(x)}\Big[p_2(x)\Big] &+ p_2'(x) + \int_0^x p_2(\upsilon)d\upsilon = 4x + \frac{7x^3}{3} - p_2'(x)p_2(x), \\ p_2(0) &= p_2'(0) = 0, \end{aligned} \tag{51}$$

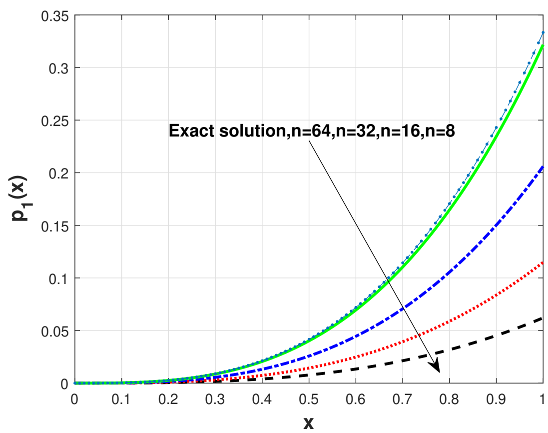

Figure 5: Numerical and exact solutions of the Example 3 with $\varrho_1(x) = e^x - x^2$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

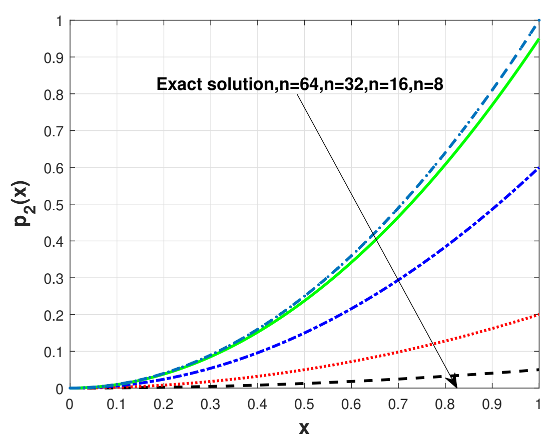

Figure 6: Numerical and exact solutions of the Example 3 with $\varrho_2(x) = \sin^2(x)$ and for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$2.18198e-15$	$2.26887e-16$	$2.35518e-17$	$2.44075e-18$
0.2	$1.83178e-15$	$1.91941e-16$	$2.00709e-17$	$2.09466e-18$
0.3	$1.48495e-15$	$1.57079e-16$	$1.65732e-17$	$1.74436e-18$
0.4	$1.15160e-15$	$1.23312e-16$	$1.31596e-17$	$1.39996e-18$
0.5	$0.84181e-15$	$0.91649e-16$	$0.99312e-17$	$1.07154e-18$
0.6	$0.56569e-15$	$0.63101e-16$	$0.69891e-17$	$0.76923e-18$
0.7	$0.33335e-15$	$0.38678e-16$	$0.44342e-17$	$0.50311e-18$
0.8	$0.15489e-15$	$0.19390e-16$	$0.23676e-17$	$0.28329e-18$
0.9	$0.04040e-15$	$0.06247e-16$	$0.08902e-17$	$0.11987e-18$

Table 4: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = \frac{x}{3}$ for Example 2.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$1.35242e-15$	$1.42118e-16$	$1.49223e-17$	$1.56560e-18$
0.2	$1.09957e-15$	$1.15953e-16$	$1.22164e-17$	$1.28592e-18$
0.3	$0.88038e-15$	$0.93215e-16$	$0.98592e-17$	$1.04171e-18$
0.4	$0.69244e-15$	$0.73662e-16$	$0.78265e-17$	$0.83056e-18$
0.5	$0.53335e-15$	$0.57055e-16$	$0.60944e-17$	$0.65006e-18$
0.6	$0.40071e-15$	$0.43153e-16$	$0.46388e-17$	$0.49781e-18$
0.7	$0.29212e-15$	$0.31715e-16$	$0.34356e-17$	$0.37141e-18$
0.8	$0.20516e-15$	$0.22501e-16$	$0.24609e-17$	$0.26845e-18$
0.9	$0.13744e-15$	$0.15270e-16$	$0.16905e-17$	$0.18652e-18$

Table 5: The absolute error for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\varrho_1(x) = e^x - x^2$ for Example 3.

with $\varrho_1(x) = e^x - x^2$ and $\varrho_2(x) = \sin^2(x)$. The exact solutions of Eqs.(50),(51) are $p_1(x) = \frac{1}{3}x^3, p_2(x) = x^2$. This example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 5 and 6. In the Tables 5 and 6, the absolute error between the approximate and analytical solutions are displayed.

5 Conclusion

This article is focused on the coupled differential-integral equation including the Caputo fractional operator of variable-orders and it is proposed a numerical method based on the shifted Jacobi–Gauss collocation scheme to obtain the solution of the coupled differential-integral equation. Also, in this paper about the convergence and an upper bound on the error are discussed. Some numerical Examples have been showed in order to display the high exactness of the suggested method.

	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
x	**n=8**	**n=16**	**n=32**	**n=64**
0.1	$1.96991e-15$	$2.02502e-16$	$2.08087e-17$	$2.13749e-18$
0.2	$1.75711e-15$	$1.80917e-16$	$1.86199e-17$	$1.91558e-18$
0.3	$1.55647e-15$	$1.60549e-16$	$1.65527e-17$	$1.70581e-18$
0.4	$1.36799e-15$	$1.41398e-16$	$1.46072e-17$	$1.50822e-18$
0.5	$1.19168e-15$	$1.23462e-16$	$1.27832e-17$	$1.32277e-18$
0.6	$1.02752e-15$	$1.06742e-16$	$1.10808e-17$	$1.14950e-18$
0.7	$0.87552e-15$	$0.91238e-16$	$0.95000e-17$	$0.98838e-18$
0.8	$0.73568e-15$	$0.76950e-16$	$0.80408e-17$	$0.83942e-18$
0.9	$0.60800e-15$	$0.63878e-16$	$0.67032e-17$	$0.70262e-18$

Table 6: The absolute error for the values of the parameters $n = \mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\varrho_2(x) = \sin^2(x)$ for Example 3.

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Received: May 15, 2021
Accepted for publication: August 14, 2021
Communicated by: Yong-Xin Guo