Communications in Mathematics **32** (2024), no. 1, 103–115
DOI: https://doi.org/10.46298/cm.10869

Generalized curvature tensor and the hypersurfaces of the Hermitian manifold for the class of Kenmotsu type

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Abstract. This paper determines the components of the generalized curvature tensor for the class of Kenmotsu type and establishes the mentioned class is η-Einstein manifold when the generalized curvature tensor is flat; the converse holds true under suitable conditions. It also introduces the notion of generalized Φ-holomorphic sectional ($G\Phi SH$-) curvature tensor and thus finds the necessary and sufficient conditions for the class of Kenmotsu type to be of constant $G\Phi SH$-curvature. In addition, the notion of Φ-generalized semi-symmetric is introduced and its relationship with the class of Kenmotsu type and η-Einstein manifold established. Furthermore, this paper generalizes the notion of the manifold of constant curvature and deduces its relationship with the aforementioned ideas. It finally shows that the class of Kenmotsu type exists as a hypersurface of the Hermitian manifold and derives a relation between the components of the Riemannian curvature tensors of the almost Hermitian manifold and its hypersurfaces.

1 Introduction

The notion of generalized curvature tensor was introduced by Shaikh and Kundu [19] to generalize well-known curvature tensors such as the conformal curvature tensor, the concircular tensor, and the conharmonic tensor. Yildiz and De [22] introduced and studied Φ-projectively semisymmetric and Φ-Weyl semisymmetric non-Sasakian (k, μ)-contact metric manifolds while Kenmotsu [13] and Kirichenko and Khari-tonova [16] discussed the

MSC 2020: AMS classification 53C25, 53D10, 53D15.
Keywords: Almost contact metric manifolds; Einstein manifold; Kenmotsu manifold; Φ-holomorphic sectional curvature tensor; hypersurfaces on Hermitian manifolds.
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Φ-holomorphic sectional curvature tensor. On the other hand, investigation of the geometry of the submanifolds of some Riemannian manifolds has captured the interest of authors such as Alegre and Carriazo [3], Sular and Özgür [20] and Chen [6]. The special subject in the study of the geometry of submanifolds is the hypersurface of the Riemannian manifolds, which has been discussed by Goldberg [9]. We concentrated on the geometry of the hypersurfaces of the almost Hermitian manifolds that have almost contact structures on the associated G-structure space. The last mentioned topic was studied by Banaru and Kirichenko [5]. Moreover, Ignatochkina [12], Ignatochkina and Morozov [11], and Nikiforova and Ignatochkina [17] studied the transformations and conformal transformations on hypersurfaces induced from almost Hermitian manifolds.

The aim of this article is organized according to the differential geometry of the generalized curvature tensor of the almost contact metric manifolds, especially the class of Kenmotsu type and the class of Kenmotsu type as a hypersurface of the Hermitian manifold.

2 Preliminaries

We use the notations M^{2n+1}, $X(M)$ and ∇ to denote the smooth manifold M of dimension $2n+1$, the Lie algebra of smooth vector fields of M, and the Riemannian connection respectively.

Definition 2.1 ([14]). A smooth manifold M^{2n+1} with the quadruple (Φ, ξ, η, g) is called an almost contact metric manifold or briefly ACR-manifold, where $\Phi : X(M) \to X(M)$, $\xi \in X(M)$, g is the Riemannian metric and $\eta(\cdot) = g(\cdot, \xi)$, are such that

$$\Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi^2 = -\mathrm{id} + \eta \otimes \xi;$$
$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y); \quad \forall X, Y \in X(M).$$

In the present article, we fix the components of the Riemannian metric g of an ACR-manifold M^{2n+1} as follows:

$$g_{00} = 1; \quad g_{a0} = g_{ab} = g_{\hat{a}\hat{b}} = 0; \quad g_{\hat{a}b} = \delta_b^a; \quad g_{ij} = g_{ji}, \tag{1}$$

where $a, b = 1, 2, \dots, n$, $\hat{a} = a + n$ and $i, j = 0, 1, \dots, 2n$. Moreover, the components of the endomorphism Φ are given by

$$\Phi_0^0 = \Phi_{\hat{b}}^a = 0; \quad \Phi_b^a = \sqrt{-1}\delta_b^a; \quad \Phi_i^j = -\Phi_{\hat{j}}^{\hat{i}}, \tag{2}$$

where $\hat{\hat{i}} = i$. So, for all $X, Y \in X(M)$, we have

$$X = X^i \varepsilon_i; \quad g(X, Y) = g_{ij} X^i Y^j; \quad \Phi(X) = \Phi_j^i X^j \varepsilon_i,$$

where $X^i \in C^\infty(M)$ and $(p; \varepsilon_0 = \xi, \varepsilon_1, \dots, \varepsilon_{2n})$ is an A-frame over M^{2n+1} such that $p \in M$, $\varepsilon_a = \frac{1}{\sqrt{2}}(id - \sqrt{-1}\Phi)e_a$, $\varepsilon_{\hat{a}} = \frac{1}{\sqrt{2}}(id + \sqrt{-1}\Phi)e_a$, and $\{\xi, e_1, \dots, e_n, \Phi e_1, \dots, \Phi e_n\}$ is a basis of $X(M)$. The set of all A-frames as given above is called an associated G-structure space (AG-structure space). For more details, we refer to [14].

Definition 2.2 ([1]). A class of ACR-manifold such that the following identity:

$$\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X, \quad \forall\, X, Y \in X(M)$$

holds is called a class of Kenmotsu type.

Lemma 2.3 ([1]). *On the AG-structure space, the class of Kenmotsu type satisfies the following relations:*

$$A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\, B_{|h|c]}{}^{d} = 0; \quad A^{acd}_{b} - B^{a[cd]}{}_{b} + B^{a[c}{}_{h} B^{|h|d]}{}_{b} = 0; \quad A^{a}_{[bcd]} = 0$$
$$A^{[bc]}_{ad} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\, B^{|h|c]}{}_{d} = 0; \quad A^{b}_{acd} + B_{a[cd]}{}^{b} - B_{a[c}{}^{h} B_{|h|d]}{}^{b} = 0; \quad A^{[bcd]}_{a} = 0;$$

where $[\cdot|\cdot|\cdot]$ denotes the anti-symmetric operator of the involved indices except $|\cdot|$ and $c, d, h \in \{1, 2, \ldots, n\}$.

We denote by R, r, Q the Riemann curvature tensor, the Ricci tensor and the Ricci operator of and ACR-manifold respectively.

Theorem 2.4 ([1]). *The components of R for the class of Kenmotsu type over the AG-structure space are given by*

1. $R^{a}_{0c0} = -\delta^{a}_{c}; \quad R^{a}_{\hat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^{a}_{[c}\, \delta^{b}_{d]}); \quad R^{a}_{\hat{b}c\hat{d}} = B^{abd}{}_{c} - B^{ab}{}_{h}\, B^{hd}{}_{c};$
2. $R^{a}_{bcd} = 2A^{a}_{bcd}; \quad R^{a}_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_{c}\, B_{bh}{}^{d} - \delta^{a}_{c}\, \delta^{d}_{b},$

where $R(X,Y)Z = R^{i}_{jkl} X^k Y^l Z^j \varepsilon_i$, $k, l = 0, 1, \ldots, 2n$ and the remaining components of R are given by the first Bianchi identity or by the conjugate (i.e. $R^{i}_{jkl} = \overline{R^{\hat{i}}_{\hat{j}\hat{k}\hat{l}}}$; $\hat{0} = 0$) of the above components or are identically zero.

Theorem 2.5 ([1]). *The components of r of the class for Kenmotsu type over the AG-structure space are as follows:*

1. $r_{00} = -2n; \quad r_{ab} = -2A^{c}_{abc} + B_{cab}{}^{c} - B_{ca}{}^{h}\, B_{hb}{}^{c};$
2. $r_{a0} = 0; \quad r_{\hat{a}b} = -2(n\delta^{a}_{b} + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_{b}\, B_{ch}{}^{c},$

where $r(X,Y) = r_{ij} X^i Y^j$, $r_{ij} = r_{ji}$ and the remaining components of r are conjugate to the above components.

Definition 2.6 ([1]). An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ with Ricci tensor r,

1. is called an Einstein manifold, if $r_{ij} = \lambda g_{ij}$, where λ is an Einstein constant.
2. is called an η-Einstein manifold, if $r_{ij} = \lambda g_{ij} + \mu \eta_i \eta_j$, where λ, μ are scalars.
3. is said to have Φ-invariant property, if $r_{a0} = r_{ab} = 0$.

Definition 2.7 **([19]).** The projective, concircular and generalized curvature tensors of type (4, 0) on the ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ are defined by the following formulas respectively:

$$\begin{aligned} P(X,Y,Z,W) &= R(X,Y,Z,W) - \frac{1}{2n}\{g(X,Z)r(Y,W) - g(X,W)r(Y,Z)\}; \\ C(X,Y,Z,W) &= R(X,Y,Z,W) - \frac{s}{2n(2n+1)}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}; \\ B(X,Y,Z,W) &= a_0 R(X,Y,Z,W) + a_1\{g(X,Z)r(Y,W) - g(X,W)r(Y,Z) \\ &\qquad + r(X,Z)g(Y,W) - r(X,W)g(Y,Z)\} \\ &\quad + 2a_2 s\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}; \end{aligned}$$

for all $X, Y, Z, W \in X(M)$, where s is the scalar curvature, a_0, a_1, a_2 are scalars and for any tensor T of type $(3,1)$, we get $T(X,Y,Z,W) = g(T(Z,W)Y,X)$, which is a tensor of type $(4,0)$.

We can rewrite the above tensors on AG-structure space as follows:

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}\{g_{ik}\ r_{jl} - g_{il}\ r_{jk}\}; \tag{3}$$

$$C_{ijkl} = R_{ijkl} - \frac{s}{2n(2n+1)}\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}; \tag{4}$$

$$B_{ijkl} = a_0 R_{ijkl} + a_1\{g_{ik}\ r_{jl} - g_{il}\ r_{jk} + r_{ik}\ g_{jl} - r_{il}\ g_{jk}\} + 2a_2 s\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}. \tag{5}$$

We note that the generalized curvature tensor B satisfies the first Bianchi identity.

3 Properties of the Generalized Curvature Tensor

In this section, we shall investigate some properties of the generalized curvature tensor on the class of Kenmotsu type.

Theorem 3.1. *On the AG-structure space, the components of the generalized curvature tensor are given by*

1. $B_{a0b0} = a_1\ r_{ab}$;
2. $B_{\hat{a}0b0} = -(a_0 + 2na_1 - 2a_2 s)\delta^a_b + a_1\ r_{\hat{a}b}$;
3. $B_{\hat{a}bcd} = 2a_0\ A^a_{bcd} + a_1\{\delta^a_c\ r_{bd} - \delta^a_d\ r_{bc}\}$;
4. $B_{\hat{a}bc\hat{d}} = a_0(A^{ad}_{bc} - B^{ah}{}_c\ B_{bh}{}^d) + a_1\{\delta^a_c\ Q^d_b + \delta^d_b\ Q^a_c\} + (2a_2 s - a_0)\delta^a_c\ \delta^d_b$;
5. $B_{\hat{a}\hat{b}cd} = 2a_0\ B^{ab}{}_{[cd]} + 4a_1\ \delta^{[a}_{[c}\ Q^{b]}_{d]} + 2(2a_2 s - a_0)\ \delta^{[a}_{[c}\ \delta^{b]}_{d]}$;

and the remaining components are identically zero, given by the first Bianchi identity or conjugate to the above components.

Proof. Since $r(X,Y) = g(X,QY)$, then $r_{ij} = g_{ik}Q^k_j$. Consquently, regarding the Equation (1), we have

$$r_{\hat{a}b} = g_{\hat{a}k}Q^k_b = g_{\hat{a}0}Q^0_b + g_{\hat{a}c}Q^c_b + g_{\hat{a}\hat{c}}Q^{\hat{c}}_b = Q^a_b.$$

Since B is defined on the class of Kenmotsu type, then we may substitute the values of $R_{ijkl} = R^{\hat{i}}_{jkl}$ according to Theorem 2.4 and the values of g_{ij} according to Equation (1) in Equation (5), obtaining the desired result. □

Theorem 3.2. *The class of Kenmotsu type $(M^{2n+1}, \Phi, \xi, \eta, g)$ has flat generalized curvature tensor if and only if M is η-Einstein manifold with:*

$$\lambda = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2s), \quad A^a_{bcd} = 0, \quad \mu = -(2n+\lambda),$$
$$A^{ad}_{bc} = B^{ah}_c B^d_{bh} + \frac{a_1}{a_0}\mu\delta^a_c\delta^d_b \quad \text{and} \quad B^{ab}_{[cd]} = \frac{a_1}{a_0}\mu\delta^a_{[c}\delta^b_{d]},$$

provided that $a_0, a_1 \neq 0$.

Proof. Suppose that M^{2n+1} has flat generalized curvature tensor with $a_0 \neq 0$ and $a_1 \neq 0$, then $B_{ijkl} = 0$ and from Theorem 3.1, we have

$$r_{ab} = 0; \quad r_{\hat{a}b} = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2s)\delta^a_b; \quad A^a_{bcd} = 0.$$

Then, according to the Definition 2.6, we get $\lambda = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2s)$. Since M is the class of Kenmotsu type, then from the Theorem 2.5, we have $r_{00} = -2n = \lambda + \mu$ and this gives us μ. Again, Theorem 3.1, item 4 gives $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d + \frac{a_1}{a_0}\mu\ \delta^a_c\ \delta^d_b$. Moreover, Theorem 3.1, item 5 gives $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\mu\ \delta^a_{[c}\ \delta^b_{d]}$. The converse is also true. □

Now, we introduce the notion of generalized Φ-holomorphic sectional $G\Phi HS$-curvature tensor as follows:

Definition 3.3. A GΦHS-curvature tensor S of an ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is a map defined by

$$S(X) = \frac{B(\Phi X, X, X, \Phi X)}{(g(X,X))^2}; \quad \forall\ X \in \ker(\eta); \quad X \neq 0.$$

Moreover, M is called of pointwise constant GΦHS-curvature if $S(X) = \gamma$ and γ does not depend on X.

Clearly, a $G\Phi HS$-curvature tensor is a Φ-holomorphic sectional (ΦHS-)curvature tensor if and only if $a_0 = 1$ and $a_1 = a_2 = 0$. Therefore, we can derive the necessary and sufficient condition for an ACR-manifold to have pointwise constant $G\Phi HS$-curvature on AG-structure space.

Theorem 3.4. *An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ has pointwise constant GΦHS-curvature if and only if, on the AG-structure space, the generalized curvature tensor B of M satisfies*

$$B^{(a\ d)}_{(bc)} = \frac{\gamma}{2}\widetilde{\delta}^{ad}_{bc},$$

where $\widetilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$ and $(\cdot\cdot)$ denotes the symmetric operator of the included indices.

Proof. Since the tensor B has the same properties as the Riemannian curvature tensor R, then we can follow the same proof was found in [14] or equivalently in [21]. □

Theorem 3.5. *The class of Kenmotsu type $(M^{2n+1}, \Phi, \xi, \eta, g)$ has pointwise constant GΦHS--curvature if and only if, on the AG-structure space, M satisfies the following equality:*

$$A^{ad}_{bc} = B_{bc}{}^{[ad]} - B_{hb}{}^{a}\ B^{dh}{}_{c} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

Proof. Suppose that M is the class of Kenmotsu type and has pointwise constant GΦHS-curvature. Using Theorem 3.4 and Theorem 3.1, item 4, we get

$$A^{(ad)}_{(bc)} = B^{(a|h|}{}_{(b}\ B_{c)h}{}^{d)} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

The above equation can be rewritten as follows:

$$A^{(ad)}_{(bc)} = -B_{h(b}{}^{(a}\ B^{d)h}{}_{c)} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

Since $A^{ad}_{bc} = A^{[ad]}_{[bc]} + A^{[ad]}_{(bc)} + A^{(ad)}_{[bc]} + A^{(ad)}_{(bc)}$, then taking into account Lemma 2.3 and the above result, we conclude the proof. □

Recently, Yıldız and De [22] introduced the notions of Φ-projectively semisymmetric and Φ-Weyl semisymmetric. Regarding these ideas, we can introduce the following definition:

Definition 3.6. An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is called Φ-generalized semisymmetric if $B(Z, W) \cdot \Phi = 0$, for all $Z, W \in X(M)$, or equivalently

$$B(X, \Phi Y, Z, W) + B(\Phi X, Y, Z, W) = 0; \quad \forall\ X, Y, Z, W \in X(M).$$

Lemma 3.7. *On AG-structure space, the ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is Φ-generalized semi (ΦGS-)symmetric if and only if*

$$B_{a0b0} = B_{\hat{a}0b0} = B_{a0bc} = B_{\widehat{a}0bc} = B_{a0\widehat{b}\widehat{c}} = B_{abcd} = B_{\hat{a}\hat{b}cd} = 0.$$

Proof. According to the Definition 3.6, we have that M is Φ-generalized semi-symmetric if and only if

$$B(X, \Phi Y, Z, W) + B(\Phi X, Y, Z, W) = 0; \quad \forall\ X, Y, Z, W \in X(M).$$

On the AG-structure space, the above identity is equivalent to the following:

$$B_{iqkl}\ \Phi^q_j + B_{tjkl}\ \Phi^t_i = 0; \quad q, t = 0, 1, \dots, 2n.$$

If we take

$$(i, j, k, l) = (a, 0, b, 0), (\hat{a}, 0, b, 0), (a, 0, b, c), (\widehat{a}, 0, b, c), (a, 0, \widehat{b}, c), (a, b, c, d), (\hat{a}, \hat{b}, c, d),$$

and using Equation (2), we obtain the result. □

It is not hard to conclude the following:

Corollary 3.8. *The ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ of flat generalized curvature tensor is usually ΦGS-symmetric.*

Corollary 3.9. *The class of Kenmotsu type $(M^{2n+1}, \Phi, \xi, \eta, g)$ has flat generalized curvature tensor if and only if M is ΦGS-symmetric with $A^a_{bcd} = 0$ and $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d + \frac{a_1}{a_0}\mu\ \delta^a_c\delta^d_b$, where $\mu = -\frac{1}{a_1}(a_0 + 4na_1 - 2a_2 s)$, provided that $a_0, a_1 \neq 0$.*

Proof. Suppose that M is the class of Kenmotsu type and it has flat generalized curvature tensor, then from the Corollary 3.8, M is ΦGS-symmetric and from the Theorem 3.1, we get the other conditions.

Conversely, If M is ΦGS-symmetric with the above conditions then according to Lemma 3.7 and Theorem 3.1, M has flat generalized curvature tensor. □

Theorem 3.10. *The class of Kenmotsu type $(M^{2n+1}, \Phi, \xi, \eta, g)$ is ΦGS-symmetric if and only if M is η-Einstein manifold with $\lambda = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)$, $\mu = -(2n + \lambda)$ and $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\mu\ \delta^a_{[c}\delta^b_{d]}$, provided that $a_0, a_1 \neq 0$.*

Proof. Suppose that M is Φ-generalized semi-symmetric class of Kenmotsu type, then from Lemma 3.7 and Theorem 3.1, we have

$$r_{ab} = 0; \quad r_{\hat{a}b} = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)\delta^a_b; \quad B^{ab}{}_{[cd]} = -\frac{1}{a_0}(a_0 + 4na_1 - 2a_2 s)\delta^a_{[c}\delta^b_{d]}.$$

Regarding the Definition 2.6 and Theorem 2.5, we obtain the values of λ and μ.

The converse is verified directly. □

Corollary 3.11. *The class of Kenmotsu type $(M^{2n+1}, \Phi, \xi, \eta, g)$ is ΦGS-symmetric and has $G\Phi HS$-curvature if and only if, M is η-Einstein manifold with $\lambda = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)$, $\mu = -(2n + \lambda)$, $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\mu\delta^a_{[c}\delta^b_{d]}$, and $A^{ad}_{bc} = \frac{\gamma}{2a_0}\widetilde{\delta}^{ad}_{bc} - B_{hb}{}^a\ B^{dh}{}_c + \frac{a_1}{a_0}\mu\delta^a_b\delta^d_c$, provided that $a_0, a_1 \neq 0$.*

Proof. Suppose that M is the class of Kenmotsu type, then the necessary and sufficient conditions of the present corollary are satisfied from Theorems 3.5 and 3.10. □

4 Generalized Curvature Tensor Related with Another Tensors

In this section, we introduce a generalization of the notion of ACR-manifold of constant curvature that is used by Abood and Al-Hussaini [2]. We shall present this idea in the following definition:

Definition 4.1. An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is said to have constant generalized curvature κ if the following identity holds:

$$B(X,Y,Z,W) = \kappa\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}; \quad \forall\ X,Y,Z,W \in X(M).$$

On the AG-structure space, Definition 4.1 equivalent to the identity below.

$$B_{ijkl} = \kappa\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}. \tag{6}$$

Directly, regarding the Definition 4.1, Definition 2.7 and the definition of the conharmonic curvature tensor (see [8]), we have the following result:

Theorem 4.2. *Suppose that M^{2n+1} is an ACR-manifold of constant generalized curvature $\kappa = 2a_2 s$. Then M has flat conharmonic curvature tensor if and only if, $a_0 = 1$ and $a_1 = -\frac{1}{2n-1}$.*

Theorem 4.3. *An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ has constant generalized curvature κ if and only if, on the AG-structure space, B has the following components:*

1. $B_{\hat{a}0b0} = \kappa\ \delta^a_b$;
2. $B_{\hat{a}bc\hat{d}} = \kappa\ \delta^a_c \delta^d_b$;
3. $B_{\hat{a}\hat{b}cd} = 2\kappa\ \delta^a_{[c}\delta^b_{d]}$;

and the remaining components are identically zero, obtained from the above components by the first Bianchi identity or by taking the conjugate operation.

Proof. The result follows from Equation (6) by taking

$$(i,j,k,l) = (\hat{a},0,b,0), (\hat{a},b,c,\hat{d}), (\hat{a},\hat{b},c,d);$$

and using the Equation (1). □

Theorem 4.4. *The ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is ΦGS-symmetric if and only if, M has constant generalized curvature $\kappa = 0$.*

Proof. The claim of this theorem is obtained from Lemma 3.7 and Theorem 4.3. □

Theorem 4.5. *If an ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ has constant generalized curvature κ, then M has pointwise constant $G\Phi HS$-curvature equal to $\gamma = \kappa$.*

Proof. The result follows from Theorems 3.4 and 4.3. □

Theorem 4.6. *The class of Kenmotsu type* $(M^{2n+1}, \Phi, \xi, \eta, g)$ *has constant generalized curvature* κ *if and only if,* M *is an* η*-Einstein manifold with:*

$$\lambda = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s + \kappa), \quad A^a_{bcd} = 0, \quad \mu = -(2n + \lambda),$$
$$A^{ad}_{bc} = B^{ah}_c B^d_{bh} + \frac{a_1}{a_0}\mu\delta^a_c\delta^d_b \quad \text{and} \quad B^{ab}_{[cd]} = \frac{a_1}{a_0}\mu\delta^a_{[c}\delta^b_{d]},$$

provided that $a_0, a_1 \neq 0$.

Proof. The assertion of this theorem is obtained by combining the results of the Theorems 3.1 and 4.3. □

Now, we find the geometric properties of ACR-manifold if the generalized curvature tensor, the concircular tensor and the projective tensor are related.

Suppose that $(M^{2n+1}, \Phi, \xi, \eta, g)$ is an ACR-manifold satisfies the following condition:

$$B(X,Y,Z,W) = \frac{a_0}{3}\{P(X,Y,Z,W) - P(Y,X,Z,W) + C(X,Y,Z,W)\}. \tag{7}$$

Regarding the Equations (3), (4) and (5), we can write the Equation (7) on the AG-structure space as follows:

$$(a_1 + \frac{a_0}{6n})\{g_{ik}\, r_{jl} - g_{il}\, r_{jk} + r_{ik}\, g_{jl} - r_{il}\, g_{jk}\} + (2a_2 + \frac{a_0}{6n(2n+1)})s\{g_{ik}\, g_{jl} - g_{il}\, g_{jk}\} = 0. \tag{8}$$

The contracting of the Equation (8), that is, multiplying it by g^{ik} (the components of g^{-1} on AG-structure space), we can deduce that

$$r_{jl} = -\frac{(\alpha + 2n\beta)s}{(2n-1)\alpha} g_{jl}, \tag{9}$$

where $\alpha = a_1 + \frac{a_0}{6n}$ and $\beta = 2a_2 + \frac{a_0}{6n(2n+1)}$. Moreover, the contracting of Equation (9) gives $a_0 + 4na_1 + 4n(2n+1)a_2 = 0$. Then we can state the following theorem:

Theorem 4.7. *Any* ACR*-manifold* $(M^{2n+1}, \Phi, \xi, \eta, g)$ *which satisfies the identity* (7) *is an Einstein manifold with* $a_0 + 4na_1 + 4n(2n+1)a_2 = 0$*, provided that* $\alpha \neq 0$*. Moreover, if* M *is the class of Kenmotsu type then* $s = \frac{2n(2n-1)\alpha}{\alpha+2n\beta}$*, provided that* $\alpha + 2n\beta \neq 0$*.*

Proof. The first part of this theorem is obvious from the above discussion. Now, if M is the class of Kenmotsu type then from the Theorem 2.5, we have $r_{00} = -2n$. Then the result is established from the Equations (1) and (9). □

5 The Hypersurfaces of the Hermitian Manifold

Suppose that $(M^{2n-1}, \Phi, \xi, \eta, g)$ is an ACR-manifold, then there exists an almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\Phi X - f\xi, \eta(X)\frac{d}{dt})$, where $X \in X(M)$, $t \in \mathbb{R}$ and f is a smooth function on $\mathbb{R}$. The Riemannian metric h on $M \times \mathbb{R}$ is defined by

$$h((X, f_1\frac{d}{dt}), (Y, f_2\frac{d}{dt})) = g(X, Y) + f_1\ f_2; \quad \forall\ X, Y \in X(M); \quad f_1, f_2 \in C^{\infty}(\mathbb{R}).$$

The structure on $M \times \mathbb{R}$ is Hermitian if and only if the structure on M is normal (see [18]). Since the class of Kenmotsu type is normal because it is the class $C_3 \oplus C_4 \oplus C_5$, where C_5 is taken here to be α-Kenmotsu manifold with $\alpha = 1$ (see [7] for more detail about the classes C_3 and C_4). Then the product manifold of the class of Kenmotsu type and the real line is Hermitian (i.e. $W_3 \oplus W_4$, see [10]).

Now, we discuss the opposite problem, that is, if (N^{2n}, J, h) is an Hermitian manifold, then can we find a hypersuface of N which is the class of Kenmotsu type? We rely on the citation [5] for the background.

Suppose that $a, b, c = 1, 2, \ldots, n-1$ and $\sigma_{ij} = \sigma_{ji}$; $i, j = 1, 2, \ldots, 2n-1$ are the components of the second quadratic form as mentioned in [5].

Theorem 5.1 ([5]). *An ACR-manifold which a hypersuface of an almost Hermitian manifold has the following first family of the Cartan structure equations:*

$$\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}_c\ \omega^c \wedge \omega_b + B^{abc}\ \omega_b \wedge \omega_c + (\sqrt{2}B^{an}_b + \sqrt{-1}\sigma^a_b)\omega^b \wedge \omega \\
&\quad + (\sqrt{-1}\sigma^{ab} - \sqrt{2}\widetilde{B}^{nab} - \frac{1}{\sqrt{2}}B^{ab}_n - \frac{1}{\sqrt{2}}\widetilde{B}^{abn})\omega_b \wedge \omega; \\
d\omega_a &= -\omega^b_a \wedge \omega_b + B^c_{ab}\ \omega_c \wedge \omega^b + B_{abc}\ \omega^b \wedge \omega^c + (\sqrt{2}B^b_{an} - \sqrt{-1}\sigma^b_a)\omega_b \wedge \omega \\
&\quad - (\sqrt{-1}\sigma_{ab} + \sqrt{2}\widetilde{B}_{nab} + \frac{1}{\sqrt{2}}B^n_{ab} + \frac{1}{\sqrt{2}}\widetilde{B}_{abn})\omega^b \wedge \omega; \\
d\omega &= \sqrt{2}B_{nab}\ \omega^a \wedge \omega^b + \sqrt{2}B^{nab}\ \omega_a \wedge \omega_b + (\sqrt{2}B^{na}_b - \sqrt{2}B^a_{nb} - 2\sqrt{-1}\sigma^a_b)\omega^b \wedge \omega_a \\
&\quad + (\widetilde{B}_{nbn} + B^n_{nb} + \sqrt{-1}\sigma_{nb})\omega \wedge \omega^b + (\widetilde{B}^{nbn} + B^{nb}_n - \sqrt{-1}\sigma^b_n)\omega \wedge \omega_b.
\end{aligned}$$

From Banaru [4], we see that the Hermitian manifold N satisfies $B^{\alpha\beta\gamma} = B_{\alpha\beta\gamma} = 0$, where $\alpha, \beta, \gamma = 1, 2, \ldots, n$, and thus the Theorem 5.1 reduce to the following form:

Theorem 5.2. *An ACR-manifold which a hypersuface of the Hermitian manifold has the following first family of the Cartan structure equations:*

$$\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}_c\ \omega^c \wedge \omega_b + (\sqrt{2}B^{an}_b + \sqrt{-1}\sigma^a_b)\omega^b \wedge \omega + (\sqrt{-1}\sigma^{ab} - \frac{1}{\sqrt{2}}B^{ab}_n)\omega_b \wedge \omega; \\
d\omega_a &= -\omega^b_a \wedge \omega_b + B^c_{ab}\ \omega_c \wedge \omega^b + (\sqrt{2}B^b_{an} - \sqrt{-1}\sigma^b_a)\omega_b \wedge \omega - (\sqrt{-1}\sigma_{ab} + \frac{1}{\sqrt{2}}B^n_{ab})\omega^b \wedge \omega; \\
d\omega &= (\sqrt{2}B^{na}_b - \sqrt{2}B^a_{nb} - 2\sqrt{-1}\sigma^a_b)\omega^b \wedge \omega_a + (B^n_{nb} + \sqrt{-1}\sigma_{nb})\omega \wedge \omega^b \\
&\quad + (B^{nb}_n - \sqrt{-1}\sigma^b_n)\omega \wedge \omega_b.
\end{aligned}$$

Regarding Abood and Abass [1], we note that the class of Kenmotsu type satisfies the following theorem:

Theorem 5.3 ([1]). *The class of Kenmotsu type M^{2n-1} has the following first group of Cartan structure equations:*

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b - \omega^a \wedge \omega; \\ d\omega_a &= -\omega^b_a \wedge \omega_b + B_{ab}{}^c\, \omega_c \wedge \omega^b - \omega_a \wedge \omega; \\ d\omega &= 0, \end{aligned}$$

where $B^{ab}{}_c$ and $B_{ab}{}^c$ are the components of the first Kirichenko's tensor as explained in [15].

Now, if the class of Kenmotsu type M^{2n-1} is a hypersurface of the Hermitian manifold N^{2n}, then the cartan structure equations that mentioned in the Theorems 5.2 and 5.3 must be equal. Then we get

$$\begin{gathered} B^{ab}_c = B^{ab}{}_c; \quad \sqrt{2}B^{an}_b + \sqrt{-1}\sigma^a_b = -\delta^a_b; \quad \sqrt{-1}\sigma^{ab} - \frac{1}{\sqrt{2}}B^{ab}_n = 0; \\ B^c_{ab} = B_{ab}{}^c; \quad \sqrt{2}B^b_{an} - \sqrt{-1}\sigma^b_a = -\delta^b_a; \quad \sqrt{-1}\sigma_{ab} + \frac{1}{\sqrt{2}}B^n_{ab} = 0; \\ \sqrt{2}B^{na}_b - \sqrt{2}B^a_{nb} - 2\sqrt{-1}\sigma^a_b = 0; \quad B^n_{nb} + \sqrt{-1}\sigma_{nb} = 0; \quad B^{nb}_n - \sqrt{-1}\sigma^b_n = 0. \end{gathered} \tag{10}$$

Since $\sigma_{[\alpha\beta]} = 0$ and $B^\gamma_{[\alpha\beta]} = B^\gamma_{\alpha\beta}$, then Equation (10) gives the following relations:

$$\sigma_{ab} = 0; \quad \sigma_{nb} = 0; \quad \sigma^a_b = \sqrt{-1}(\sqrt{2}B^{an}_b + \delta^a_b). \tag{11}$$

Then from the above discussion, we can establishing the theorem below.

Theorem 5.4. *If the Hermitian manifold has the class of Kenmotsu type as a hypersurface, then the second quadratic form has components agree with the Equation* (11).

On the other hand, we can establish a relation between the components of the Riemannian curvature tensors of the almost Hermitian manifold and its hypersurfaces.

Suppose that $\mathcal{R}^i_{jkl}$ are the components of the Riemannian curvature tensor of the almost Hermitian manifold, N^{2n} and $\widetilde{\mathcal{R}}^i_{jkl}$ are the components of the Riemannian curvature tensor of its hypersurface M^{2n-1}. Then from the second group of cartan structure equations, we have

$$\begin{aligned} d\omega^i_j &= \omega^i_k \wedge \omega^k_j + \frac{1}{2}\mathcal{R}^i_{jkl}\, \omega^k \wedge \omega^l; \\ d\theta^i_j &= \theta^i_k \wedge \theta^k_j + \frac{1}{2}\widetilde{\mathcal{R}}^i_{jkl}\, \theta^k \wedge \theta^l; \end{aligned}$$

where ω^i_j and θ^i_j are the Riemannian connection forms of N and M respectively. Whereas, ω^k and θ^k are the dual A-frames on AG-structure spaces of N and M respectively. Moreover, from [5], we have

$$\theta^i = C^i_j\, \omega^j; \quad \omega^i = \widetilde{C}^i_j\, \theta^j; \quad \theta^i_j = C^i_k\, \omega^k_r\, \widetilde{C}^r_j; \quad \omega^i_j = \widetilde{C}^i_k\, \theta^k_r\, C^r_j;$$

where $C = (C^i_j)$ and $C^{-1} = (\widetilde{C}^i_j)$ were defined in [5]. Then the substitution of the above relations in the second group of cartan structure equations, we conclude the following theorem:

Theorem 5.5. *If $\mathcal{R}^i_{jkl}$ and $\widetilde{\mathcal{R}}^q_{rst}$ are the components of the Riemannian curvature tensor of the almost Hermitian manifold (N^{2n}, J, g) and its hypersurface $(M^{2n-1}, \Phi, \xi, \eta, g)$ respectively, then they are related as follow:*

$$\mathcal{R}^i_{jkl} = \widetilde{C}^i_q \ \widetilde{\mathcal{R}}^q_{rst} \ C^r_j \ C^s_k \ C^t_l.$$

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Received: June 28, 2021
Accepted for publication: December 31, 2021
Communicated by: Haizhong Li