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# Gradient estimates for a nonlinear elliptic equation on a smooth metric measure space

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**Abstract.** Let $(M, g, e^{-f}dv)$ be a smooth metric measure space. We consider local gradient estimates for positive solutions to the following elliptic equation

$$\Delta_f u + au \log u + bu = 0$$

where $a, b$ are two real constants and $f$ be a smooth function defined on $M$. As an application, we obtain a Liouville type result for such equation in the case $a < 0$ under the $m$-dimensions Bakry-Émery Ricci curvature.

In this paper, we study the local gradient estimate for the positive solution to the following weighted nonlinear elliptic equation

$$\Delta_f u + au \log u + bu = 0 \tag{1}$$

on a smooth metric measure space $(M, g, e^{-f}dv)$, where $a, b$ are two real constants and $f$ be a smooth function defined on $M$. The motivation to study (1) comes from understanding the Ricci flow. Moreover, the (1) is closely related to the famous Gross Logarithmic Sobolev inequality, see [2]. It is well known that Yau has proved in [14] that every positive or bounded harmonic function is constant if $M$ has nonnegative Ricci curvature by establishing gradient estimates for the solutions to Laplacian equation, see also [6], [8], [11].

It is natural to consider similar Liouville type results for positive solutions to the nonlinear elliptic equation (1). In [10], Qian considered positive solutions to

$$\Delta u + au \log u = 0 \tag{2}$$

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and proved the following

**Theorem 0.1.** *(B. Qian) Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with the Ricci curvature $Ric(B(x,R)) \geq -K$, where $K \geq 0$ is a constant. Let $u$ be a positive solution to (2) on $B(x,R)$, then for any $\alpha > 0$,*

$$\sup_{y\in B(x,\frac{R}{2})} \frac{|\nabla u|}{u} \leq C(n)\sqrt{(1+\alpha)((a+K)+\frac{(1+\alpha)}{R^2})+\sqrt{\frac{1+\alpha}{\alpha}}|a|L(x,R)}, \tag{3}$$

*where $L(x,R) = sup_{y\in B(x,R)}|\log u| < \infty$ and $C(n)$ is a constant depending only on the dimension $n$.*

*In particular, by letting $R \to \infty$ in (3), we obtain the following gradient estimates on complete non-compact Riemannian manifolds:*

$$\frac{|\nabla u|}{u} \leq C(n)\sqrt{(1+\alpha)(a+K)+\sqrt{\frac{1+\alpha}{\alpha}}|a|L}, \tag{4}$$

*where $L = sup_M|\log u|$.*

**Remark 0.2.** Clearly, from 4, it is easy to see that if $a + K < 0$ and $a < 0$, then any bounded positive solution to (2) must be a constant $u \equiv 1$. On the other hand, in [5], Huang and Ma also obtained the similar Liouville type result by a different method.

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold and $f$ be a smooth function defined on $M$. In general, the triple $(M, g, e^{-f}dv)$ is called a smooth metric measure space. The $f$-Laplacian operator is defined by

$$\Delta_f = e^f div(e^{-f}\nabla) = \Delta - \nabla f\nabla,$$

which is symmetric in $L^2(M, g, e^{-f}dv)$.

It is well-known that the $m$-dimensions Bakry-Émery Ricci curvature associated with the $f$-Laplacian is defined by (see [3], [4], [13] )

$$Ric_f^m = Ric + \nabla^2 f - \frac{df\otimes df}{m-n},$$

where $m \geq n$ is a constant and $m = n$ if and only if $f$ is a constant. Define

$$Ric_f = Ric + \nabla^2 f.$$

Then $Ric_f$ can be seen as $\infty$-dimensions Bakry-Émery Ricci curvature. Recently, the $\infty$-dimensions Bakry-Émery Ricci curvature has become an important object of study in Riemannian geometry. The equation $\mathrm{Ric}_f = \rho\langle,\rangle$ for some constant $\rho$ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow (see [1]).

The aim of this paper is to generalize the results of Qian in [10] to the weighted nonlinear elliptic equation (1) under the assumption that the $m$-dimensions Bakry-Émery Ricci curvature is bounded from blow. Our main results are as follows:

**Theorem 0.3.** *Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space with $m$-dimensions Bakry-Émery Ricci curvature $Ric_f^m(B(x,R)) \geq -K$ for some constant $K \geq 0$. Let $u$ be a positive solution to the nonlinear equation* (1) *on $B(x,R)$, then there exists a constant $C = C(m)$, such that*

$$\sup_{y\in B(x,\frac{R}{2})} \frac{|\nabla u|}{u} \leq C\sqrt{a + K + \frac{1}{R^2} + |b| + |a|L}. \tag{5}$$

*where $L(x,R) := \sup_{y\in B(x,R)} |\log u| < \infty$.*

*In particular, by letting $R \to \infty$ in* (5)*, we obtain the following gradient estimates on complete non-compact Riemannian manifolds:*

$$\frac{|\nabla u|}{u} \leq C\sqrt{a + K + |b| + |a|L}, \tag{6}$$

*where $L = sup_M |\log u|$.*

From (6), it is easy to obtain the following results:

**Corollary 0.4.** *Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space with $Ric_f^m \geq -K$. If $u$ is a bounded positive solution to* (1) *with $a + K < 0$ and $a < 0$, then $u \equiv 1$. Furthermore, if $Ric_f^m \geq 0$ and $a \leq 0$, then any bounded positive solution to* (1) *must be $u \equiv 1$.*

**Remark 0.5.** When $m = n$ in Theorem 0.3, we have $Ric_f^m = Ric$ and $\Delta_f = \Delta$. Hence, in this case, our Theorem 0.3 becomes Theorem 0.1 of Qian. That is, our results of this paper generalize those of Qian in [10].

## 1 Proof of Theorem 0.3

**Lemma 1.1.** *Let $u$ be a bounded positive solution to the nonlinear equation* (1)*, then we have,*

$$\begin{aligned} |\nabla u|\Delta_f|\nabla u| \geq &\frac{|\nabla(|\nabla u|)|^2}{m} - (au\log u + bu)^2 + Ric_f^m(\nabla u, \nabla u) \\ &- (a + a\log u + b)|\nabla u|^2. \end{aligned} \tag{7}$$

*Proof.* Since we have the Bochner-weitzenböck formula with respect to $f$-Laplacian, for any $u \in C^2(M)$, we have

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + (\nabla u, \nabla\Delta_f u).$$

On the other hand,

$$\Delta_f|\nabla u|^2 = 2|\nabla u|\Delta_f|\nabla u| + 2|\nabla(|\nabla u|)|^2,$$

hence

$$|\nabla u|\Delta_f|\nabla u| = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + (\nabla u, \nabla\Delta_f u) - |\nabla(|\nabla u|)|^2.$$

Since $u$ is a solution to (1), we obtain

$$|\nabla u|\Delta_f|\nabla u| = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) - (a + a\log u + b)|\nabla u|^2 - |\nabla(|\nabla u|)|^2.$$

If we consider a local normal chart at $x$ in which $u_1(x) = |\nabla u|(x)$ and $u_j(x) = 0$ for $j \geq 2$, then $\nabla_i(|\nabla u|) = u_{1i}$, hence $|\nabla(|\nabla u|)|^2 = \Sigma_i u_{1i}^2$. Since $u$ is a solution to (1), in the above local chart we have at $x$

$$\sum_{i\geq 2} u_{ii} = -u_{11} - au\log u - bu + \nabla f \cdot \nabla u.$$

Therefore,

$$\begin{aligned}
|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 =& \sum_{i\geq 1, j\geq 1} u_{ij}^2 - \sum_{j\geq 1} u_{1j}^2 = \sum_{i\geq 2, j\geq 1} u_{ij}^2 \geq \sum_{i\geq 2} u_{i1}^2 + \sum_{i\geq 2} u_{ii}^2 \\
\geq& \sum_{i\geq 2} u_{i1}^2 + \frac{1}{n-1}(\sum_{i\geq 2} u_{ii})^2 \\
=& \sum_{i\geq 2} u_{i1}^2 + \frac{1}{n-1}(-u_{11} - au\log u - bu + \nabla f \cdot \nabla u)^2 \\
\geq& \frac{1}{(n-1)(1+\alpha)} \sum_{i\geq 1} u_{i1}^2 - \frac{1}{(n-1)\alpha}(au\log u + bu - \nabla f\cdot\nabla u)^2 \\
\geq& -\frac{1}{(n-1)\alpha}[(1+\frac{1}{\epsilon})(au\log u + bu)^2 + (1+\epsilon)(\nabla f\cdot \nabla u)^2] \\
&+ \frac{1}{(n-1)(1+\alpha)} \sum_{i\geq 1} u_{i1}^2 \\
=& \frac{1}{m} \sum_{i\geq 1} u_{i1}^2 - (au\log u + bu)^2 - \frac{(\nabla f\cdot\nabla u)^2}{m-n},
\end{aligned}$$

where we use the elementary inequalities (see [12]):

$$(a+b)^2 \geq \frac{1}{1+\alpha}a^2 - \frac{1}{\alpha}b^2 \quad \text{and} \quad (a+b)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})b^2,$$

which holds for any $\alpha > 0$, $\epsilon > 0$. The last equality we choose $\alpha = \frac{m-n+1}{n-1}$ and $\epsilon = \frac{1}{m-n}$.

Hence

$$\begin{aligned}
|\nabla u|\Delta_f|\nabla u| \geq& \frac{|\nabla(|\nabla u|)|^2}{m} - (au\log u + bu)^2 - \frac{(\nabla f\cdot\nabla u)^2}{m-n} \\
&+ Ric_f(\nabla u, \nabla u) - (a + a\log u + b)|\nabla u|^2 \\
=& \frac{|\nabla(|\nabla u|)|^2}{m} - (au\log u + bu)^2 + (Ric_f - \frac{df\otimes df}{m-n})u_i u_j \\
&- (a + a\log u + b)|\nabla u|^2 \\
=& \frac{|\nabla(|\nabla u|)|^2}{m} - (au\log u + bu)^2 + Ric_f^m(\nabla u, \nabla u) \\
&- (a + a\log u + b)|\nabla u|^2.
\end{aligned} \tag{8}$$

This completes the prove of Lemma 1.1. □

*Proof of Theorem 0.3.* Denote $\psi = |\nabla \log u| = \frac{|\nabla u|}{u}$, i.e.,

$$|\nabla u| = \psi u.$$

Direct computation gives

$$\nabla\psi = \frac{\nabla(|\nabla u|)}{u} - \frac{|\nabla u|\nabla u}{u^2}. \tag{9}$$

At the point where $\nabla u \neq 0$, we have

$$\begin{aligned}\Delta_f|\nabla u| &= \Delta_f(\psi u) = u\Delta_f\psi + \psi\Delta_f u + 2\nabla\psi\cdot\nabla u\\ &= u\Delta_f\psi - \psi(au\log u + bu) + 2\nabla\psi\cdot\nabla u.\end{aligned}$$

This yields

$$\begin{aligned}\Delta_f\psi &= \frac{\Delta_f|\nabla u|}{u} + \psi(a\log u + b) - 2\frac{\nabla\psi\cdot\nabla u}{u}\\ &= \frac{|\nabla u|\Delta_f|\nabla u|}{|\nabla u|u} + \psi(a\log u + b) - 2\frac{\nabla\psi\cdot\nabla u}{u}.\end{aligned}$$

By Lemma 1.1, we can derive,

$$\begin{aligned}\Delta_f\psi \geq& \frac{\frac{|\nabla(|\nabla u|)|^2}{m} - (au\log u + bu)^2 + Ric_f^m(\nabla u, \nabla u) - (a + a\log u + b)|\nabla u|^2}{|\nabla u|u}\\ &+ \psi(a\log u + b) - 2\frac{\nabla\psi\cdot\nabla u}{u}\\ \geq& \frac{|\nabla(|\nabla u|)|^2}{m|\nabla u|u} - \frac{(a\log u + b)^2}{\psi} - (a+K)\psi - 2\frac{\nabla\psi\cdot\nabla u}{u}.\end{aligned} \tag{10}$$

For any $\delta > 0$, by (9)

$$\begin{aligned}2\frac{\nabla\psi\cdot\nabla u}{u} &= (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \delta\frac{\nabla\psi\cdot\nabla u}{u}\\ &= (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \delta\frac{\nabla u}{u}\Big(\frac{\nabla(|\nabla u|)}{u} - \frac{|\nabla u|\nabla u}{u^2}\Big)\\ &= (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \delta\frac{\nabla(|\nabla u|)\cdot\nabla u}{u^2} - \delta\psi^3\\ &\leq (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \delta\frac{|\nabla(|\nabla u|)|\cdot|\nabla u|}{u^2} - \delta\psi^3\\ &\leq (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \frac{\delta}{2}\Big(\frac{|\nabla(|\nabla u|)|^2}{|\nabla u|u} + \frac{|\nabla u|^3}{u^3}\Big) - \delta\psi^3\\ &= (2-\delta)\frac{\nabla\psi\cdot\nabla u}{u} + \frac{\delta}{2}\frac{|\nabla(|\nabla u|)|^2}{|\nabla u|u} - \frac{\delta}{2}\psi^3.\end{aligned}$$

Choosing $\delta = \frac{2}{m}$ and substituting into (10), we obtain

$$\begin{aligned}\Delta_f \psi \geq & -(a+K)\psi - (2-\frac{2}{m})\frac{\nabla\psi\cdot\nabla u}{u} + \frac{\psi^3}{m} \\ & - \frac{(a\log u + b)^2}{\psi}.\end{aligned} \tag{11}$$

Now we define

$$F(y) := (R^2 - d^2(x,y))\frac{|\nabla u|(y)}{u(y)} = (R^2 - d^2)\psi(y),$$

and

$$\psi(y) = \frac{|\nabla u|(y)}{u(y)}, \quad y \in B(x,R).$$

Since $F|_{\partial B(x,R)} = 0$, if $\nabla u = 0$, then $F$ can only achieve its maximum at some point $x_0 \in B(x,R)$, if $|\nabla u|(x_0) = 0$, the desired result holds. Then, without loss of generality, we can suppose $|\nabla u|(x_0) \neq 0$. Assume $x_0 \notin cut(x)$, by the maximum principle we have $\Delta_f F(x_0) \leq 0$ and $\nabla F(x_0) = 0$. It yields, at $x_0$,

$$\nabla F = -\psi\nabla d^2 + (R^2 - d^2)\nabla\psi = 0.$$

It holds,

$$\frac{\nabla\psi}{\psi} = \frac{\nabla d^2}{R^2 - d^2} = \frac{2d\nabla d}{R^2 - d^2} \tag{12}$$

and

$$\Delta_f F = \Delta_f((R^2 - d^2)\psi) = (R^2 - d^2)\Delta_f\psi - \psi\Delta_f d^2 - 2\nabla d^2\nabla\psi \leq 0. \tag{13}$$

Hence, dividing by $(R^2 - d^2)\psi$ to both sides of (13) and combining (12), we have at $x_0$,

$$0 \geq \frac{\Delta_f\psi}{\psi} - \frac{\Delta_f d^2}{R^2 - d^2} - \frac{8d^2}{(R^2 - d^2)^2}.$$

By the $f$-Laplacian comparison theorem in [12] (see also [7] or [9]), we have

$$\Delta_f d^2 \leq C\sqrt{K}d\coth(\sqrt{K}d) \leq C\sqrt{K}d,$$

where $C$ only depends on $m$. Together with (11), we have at $x_0$,

$$\begin{aligned}0 \geq & -(a+K) - (2-\frac{2}{m})\frac{\nabla\psi\cdot\nabla u}{\psi u} + \frac{\psi^2}{m} \\ & - \frac{(a\log u + b)^2}{\psi^2} - \frac{C\sqrt{K}d}{R^2 - d^2} - \frac{8d^2}{(R^2 - d^2)^2}.\end{aligned} \tag{14}$$

By (12),

$$\frac{\nabla\psi\cdot\nabla u}{\psi u} = \frac{2d}{R^2 - d^2}\frac{\nabla d\cdot\nabla u}{u} \leq \frac{2d\psi}{R^2 - d^2},$$

then multiplying both sides of (14) by $\psi^2(R^2-d^2)^4$, combining with $\psi = \frac{F}{R^2-d^2}$, we can derive

$$\begin{aligned} 0 \geq & \frac{1}{m}F^4 - \frac{4(m-1)}{m}dF^3 - (R^2-d^2)^4(a\log u + b)^2 \\ & - ((a+K)(R^2-d^2)^2 + 8d^2 + C\sqrt{K}d(R^2-d^2))F^2 \\ \geq & \frac{1}{m}F^4 - 4RF^3 - ((a+K)R^2 + C\sqrt{K}R + 8)R^2F^2 \\ & - R^8(a\log u + b)^2. \end{aligned}$$

Since $\frac{1}{2m}F^4 - 4RF^3 \geq -8mR^2F^2$, it holds

$$\begin{aligned} \frac{1}{2m}F^4 - & ((a+K)R^2 + C\sqrt{K}R + C)R^2F^2 \\ & - R^8(a\log u + b)^2 \leq 0. \end{aligned}$$

It follows

$$\begin{aligned} & \sup_{y\in B(x,R)} (R^2 - d^2(x,y))|\nabla \log u| \leq F(x_0) \\ & \leq \sqrt{2m((a+K)R^2 + C\sqrt{K}R + C)R^2 + \sqrt{2m}(|b| + |a|L)R^4}. \end{aligned}$$

Restricting on the ball $B(x, \frac{R}{2})$, we have

$$\begin{aligned} & \sup_{y\in B(x,\frac{R}{2})} \frac{3R^2}{4}|\nabla \log u| \leq \sup_{y\in B(x,\frac{R}{2})} (R^2 - d^2(x,y))|\nabla \log u| \leq F(x_0) \\ & \leq \sqrt{2m((a+K)R^2 + C\sqrt{K}R + C)R^2 + \sqrt{2m}(|b| + |a|L)R^4}. \end{aligned}$$

Therefor, we can derive

$$\sup_{y\in B(x,\frac{R}{2})} \frac{|\nabla u|}{u} \leq \sqrt{4m(a + K + C\frac{\sqrt{K}}{R} + \frac{C}{R^2}) + \sqrt{8m}(|b| + |a|L)}.$$

Then use the Cauchy-Schwarz inequality, we obtain

$$\sup_{y\in B(x,\frac{R}{2})} \frac{|\nabla u|}{u} \leq C\sqrt{a + K + \frac{1}{R^2} + |b| + |a|L}.$$

Now let $R \to \infty$, this yields, for any $x \in M$,

$$\psi(x) \leq \psi(x_0) \leq C\sqrt{a + K + |b| + |a|L}.$$

This completes the prove of Theorem 0.3. □

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