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On λ-pseudo bi-starlike functions related with Fibonacci numbers

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Abstract. In this paper we define a new subclass λ-bi-pseudo-starlike functions of Σ related to shell-like curves connected with Fibonacci numbers and determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for $f \in \mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$ and for special cases, corollaries are stated which some of them are new and have not been studied so far.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk

$$\mathbb{U} = \{z \,:\, z \in \mathbb{C} \,\text{and}\, |z| < 1\}.$$

Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalised by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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The Koebe one quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } \ f(f^{-1}(w)) = w \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurian series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{2}$$

One can see a short history and examples of functions in the class Σ in [15]. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2], [3], [9], [15], [16], [10]). For a brief historical account and for several interesting examples of functions in the class Σ; see the pioneering work on this subject by Srivastava *et al.* [15], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [15], we choose to recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the Koebe function is not a member of Σ.

An analytic function f is said to be subordinate to an analytic function F in $\mathbb{U}$, which we denote by $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \quad \Longleftrightarrow f(0) = F(0) \text{ and } \ f(\mathbb{U}) \subset F(\mathbb{U}) \, , z \in \mathbb{U}$$

(for details see [4], [11]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z),$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently, in [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

Definition 1.1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with
$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$
where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc of radius $(3-\sqrt{5})/2$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that
$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$
which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by
$$(10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$
which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:
$$\tau^n = u_n \tau + u_{n-1}.$$

In [13] Raina and Sokół showed that
$$\begin{aligned}
\tilde{p}(z) &= \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \\
&= \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\
&= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t}-\frac{1}{1-\tau t}\right) \\
&= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n \\
&= 1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n, \qquad (3)
\end{aligned}$$
where
$$u_n = \frac{(1-\tau)^n-\tau^n}{\sqrt{5}}, \tau = \frac{1-\sqrt{5}}{2} \quad (n = 1, 2, \ldots). \qquad (4)$$
This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$. And they got

$$\begin{aligned}\tilde{p}(z) &= 1+\sum_{n=1}^{\infty}\tilde{p}_n z^n \\ &= 1+(u_0+u_2)\tau z+(u_1+u_3)\tau^2 z^2+\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n \\ &= 1+\tau z+3\tau^2 z^2+4\tau^3 z^3+7\tau^4 z^4+11\tau^5 z^5+\cdots . \end{aligned} \tag{5}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.2 ([5]). *The function* $\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ *belongs to the class* $\mathcal{P}(\sqrt{5}/10)$, *where* $\sqrt{5}/10 \approx 0.2236$.

Now we recall the following lemma which will be relevant for our study.

Lemma 1.3 ([12]). *Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$|c_n| \le 2 \qquad for \qquad n \ge 1. \tag{6}$$

In this present work, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for this function class. Also, we give bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for this class.

2 Bi-Univalent function class $\mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

Firstly, let $p(z) = 1+p_1 z+p_2 z^2+\cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \ldots \tag{7}$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{8}$$

and

$$\begin{aligned}\tilde{p}(u(z)) &= 1+\frac{\tilde{p}_1 c_1 z}{2}+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right\}z^2 \\ &\quad +\left\{\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right\}z^3+\cdots . \end{aligned} \tag{9}$$

And similarly, there exists an analytic function v such that $|v(w)| < 1$ in $\mathbb{U}$ and such that $p(w) = \tilde{p}(v(w))$. Therefore, the function

$$k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \dots \tag{10}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2}\right)\frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\frac{w^3}{2} + \cdots \tag{11}$$

and

$$\begin{aligned}\tilde{p}(v(w)) &= 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_1 + \frac{d_1^2}{4}\tilde{p}_2\right\} w^2 \\ &\quad + \left\{\frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2} d_1 \left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_2 + \frac{d_1^3}{8}\tilde{p}_3\right\} w^3 + \cdots .\end{aligned} \tag{12}$$

The class $\mathcal{L}_\lambda(\alpha)$ of λ-pseudo-starlike functions of order α was introduced and investigated by Babalola [1]. A function $f \in \mathcal{A}$ is in the class $\mathcal{L}_\lambda(\alpha)$ if it satisfies

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha, \quad (0 \le \alpha < 1)$$

where $\lambda \ge 1, \lambda \in \mathbb{R}$ and $z \in \mathbb{U}$. In [1] it was showed that all pseudo-starlike functions are Bazilevič functions of type $(1 - 1/\lambda)$ and of order $\alpha^{1/\lambda}$ and univalent in open unit disk $\mathbb{U}$.

Recently Joshi et al. [8] defined the bi-pseudo-starlike functions class and obtained the bounds for the initial coefficients $|a_2|$ and $|a_3|$. In this paper we define a new class $\mathcal{PSL}_\Sigma^\lambda(\tilde{p}(z))$,$\lambda$-bi-pseudo-starlike functions of Σ related to shell-like curves connected with Fibonacci numbers and determine the bounds for the initial Taylor-Maclaurin coefficients of $|a_2|$ and $|a_3|$. Further we consider the Fekete-Szegö problem in this class and the special cases are stated as corollaries which are new and have not been studied so far.

Definition 2.1. For $\lambda \ge 1$ and $\lambda \in \mathbb{R}$, a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{PSL}_\Sigma^\lambda(\tilde{p}(z))$ if the following subordination hold:

$$\frac{z(f'(z))^\lambda}{f(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \tag{13}$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \tag{14}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Specialising the parameter $\lambda = 1$ and $\lambda = 2$, we have the following remarks, respectively:

Remark 2.2 ([7]). For $\lambda = 1$ a function $f \in \Sigma$ is in the class $\mathcal{PSL}^1_\Sigma(\tilde{p}(z)) \equiv \mathcal{SL}_\Sigma(\tilde{p}(z))$ if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{15}$$

and

$$\frac{wg'(w)}{g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \tag{16}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Remark 2.3. For $\lambda = 2$ a function $f \in \Sigma$ is in the class $\mathcal{PSL}^2_\Sigma(\tilde{p}(z)) \equiv \mathcal{GSL}_\Sigma(\tilde{p}(z))$ if the following conditions are satisfied:

$$\left(f'(z)\frac{zf'(z)}{f(z)}\right) \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{17}$$

and

$$\left(g'(w)\frac{wg'(w)}{g(w)}\right) \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \tag{18}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{PSL}^\lambda_\Sigma(\tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2.4. *Let f given by (1) be in the class $\mathcal{PSL}^\lambda_\Sigma(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(2\lambda-1)^2-(10\lambda^2-11\lambda+3)\tau}} \tag{19}$$

and

$$|a_3| \leq \frac{|\tau|\left[(2\lambda-1)^2-2(5\lambda^2-4\lambda+1)\tau\right]}{(3\lambda-1)\left[(2\lambda-1)^2-(10\lambda^2-11\lambda+3)\tau\right]}, \tag{20}$$

where $\lambda \geq 1$.

Proof. Let $f \in \mathcal{PSL}^\lambda_\Sigma(\tilde{p}(z))$ and $g = f^{-1}$. Considering (13) and (14), we have

$$\frac{z(f'(z))^\lambda}{f(z)} = \tilde{p}(u(z)) \tag{21}$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} = \tilde{p}(v(w)), \tag{22}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2). Since

$$\frac{z(f'(z))^{\lambda}}{f(z)} = 1 + (2\lambda-1)a_2 z + [(3\lambda-1)a_3 + \left(2\lambda^2 - 4\lambda + 1\right) a_2^2]z^2 + \dots$$

and

$$\frac{w(g'(w))^{\lambda}}{g(w)} = 1 - (2\lambda-1)a_2 w + [\left(2\lambda^2 + 2\lambda - 1\right) a_2^2 - (3\lambda-1)a_3]w^2 + \dots.$$

Thus we have

$$1 + (2\lambda-1)a_2 z + [(3\lambda-1)a_3 + \left(2\lambda^2 - 4\lambda + 1\right) a_2^2]z^2 + \dots \tag{23}$$

$$= 1 + \frac{\tilde{p}_1 c_1}{2} z + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_1 + \frac{c_1^2}{4}\tilde{p}_2\right] z^2 + \left[\frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_2 + \frac{c_1^3}{8}\tilde{p}_3\right] z^3 + \cdots. \tag{24}$$

and

$$1 - (2\lambda-1)a_2 w + [\left(2\lambda^2 + 2\lambda - 1\right) a_2^2 - (3\lambda-1)a_3]w^2 + \cdots,$$

$$= 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left[\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_1 + \frac{d_1^2}{4}\tilde{p}_2\right] w^2 + \left[\frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_2 + \frac{d_1^3}{8}\tilde{p}_3\right] w^3 + \cdots. \tag{25}$$

It follows from (23) and (25) that

$$(2\lambda-1)a_2 = \frac{c_1\tau}{2}, \tag{26}$$

$$(3\lambda-1)a_3 + \left(2\lambda^2 - 4\lambda + 1\right) a_2^2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{c_1^2}{4}3\tau^2, \tag{27}$$

and

$$-(2\lambda-1)a_2 = \frac{d_1\tau}{2}, \tag{28}$$

$$\left(2\lambda^2 + 2\lambda - 1\right) a_2^2 - (3\lambda-1)a_3 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{d_1^2}{4}3\tau^2. \tag{29}$$

From (26) and (28), we have

$$c_1 = -d_1, \tag{30}$$

and

$$a_2^2 = \frac{(c_1^2 + d_1^2)}{8(2\lambda-1)^2}\tau^2. \tag{31}$$

Hence

$$|a_2| \le \frac{|\tau|}{2\lambda - 1}. \tag{32}$$

Now, by summing (27) and (29), we obtain

$$2(2\lambda-1)^2 a_2^2 = \frac{1}{2}(c_2+d_2)\tau - \frac{1}{4}(c_1^2+d_1^2)\tau + \frac{3}{4}(c_1^2+d_1^2)\tau^2. \tag{33}$$

Substituting (31) in (33), we have

$$2\left[(2\lambda-1)^2 - (10\lambda^2-11\lambda+3)\tau\right] a_2^2 = \frac{1}{2}(c_2+d_2)\tau^2. \tag{34}$$

Therefore, using Lemma (1.3) we obtain

$$|a_2| \le \frac{|\tau|}{\sqrt{(2\lambda-1)^2 - (10\lambda^2-11\lambda+3)\tau}}. \tag{35}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (27) and (29). So, we find

$$2(3\lambda-1)a_3 - 2(3\lambda-1)a_2^2 = \frac{1}{2}\left(c_2-d_2\right)\tau. \tag{36}$$

Hence, we get

$$2(3\lambda-1)|a_3| \le 2|\tau| + 2(3\lambda-1)|a_2|^2. \tag{37}$$

Then, in view of (35), we obtain

$$|a_3| \le \frac{|\tau|\left[(2\lambda-1)^2 - 2(5\lambda^2-4\lambda+1)\tau\right]}{(3\lambda-1)\left[(2\lambda-1)^2-(10\lambda^2-11\lambda+3)\tau\right]}. \qquad \square$$

By taking the parameter $\lambda = 1$ and $\lambda = 2$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{GSL}_\Sigma(\tilde{p}(z))$, respectively.

Corollary 2.5 ([7]). *Let f given by* (1) *be in the class $\mathcal{SL}_\Sigma(\tilde{p}(z))$, then*

$$|a_2| \le \frac{|\tau|}{\sqrt{1-2\tau}} \tag{38}$$

and

$$|a_3| \le \frac{|\tau|(1-4\tau)}{2(1-2\tau)}. \tag{39}$$

Corollary 2.6. *Let f given by* (1) *be in the class $\mathcal{GSL}_\Sigma(\tilde{p}(z))$, then*

$$|a_2| \le \frac{|\tau|}{\sqrt{9-21\tau}} \tag{40}$$

and

$$|a_3| \le \frac{|\tau|(9-26\tau)}{5(9-21\tau)}. \tag{41}$$

3 Fekete-Szegö inequality for the function class $\mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$

Fekete and Szegö [6] introduced the generalised functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [17],in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$.

Theorem 3.1. *Let f given by (1) be in the class $\mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{4(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{4(3\lambda-1)}, \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{4(3\lambda-1)}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[(2\lambda-1)^2 - (10\lambda^2 - 11\lambda + 3)\tau\right]}. \tag{42}$$

Proof. From (34) and (36) we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{\tau^2(c_2+d_2)}{4\left[(2\lambda-1)^2 - (10\lambda^2 - 11\lambda + 3)\tau\right]} + \frac{\tau(c_2-d_2)}{4(3\lambda-1)} \\ &= \left(\frac{(1-\mu)\tau^2}{4\left[(2\lambda-1)^2 - (10\lambda^2 - 11\lambda + 3)\tau\right]} + \frac{\tau}{4(3\lambda-1)}\right) c_2 \\ &\quad + \left(\frac{(1-\mu)\tau^2}{4\left[(2\lambda-1)^2 - (10\lambda^2 - 11\lambda + 3)\tau\right]} - \frac{\tau}{4(3\lambda-1)}\right) d_2. \end{aligned} \tag{43}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{|\tau|}{4(3\lambda-1)}\right) c_2 + \left(h(\mu) - \frac{|\tau|}{4(3\lambda-1)}\right) d_2, \tag{44}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[(2\lambda-1)^2 - (10\lambda^2 - 11\lambda + 3)\tau\right]}. \tag{45}$$

Then, by taking modulus of (44), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{4(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{4(3\lambda-1)}, \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{4(3\lambda-1)}. \end{cases}$$ □

Taking $\mu = 1$, we have the following corollary.

Corollary 3.2. *If $f \in \mathcal{PSL}_{\Sigma}^{\lambda}(\tilde{p}(z))$, then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{4(3\lambda-1)}. \tag{46}$$

By specialising the parameter $\lambda = 1$ and $\lambda = 2$ in the above theorem, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_{\Sigma}(\tilde{p}(z))$ and $\mathcal{GSL}_{\Sigma}(\tilde{p}(z))$, respectively.

Corollary 3.3 ([7]). *Let f given by (1) be in the class $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mu \in \mathbb{R}$,. then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{8}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8}, \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[1-2\tau\right]}. \tag{47}$$

Further if $\mu = 1$ we get

$$|a_3 - a_2^2| \leq \frac{|\tau|}{8}.$$

Corollary 3.4. *Let f given by (1) be in the class $\mathcal{GSL}_\Sigma(\tilde{p}(z))$ and $\mu \in \mathbb{R}$, then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{20}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{20}, \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{20}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[9-21\tau\right]}. \tag{48}$$

Further if $\mu = 1$ we get

$$|a_3 - a_2^2| \leq \frac{|\tau|}{20}.$$

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