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Almost Kenmotsu Manifolds

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Abstract. The object of this paper is to study generalized ϕ-recurrent almost Kenmotsu manifolds with characteristic vector field ξ belonging to $(k,\mu)'$-nullity distribution. We have showed that these manifolds reduce to Kenmotsu manifolds with scalar curvature -1. Further we establish the relations among the associated 1-forms and proved the conditions under which gradient Ricci almost soliton reduce to gradient Ricci soliton.

1 Introduction

Dileo and Pastore [7] introduced the notion of $(k,\mu)'$-nullity distribution and established some classification results on almost Kenmotsu manifolds [17] with characteristic vector field ξ belonging to (k,μ)-nullity distribution. As a weaker version of local symmetry Takahashi [16] introduced the idea of local ϕ-symmetry on a Sasakian manifold and study extended to the locally ϕ-symmetric β-Kenmotsu manifolds by Shaikh and Hui [13]. As a weaker version of local ϕ symmetry, Dubey [8] introduced generalized recurrent manifolds and this notion with generalized Ricci recurrent manifolds has been extensively studied by De and Guha [3], [2]. Generalizing the idea of local ϕ-symmetry, De et al. [4] introduced the notion of ϕ-recurrent Sasakian manifolds and De et al. [5] extended the study to ϕ-recurrent Kenmotsu manifolds. The investigation of almost Ricci solitons was presented by Pigola et al. [11] by including the condition the parameter λ to be a variable and changed the meaning of Ricci solitons.

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Motivated by above studies, in this paper, we study generalized ϕ-recurrent almost Kenmotsu manifolds with characteristic vector field ξ belonging to $(k,\mu)'$-nullity distribution. In section 3, we proved the relations between associated 1-forms and found eigen value and the corresponding eigenvector of Ricci operator Q. Also we consider generalized concircularly ϕ-recurrent and generalized projective ϕ-recurrent almost Kenmotsu manifolds. In section 4, we show that almost gradient Ricci solitons in a generalized concircularly ϕ-recurrent and generalized projective ϕ-recurrent almost Kenmotsu manifolds reduce to gradient Ricci solitons.

2 Preliminaries

An almost contact manifold is an n-dimensional smooth manifold M endowed with a $(1,1)$-tensor field ϕ, a global vector field ξ and a one-form η on M such that

$$\phi^2 = -Id + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{1}$$

In this case, such a manifold will be denoted by (M, ϕ, ξ, η). A Riemannian metric g on an almost contact manifold M is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for any vector fields X, Y on M. An almost contact manifold (M, ϕ, ξ, η) with a compatible Riemannian metric g is called an almost contact metric manifold and will be denoted by (M, ϕ, ξ, η, g). With this structure, one can associate to any vector fields X and Y, a 2-form Φ such that $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be an almost Kenmotsu manifold if the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$. It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The normality of almost Kenmotsu manifold is expressed by [9]

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{3}$$

for any vector fields X, Y on M.

In [7], the authors introduced the idea of $(k,\mu)'$ - nullity distribution on an almost Kenmotsu manifold (M, ϕ, ξ, η, g), which is defined for any $p \in M$ and $k, \mu \in R$ as follows:

$$\begin{aligned} N_p(k,\mu)' =& \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \\ & + \mu(g(Y,Z)h'X - g(X,Z)h'Y)\}, \end{aligned} \tag{4}$$

where $h' = h \circ \phi$ and $h = \frac{1}{2} L_\xi \phi$ satisfying

$$h\xi = 0, \quad trh = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{5}$$

In an almost Kenmotsu manifold with the characteristic vector field ξ belonging to $(k,\mu)'$-nullity distribution, the following hold:

$$h^{'}\xi = 0, \qquad h^{'2} = (\kappa+1)\phi^2, \tag{6}$$
$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{7}$$
$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{8}$$

where $l = R(\cdot,\xi)\xi$.

From equation (4), we obtain

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h^{'}X - \eta(X)h^{'}Y]. \tag{9}$$

$$S(Y,Z) = (n-1)kg(Y,Z) - \mu g(h'Y,Z). \tag{10}$$

$$QY = (n-1)kY - \mu h^{'}Y. \tag{11}$$

$$r = n(n-1)k. \tag{12}$$

Also we have

$$(\nabla_X h^{'})Y = -g(h^{'}X + h^{'2}X, Y)\xi - \eta(Y)(h^{'}X + h^{'2}X) - (\mu+2)\eta(X)h^{'}Y. \tag{13}$$

A Riemannian manifold (M,g) is called generalized recurrent [3] if its curvature tensor R of type (1,3) satisfies

$$\nabla R = A \otimes R + B \otimes G, \tag{14}$$

and (M,g) is called a generalized Ricci-recurrent manifold [2] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\nabla S = A \otimes S + B \otimes S, \tag{15}$$

where A and B are non-vanishing 1-forms defined by $A(\cdot) = g(\cdot,\rho_1)$, $B(\cdot) = g(\cdot,\rho_2)$, where ρ_1 and ρ_2 are unit vector fields.

Specially, if the 1-form B vanishes, then (14) turns into the notion of recurrent manifold introduced by Walker [18] and (15) reduced to the notion of Ricci-recurrent manifold introduced by Patterson [10].

A Riemannian manifold (M,g) is called a super generalized Ricci-recurrent manifold [12] if its Ricci tensor S of type (0,2) satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \eta \otimes \eta, \tag{16}$$

where α, β, and γ are non-vanishing unique 1-forms. In particular, if $\beta = \gamma$, then (16) reduces to the notion of quasi-generalized Ricci-recurrent manifold introduced by Shaikh and Roy [14].

A Riemannian manifold (M,g) is said to be an almost Ricci soliton, if there exist a complete vector field X and a smooth soliton function λ: $M \to R$ satisfying

$$S(X,Y) + \frac{1}{2}\mathscr{L}_X g(X,Y) = \lambda g(X,Y), \tag{17}$$

where $\mathscr{L}_X$ is the Lie derivative with respect to vector field X. An almost Ricci soliton (M, g, X, λ) will be called expanding, steady or shrinking, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. If the vector field X is gradient of a smooth function $f\colon M \to R$, the manifold will be called a gradient almost Ricci soliton. In this case, the preceding equation becomes

$$S + \nabla^2 f = \lambda g, \tag{18}$$

where $\nabla^2 f$ represents the Hessian of f.

3 Generalized ϕ-recurrent Almost Kenmotsu manifolds

Definition 3.1. An almost Kenmotsu manifold is said to be generalized ϕ-recurrent if it satisfies the relation

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)(R(X,Y)Z) + B(W)(G(X,Y)Z), \tag{19}$$

for all $X, Y, Z, W \in \chi(M)$, where $A(\cdot) = g(\cdot, \rho_1)$ and $B(\cdot) = g(\cdot, \rho_2)$ are non-vanishing 1-forms, ρ_1 and ρ_2 are unit vector fields, and the tensor G of type (1,3) is given by

$$G(X,Y)Z = (X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{20}$$

for all $X, Y, Z \in \chi(M); \chi(M)$ being the lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-forms A and B are called the associated 1-forms of the manifolds.

Proposition 3.2. *A generalized ϕ-recurrent almost Kenmotsu manifold M with ξ belonging to $(k,\mu)'$ nullity distribution, the vector fields of associated 1-forms are co-directional. Further in M the following are equivalent:*

(i) $\phi^2((\nabla_{\rho_1} R)(X,Y)Z) = 0$

(ii) M is of constant sectional curvature k.

Proof. We suppose that the manifold M is a generalized ϕ-recurrent almost Kenmotsu manifold.

Differentiating (9) covariantly with respect to W, we obtain

$$\begin{aligned}(\nabla_W R)(\xi,Y)Z = {} & k[\eta(Z)\eta(W)Y - g(Z,h'W)Y + \eta(\nabla_W Z)h'Y] \\ & + \mu[g(Z,h'W)h'Y - \eta(Z)\eta(W)h'Y + \eta(Z)(\nabla_W h')(Y) - \eta(\nabla_W Z)h'Y].\end{aligned} \tag{21}$$

By virtue of (1) and (19), we get

$$\begin{aligned}-(\nabla_W R)(X,Y)Z & + \eta((\nabla_W R)(X,Y)Z)\xi \\ & = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{22}$$

Setting $X = \xi$ in (22), we have

$$-(\nabla_W R)(\xi, Y)Z + \eta((\nabla_W R)(\xi, Y)Z)\xi \\ = A(W)R(\xi, Y)Z + B(W)[g(Y, Z)\xi - g(\xi, Z)Y]. \tag{23}$$

Using (4), (21) in (23), we obtain

$$k\{g(Z, h^{'}W)Y - \eta(\nabla_W Z)Y - \eta(Z)\eta(W)Y\} + \mu\{\eta(Z)(\nabla_W h^{'})Y - \eta(Z)\eta(W)h^{'}Y \\ + g(Z, h^{'}W)h^{'}Y - \eta(\nabla_W Z)h^{'}Y\} + k\{\eta(Z)\eta(W)\eta(Y)\xi - g(Z, h^{'}W)\eta(Y)\xi \\ + \eta(\nabla_W Z)\eta(Y)\xi\} - \mu\{\eta(Z)\eta(\nabla_W h^{'})\eta(Y)\xi\} \\ = A(W)\big[k\{g(Y, Z)\xi - \eta(Z)Y\} - \mu\{\eta(Z)h^{'}Y\}\big] + B(W)\big[g(Y, Z)\xi - \eta(Z)Y\big]. \tag{24}$$

For any vector fields Y, Z orthogonal to ξ, (24) takes the form

$$k[g(Z, h^{'}W)Y - \eta(\nabla_W Z)Y] + \mu[g(Z, h^{'}W)h^{'}Y - \eta(\nabla_W Z)h^{'}Y] \\ = A(W)\{kg(Y, Z)\xi\} + B(W)g(Y, Z)\xi. \tag{25}$$

Contracting the above equation with ξ , we obtain

$$B(W) = -kA(W). \tag{26}$$

Putting (26) and (20) in (19), we obtain

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)\Big[R(X, Y)Z - k\{g(Y, Z)X - g(X, Z)Y\}\Big]. \tag{27}$$

Taking $W = \rho_1$ in (27), we get

$$\phi^2((\nabla_{\rho_1} R)(X, Y)Z) = R(X, Y)Z - k\{g(Y, Z)X - g(X, Z)Y\}. \tag{28}$$

The result follows from (28). □

We have the following result due to Dileo and Pastore [6]:

Theorem A. *An almost Kenmotsu manifold of constant sectional curvature K is a Kenmotsu manifold and $K = -1$.*

From Theorem A and Proposition 3.2, we have the following:

Corollary 3.3. *If a generalized ϕ-recurrent almost Kenmotsu manifold M with ξ belonging to $(k, \mu)^{'}$ nullity distribution satisfies $\phi^2((\nabla_{\rho_1} R)(X, Y)Z) = 0$ then it reduces to a Kenmotsu manifold, where ρ_1 is a vector field of associated 1-form of M. In this case sectional curvature $K = k = -1$.*

Theorem 3.4. *In a generalized ϕ-recurrent almost Kenmotsu manifold (M, g), the characteristic vector field ξ and the vector field $\rho_1 k + \rho_2$ associated to the 1-form $Ak + B$ are co-directional.*

Proof. Taking inner product of (22) with ξ and using first Bianchi's identity, we get

$$\begin{aligned} &A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\ &+ B(W)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)] + B(X)[\eta(Y)g(W,Z) \\ &- \eta(W)g(Y,Z)] + B(Y)[\eta(W)g(X,Z) - \eta(X)g(W,Z)] = 0. \end{aligned} \tag{29}$$

Using (4) in (29), we obtain

$$\begin{aligned} &(A(W)k + B(W))[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ (A(X)k + B(X))[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\ &+ (A(Y)k + B(Y))[g(X,Z)\eta(W) - g(W,Z)\eta(X)] = 0, \end{aligned} \tag{30}$$

for any vector fields X, Y, Z, W. Let $\{e_i, i = 1, 2, 3....n\}$ be a local orthonormal basis of tangent space at each point of the manifold M. Setting $Y = U = e_i$ in the above equation and taking summation over $i : 1 \leq i \leq n$, we get

$$[A(W)k + B(W)]\eta(X) = [A(X)k + B(X)]\eta(W). \tag{31}$$

Putting $X = \xi$ in (31), we obtain

$$[A(W)k + B(W)] = [\eta(\rho_1)k + \eta(\rho_2)]\eta(W), \tag{32}$$

for any vector field W. The result follows from equations (31) and (32). □

Theorem 3.5. *In a generalized ϕ-recurrent almost Kenmotsu manifold ρ_1 is an eigen vector of the ricci operator Q corresponding to the eigen value $\left(\frac{-(n-1)}{2}\right)$.*

Proof. Taking cyclic sum of (22) in W, Y, X and then by virtue of Bianchi's second identity we have

$$\begin{aligned} A(X)R(Y,W)Z + A(Y)R(W,X)Z &+ A(W)R(X,Y)Z \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y] \\ &+ B(X)[g(W,Z)Y - g(Y,Z)W] \\ &+ B(Y)[g(X,Z)W - g(W,Z)X] = 0. \end{aligned} \tag{33}$$

Contracting (33) with U, we obtain

$$\begin{aligned} A(X)g(R(Y,W)Z,U) + A(Y)g(R(W,X)Z,U) &+ A(W)g(R(X,Y)Z,U) \\ &+ B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\ &+ B(X)[g(W,Z)g(Y,U) - g(Y,Z)g(W,U)] \\ &+ B(Y)[g(X,Z)g(W,U) - g(W,Z)g(X,U)] = 0. \end{aligned} \tag{34}$$

Setting $Y = U = e_i$ in (34), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i = 1, 2, 3...n$, we get

$$\begin{aligned} &A(X)S(W,Z) + A(e_i)g(R(W,X)Z,e_i) + A(W)g(R(X,e_i)Z,e_i) \\ &+ B(W)[g(X,Z)(1-n)] + B(X)[(n-1)g(W,Z)] \\ &+ B(e_i)[g(X,Z)g(W,e_i) - g(W,Z)g(X,e_i)] = 0. \end{aligned} \tag{35}$$

Again taking $Z = X = e_i$ in (35) and then taking summation over i, $1 \leq i \leq n$, we get

$$S(W, \rho_1) = \frac{(n-1)^2}{2} B(W) + \frac{r}{2} A(W). \tag{36}$$

Using (26), (12) in (36), we obtain

$$Q\rho_1 = \Big(\frac{-(n-1)}{2}\Big)\rho_1.$$

□

In an almost Kenmotsu manifold for any (1,3) tensor K, we define

$$\phi^2((\nabla_W K)(X, Y)Z) = A(W)(K(X, Y)Z) + B(W)(G(X, Y)Z), \tag{37}$$

for all $X, Y, Z, W \in \chi(M)$, where A and B are defined as in (19).

The concircular curvature and projective curvature tensors of type (1,3) are respectively given by

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)} G(X, Y)Z, \tag{38}$$

and

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}(S(Y, Z)X - S(X, Z)Y), \tag{39}$$

for all X, Y, $Z \in \chi(M)$.

Theorem 3.6. *A generalized concircularly ϕ-recurrent almost Kenmotsu manifold M is a super generalized Ricci recurrent manifold.*

Proof. If M is an almost Kenmotsu manifold where (37) holds, then for a concircular curvature tensor $\tilde{C}$, we have

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)(\tilde{C}(X, Y)Z) + B(W)(G(X, Y)Z), \tag{40}$$

where A and B are defined as in (19).

Using (1) in (38), we get

$$\begin{aligned} -(\nabla_W \tilde{C})(X, Y)Z + \eta((\nabla_W \tilde{C})(X, Y)Z)\xi = {} & A(W)(\tilde{C}(X, Y)Z) \\ & + B(W)[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{41}$$

Contracting with U in (41), we obtain

$$\begin{aligned} & - g((\nabla_W \tilde{C})(X, Y)Z, U) + \eta((\nabla_W \tilde{C})(X, Y)Z)\eta(U) \\ & = A(W)\big[g(\tilde{C}(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \end{aligned} \tag{42}$$

Setting $X = U = e_i$ in (42), and using (38), we get

$$\begin{aligned} (\nabla_W S)(Y, Z) = -A(W)S(Y, Z) + {} & \left[\frac{(Wr) - n(n-1)^2 B(W)}{n(n-1)}\right] g(Y, Z) \\ & - \left[\frac{(Wr)}{n(n-1)}\right] \eta(Y)\eta(Z). \end{aligned} \tag{43}$$

Above equation can be written as

$$\nabla S = A_1 \otimes S + \psi \otimes g + H \otimes \eta \otimes \eta, \tag{44}$$

where

$$A_1(W) = -A(W),$$
$$\psi(W) = \frac{(Wr) - n(n-1)^2 B(W)}{n(n-1)},$$
$$H(W) = -\left[\frac{(Wr)}{n(n-1)}\right]. \qquad \square$$

Theorem 3.7. *In a generalized projectively ϕ-recurrent almost Kenmotsu manifold M with constant scalar curvature we have*

$$B(W) = \frac{-r}{n(n-1)} A(W).$$

Proof. We assume that an almost Kenmotsu manifold M is generalized projectively ϕ-recurrent. Then from (37), we have

$$\phi^2((\nabla_W P)(X,Y)Z) = A(W)(P(X,Y)Z) + B(W)(G(X,Y)Z), \tag{45}$$

where A and B are defined as in (19).

By virtue of (1), it follows from (45), that

$$\begin{aligned} -(\nabla_W P)(X,Y)Z + \eta((\nabla_W P)(X,Y)Z)\xi = A(W)(P(X,Y)Z) \\ + B(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{46}$$

Taking inner product of (46) with U, we obtain

$$\begin{aligned} -g((\nabla_W P)(X,Y)Z,U) + \eta((\nabla_W P)(X,Y)Z)\eta(U) = A(W)(g(P(X,Y)Z,U)) \\ + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{47}$$

Setting $X = U = e_i$ in (47) and summing over $i = 1, 2, 3....n$, we get

$$(\nabla_W S)(Y,Z) = -A(W)S(Y,Z) - [(n-1)B(W)]g(Y,Z). \tag{48}$$

Letting $Y = Z = e_i$ in (48), we obtain

$$(Wr) = -\big(rA(W) + n(n-1)B(W)\big). \qquad \square$$

4 Gradient Almost Ricci Solitons on Almost Kenmotsu manifolds

In this section we consider gradient almost Ricci soliton (M, g, X, λ). i.e. an almost Ricci soliton (M, g, X, λ) when the vector field X is the gradient of a smooth function $f \in C^\infty(M)$. Accordingly equation (18) becomes

$$\nabla_Y Df = QY + \lambda Y, \tag{49}$$

where D is the gradient operator of g and Q is Ricci operator.

Theorem 4.1. *Let M be an almost Kenmotsu manifold admitting gradient almost Ricci soliton. If M is generalized concircularly ϕ-recurrent or generalized projectively ϕ-recurrent then the following are equivalent:*

(i) $X\lambda = (\xi\lambda)\eta(X)$ for any vector field X.

(ii) f is constant along $k\phi^2 X - \mu h' X$.

Proof. By virtue of (49), we obtain

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X - (Y\lambda)X + (X\lambda)Y, \tag{50}$$

Case (i): Suppose M is generalized concircularly ϕ-recurrent. From (43), we get

$$\begin{aligned} (\nabla_Y Q)X = A(Y)QX + \Big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)A(Y)} + (n-2)B(Y)\Big]X \\ - \Big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)}A(Y) - B(Y)\Big]\eta(Y)\xi. \end{aligned} \tag{51}$$

Substituting (51) in (50), we get

$$\begin{aligned} R(X, Y)Df = {} & A(Y)QX - A(X)QY \\ & + \big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(Y) + (n-2)B(Y)\big]X \\ & - \big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(Y) - B(Y)\big]\eta(X)\xi \\ & - \big[\frac{(Xr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(X) + (n-2)B(X)\big]Y \\ & + \big[\frac{(Xr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(X) - B(X)\big]\eta(Y)\xi \\ & - (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{52}$$

Taking inner product with ξ in (52), we obtain

$$\begin{aligned} g(R(X,Y)Df,\xi) &= A(Y)g(X,Q\xi) - A(X)g(Y,Q\xi) \\ &+ \big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(Y) + (n-2)B(Y)\big]\eta(X) \\ &- \big[\frac{(Yr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(Y) - B(Y)\big]\eta(X) \\ &- \big[\frac{(Xr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(X) + (n-2)B(X)\big]\eta(Y) \\ &+ \big[\frac{(Xr)}{n(n-1)} - (k - \frac{r}{n(n-1)})A(X) - B(X)\big]\eta(Y) \\ &- (Y\lambda)\eta(X) + (X\lambda)\eta(Y). \end{aligned} \tag{53}$$

Interchange Df by ξ and using (10) in (53), we get

$$g(R(X,Y)\xi, Df) = (Y\lambda)X - (X\lambda)Y. \tag{54}$$

Using (9), (11) in (54), we obtain

$$\begin{aligned} &k\big[g(X,Df)\eta(Y) - g(Y,Df)\eta(X)\big] + \mu\Big[g(h'X,Df)\eta(Y) - g(h'Y,Df)\eta(X)\Big] \\ &= (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \end{aligned} \tag{55}$$

Putting $Y = \xi$ in (55), we get

$$k\big[g(X,Df) - \eta(X)\eta(Df)\big] + \mu\big[g(h'X,Df)\big] = (\xi\lambda)\eta(X) - (X\lambda). \tag{56}$$

The above equation may be rewritten in the form

$$(k\phi^2 X - \mu h'X)f = X\lambda - (\xi\lambda)\eta(X). \tag{57}$$

Case (ii): Suppose M is generalized projectively ϕ-recurrent. Then from (48), we have

$$\begin{aligned} (\nabla_Y Q)X = A(Y)QX &- \big[A(Y)k(\frac{2n-3}{n-1}) + (n-2)B(Y)\big]X \\ &+ \big[A(Y)k(\frac{2n-3}{n-1}) + B(Y)\big]\eta(X)\xi \\ &- (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{58}$$

Using (58) in (50) and proceeding as in case (i), we obtain (57). This completes the proof. □

Remark 4.2. In view of Theorem 4.1, we note that gradient almost Ricci soliton on a generalized concircularly ϕ-recurrent (or generalized projectively ϕ-recurrent) almost Kenmotsu manifold M reduces to gradient Ricci soliton if f is constant along $k\phi^2 X - \mu h'X$.

References

[1] T.Q. Binh, U.C. De, L. Tamassy: On partially pseudo symmetric K-contact Riemannian manifolds. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* 18 (1) (2002) 19–25.

[2] U.C. De, N. Guha, D. Kamilya: On generalized Ricci-recurrent manifolds. *Tensor (N.S.)* 56 (3) (1995) 312–317.

[3] U.C. De, A. Konar, N. Guha: On generalised projective 2-recurrent Riemannian manifold. *J. Nat. Acad. Math. India* 9 (1) (1991) 1–4.

[4] U.C. De, A.A. Shaikh, S. Biswas: On ϕ-recurrent Sasakian manifolds. *Novi Sad J. Math.* 33 (2) (2003) 13–48.

[5] U.C. De, A. Yıldız, A.F. Yalınız: On ϕ-recurrent Kenmotsu manifolds. *Turkish J. Math.* 33 (1) (2009) 17–25.

[6] G. Dileo, A.M. Pastore: Almost Kenmotsu manifolds and local symmetry. *Bull. Belg. Math. Soc. Simon Stevin* 14 (2) (2007) 343–354.

[7] G. Dileo, A.M. Pastore: Almost Kenmotsu manifolds and nullity distributions. *J. Geom.* 93 (1-2) (2009) 46–61.

[8] R.S.D. Dubey: Generalized recurrent spaces. *Indian J. Pure Appl. Math.* 10 (12) (1979) 1508–1513.

[9] K. Kenmotsu: A class of almost contact Riemannian manifolds. *Tohoku Math. J. (2)* 24 (1972) 93–103.

[10] E.M. Patterson: Some theorems on Ricci-recurrent spaces. *J. London Math. Soc.* 27 (1952) 287–295.

[11] S. Pigola, M. Rigoli, M. Rimoldi, A.G. Setti: Ricci almost solitons. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5)* 10 (4) (2011) 757–799.

[12] A.A. Shaikh, H. Ahmad: On generalized ϕ-recurrent Sasakian manifolds. *Appl. Math.* 2 (11) (2011) 1317–1322.

[13] A.A. Shaikh, S.K. Hui: On locally ϕ-symmetric β-Kenmotsu manifolds. *Extr. Math.* 24 (3) (2009) 301–316.

[14] A.A. Shaikh, I. Roy: On quasi-generalized recurrent manifolds. *Math. Pannon.* 21 (2) (2010) 251–263.

[15] Z.I. Szabo: Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$. I. The local version. *J. Differential Geometry* 17 (4) (1983) 531–582.

[16] T. Takahashi: Sasakian ϕ-symmetric spaces. *Tohoku Math. J. (2)* 29 (1) (1977) 91–113.

[17] L. Vanhecke, D. Janssens: Almost contact structures and curvature tensors. *Kodai Math. J.* 4 (1) (1981) 1–27.

[18] A.G. Walker: On Ruse's spaces of recurrent curvature. *Proc. London Math. Soc. (2)* 52 (1950) 36–64.

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