Communications in Mathematics **32** (2024), no. 1, 157–173
DOI: https://doi.org/10.46298/cm.10983






# Cyclicity of the 2-class group of the first Hilbert 2-class field of some number fields

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**Abstract.** Let $\Bbbk$ be a real quadratic number field. Denote by $\mathrm{Cl}_2(\Bbbk)$ its 2-class group and by $\Bbbk_2^{(1)}$ (resp. $\Bbbk_2^{(2)}$) its first (resp. second) Hilbert 2-class field. The aim of this paper is to study, for a real quadratic number field whose discriminant is divisible by one prime number congruent to 3 modulo 4, the metacyclicity of $G = \mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk)$ and the cyclicity of $\mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk_2^{(1)})$ whenever the rank of $\mathrm{Cl}_2(\Bbbk)$ is 2, and the 4-rank of $\mathrm{Cl}_2(\Bbbk)$ is 1.

## 1 Introduction

Let $\Bbbk$ be an algebraic number field and $\mathrm{Cl}_2(\Bbbk)$ its 2-class group, that is, the 2-Sylow subgroup of its ideal class group $\mathrm{Cl}(\Bbbk)$. Let $\Bbbk_2^{(1)}$ be the Hilbert 2-class field of $\Bbbk$, that is, the maximal abelian extension of $\Bbbk$ everywhere unramified of 2-power degree over $\Bbbk$. Put $\Bbbk_2^{(0)} = \Bbbk$ and let $\Bbbk_2^{(i+1)}$ denote the Hilbert 2-class field of $\Bbbk_2^{(i)}$ for any integer $i \geq 0$. Then the sequence of fields

$$\Bbbk = \Bbbk_2^{(0)} \subseteq \Bbbk_2^{(1)} \subseteq \Bbbk_2^{(2)} \subseteq \cdots \subseteq \Bbbk_2^{(i)} \subseteq \cdots,$$

is called the 2-class field tower of $\Bbbk$.

*MSC 2020:* Primary 11R29, 11R11, 11R20, 11R32, 11R37; Secondary 20D15
*Keywords:* quadratic field, Hilbert 2-class field, 2-class group, metacyclic 2-group, metabelian 2-group
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If, for all $i \geq 0$, we have $\mathbb{k}_2^{(i)} \neq \mathbb{k}_2^{(i+1)}$, the tower is said to be infinite. Otherwise, the tower is said to be finite, and the minimal integer $i$ such that $\mathbb{k}_2^{(i)} = \mathbb{k}_2^{(i+1)}$ is called the length of the tower. Unfortunately, there is no known method to decide whether or not a 2-class field tower of a number field is infinite. However, it is known that if the rank of $\mathrm{Cl}_2(\mathbb{k}_2^{(1)})$ is at most 2, then the tower is finite and its length is at most 3 (cf. [14]); it is also known that if the rank of $\mathrm{Cl}_2(\mathbb{k}_2^{(1)})$ is equal to 3, then there are fields $\mathbb{k}$ with infinite 2-class field tower (cf. [24]). Therefore, it is interesting to determine all fields such that $\mathrm{rank}(\mathrm{Cl}_2(\mathbb{k}_2^{(1)})) \leq 2$. That is why Benjamin et al. started a project which aims to characterize all quadratic fields $\mathbb{k}$ satisfying the last condition (cf. [6], [7], [8], [9], [11], [12], [13], [16]). Our present paper as well as our previous one (see [3]) are part of this project.

We aim to study the cyclicity of $\mathrm{Cl}_2(\mathbb{k}_2^{(1)})$ of real quadratic fields $\mathbb{k}$ such that $\mathrm{Cl}_2(\mathbb{k})$ is of the form $(2^n, 2^m)$ for some $n \geq 1$ and $m \geq 2$, and their discriminants $d_{\mathbb{k}}$ are divisible by primes congruent to 3 modulo 4. In this paper, which is a continuation of [3], we consider the field $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2q})$, where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are primes and $\mathrm{Cl}_2(\mathbb{k}) \simeq (2, 2^n)$, with $n \geq 2$. We determine complete criteria for $G = \mathrm{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ to be metacyclic and complete criteria for $\mathrm{Cl}_2(\mathbb{k}_2^{(1)})$ to be cyclic whenever $G$ is not metacyclic.

## 2 Preliminary results

We begin by collecting some results that will be useful later. We recall that a 2-group $G$ is said to be metacyclic if there exists a normal cyclic subgroup $N$ of $G$ such that $G/N$ is cyclic. It is known that if $G$ is metacyclic, then the minimal number of its generators is less or equal to 2; this number is called the rank of $G$ and will be denoted by $d(G)$. On the other hand, if $d(G) = 2$, then $G/G'$ is of type $(2^n, 2^m)$ with $n$ and $m \in \mathbb{N}^*$, where $G'$ is the commutator subgroup of $G$. If $n = m = 1$, then it is known that $G$ is dihedral, semi-dihedral, quaternionic or abelian of type $(2, 2)$ (cf. [20], [17]). In these cases, $G$ admits a cyclic maximal subgroup, and thus is metacyclic. By Blackburn [15], we know that the metacyclicity of a 2-group $G$ is characterized by the rank of its maximal subgroups, and we have the following lemmas.

**Lemma 2.1** ([3]). *Let $G$ be a finite 2-group such that $G/G'$ is of type $(2^n, 2^m)$, where $n \geq 1$ and $m \geq 2$. Denote by $H_i$ $(i = 1, 2, 3)$, the three maximal subgroups of $G$. Then $G$ is metacyclic if and only if $d(H_i) \leq 2$ for all $i = 1, 2, 3$.*

**Lemma 2.2** ([7]). *Let $G$ be a non-metacyclic 2-group such that $G/G'$ is of type $(2, 2^m)$, where $m \geq 2$. Then $G$ admits two maximal subgroups $H_1$ and $H_2$ such that $H_1/G'$ and $H_2/G'$ are cyclic. Moreover, if $G'$ is cyclic, then $H_1$ and $H_2$ are metacyclic.*

We continue by fixing some notation. For a number field $\mathbb{k}$, denote by $\mathrm{Cl}_2(\mathbb{k})$ its 2-class group in the ordinary sense, denote by $h_2(\mathbb{k})$ the order of $\mathrm{Cl}_2(\mathbb{k})$, denote by $\mathbb{k}_2^{(1)}$ the Hilbert 2-class field of $\mathbb{k}$, and denote by $\mathbb{k}_2^{(2)}$ the Hilbert 2-class field of $\mathbb{k}_2^{(1)}$. If $G = \mathrm{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$, then it is well known from class field theory that $G' = \mathrm{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k}_2^{(1)}) \simeq \mathrm{Cl}_2(\mathbb{k}_2^{(1)})$ and

$G/G' = \mathrm{Gal}(\mathbb{k}_2^{(1)}/\mathbb{k}) \simeq \mathrm{Cl}_2(\mathbb{k})$. Note that if $\mathrm{Cl}_2(\mathbb{k})$ is of type $(2, 2^n)$, with $n \geq 2$, then $\mathbb{k}$ admits three unramified quadratic extensions within $\mathbb{k}_2^{(1)}$, which will be denoted by $\mathbb{K}_i$ $(i = 1, 2, 3)$. We suppose that $\mathbb{K}_3$ is included in the three unramified biquadratic extensions of $\mathbb{k}$ within $\mathbb{k}_2^{(1)}$. The following result was shown in our earlier paper [4].

**Theorem 2.3.** *Keep the notations above and assume $G/G'$ is of type $(2, 2^n)$, where $n \geq 2$. Then*

1. *$G$ is abelian or modular if and only if*

$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_i)) = 1 \ (i = 1, 2) \ \textit{ and } \ \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 2.$$

2. *$G$ is metacyclic non-abelian non-modular if and only if*

$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_i)) = 2 \ \textit{ for all } \ i = 1, 2, 3.$$

3. *$G$ is non-metacyclic non-abelian if and only if*

$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_i)) = 2 \ (i = 1, 2) \ \textit{ and } \ \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 3.$$

Let $\mathbb{k} = \mathbb{Q}(\sqrt{d})$ be an arbitrary quadratic number field with a square-free integer $d$, and $d_{\mathbb{k}}$ be its discriminant. For a prime number $p$, define:

$$p^* = \begin{cases} (-1)^{\frac{p-1}{2}} p, & \text{if } p \neq 2; \\ -4, & \text{if } p = 2 \text{ and } d \equiv 3 \pmod 4; \\ 8, & \text{if } p = 2 \text{ and } d \equiv 2 \pmod 8; \\ -8, & \text{if } p = 2 \text{ and } d \equiv -2 \pmod 8). \end{cases}$$

Then, let $d_{\mathbb{k}} = p_1^* \dots p_s^* p_{s+1}^* \dots p_{s+t}^*$ such that $p_1^*, \dots, p_s^*$ are positive and $p_{s+1}^*, \dots, p_{s+t}^*$ are negative. The Rédei matrix $R_{\mathbb{k}}$ is defined to be the matrix in $M_{(s+t)\times(s+t)}(\mathbb{Z}/2\mathbb{Z})$ with entries $a_{i,j}$ given by: $(-1)^{a_{i,j}} = \left(\frac{p_i^*}{p_j}\right)$ if $i \neq j$ and $(-1)^{a_{i,j}} = \left(\frac{d_{\mathbb{k}}/p_i^*}{p_i}\right)$ if $i = j$, where $\left(\frac{\bullet}{\bullet}\right)$ is the Legendre symbol. Then the 4-rank of $\mathrm{Cl}^+(\mathbb{k})$, the class group of $\mathbb{k}$ in the narrow sense, is given by:

**Theorem 2.4** ([23]). *Let $\mathbb{k}$ be a quadratic number field, then*

$$4\text{-rank}(\mathrm{Cl}^+(\mathbb{k})) = s + t - 1 - \mathrm{rank}(R_{\mathbb{k}}).$$

**Remark 2.5.** If $d_{\mathbb{k}}$ is divisible by a prime congruent to 3 modulo 4, then

$$\mathrm{Cl}_2^+(\mathbb{k})) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathrm{Cl}_2(\mathbb{k}) \qquad \text{and} \qquad 4\text{-rank}(\mathrm{Cl}^+(\mathbb{k})) = 4\text{-rank}(\mathrm{Cl}(\mathbb{k})),$$

where $\mathrm{Cl}_2^+(\mathbb{k})$ is the 2-class group of $k$ in the narrow sense.

We make use of the well known Kuroda Class Number Formula, which we state as the following theorem.

**Theorem 2.6** ([22]). *Let $\mathbb{K}/\Bbbk$ be an arbitrary normal quartic extension of number fields with Galois group of type $(2,2)$, and let $\mathbb{K}_j$ $(j=1,2,3)$ denote the quadratic subextensions. Then the class number of $\mathbb{K}$ satisfies*

$$h(\mathbb{K}) = 2^{d-\kappa-2-\upsilon}\frac{q(\mathbb{K}/\Bbbk)h(\mathbb{K}_1)h(\mathbb{K}_2)h(\mathbb{K}_3)}{(h(\Bbbk))^2},$$

*where $q(\mathbb{K}/\Bbbk)=[E_{\mathbb{K}}:E_1E_2E_3]$ denotes the unit index of $\mathbb{K}/\Bbbk$ (with $E_j=$ the unit group of $\mathbb{K}_j$), $d$ is the number of infinite primes in $\Bbbk$ that ramify in $\mathbb{K}$, $\kappa$ is the $\mathbb{Z}$-rank of the unit group $E_\Bbbk$ of $\Bbbk$, and $\upsilon=0$ except when $\mathbb{K}\subseteq\Bbbk(\sqrt{E_\Bbbk})$, in which case $\upsilon=1$.*

To prove our main theorems, we also need the following results.

**Theorem 2.7** ([6]). *Let $\Bbbk$ be a number field such that $\mathrm{Cl}_2(\Bbbk)\simeq(2,2^n)$, where $n\geq 2$. Denote by $\mathbb{K}_i$ $(i=1,2,3)$, the three unramified quadratic extensions of $\Bbbk$. Then the 2-class group of $\Bbbk_2^{(1)}$ is a non-elementary cyclic group if and only if $h_2(\mathbb{K}_i)\geq 2h_2(\Bbbk)$ and $h_2(\mathbb{K}_j)=h_2(\mathbb{K}_m)=h_2(\Bbbk)$ for some $\{i,j,m\}=\{1,2,3\}$.*

**Lemma 2.8** ([6]). *Let $\Bbbk$ be a number field such that $\mathrm{Cl}_2(\Bbbk)\simeq(2^m,2^n)$, $m\geq 1$, $n\geq 1$. Denote by $\mathbb{K}_i$ $(i=1,2,3)$, the three unramified quadratic extensions of $\Bbbk$. Then $h_2(\Bbbk_2^{(1)})=2$ if and only if $h_2(\mathbb{K}_0)=(1/2)h_2(\Bbbk)$ where $\mathbb{K}_0=\mathbb{K}_1\mathbb{K}_2\mathbb{K}_3$.*

**Corollary 2.9** ([6]). *Let $\Bbbk$ be a real quadratic number field such that $\mathrm{Cl}_2(\Bbbk)\simeq(2^m,2^n)$, $m\geq 1$, $n\geq 1$. Denote by $\mathbb{K}_i$ $(i=1,2,3)$, the three unramified quadratic extensions of $\Bbbk$. Then $h_2(\Bbbk_2^{(1)})=2$ if and only if $h_2(\mathbb{K}_1)=h_2(\mathbb{K}_2)=h_2(\mathbb{K}_3)=h_2(\Bbbk)$ and $q(\mathbb{K}_0/\Bbbk)=4$, where $\mathbb{K}_0=\mathbb{K}_1\mathbb{K}_2\mathbb{K}_3$.*

**Theorem 2.10** ([6]). *Let $\Bbbk$ be a real quadratic number field with $\mathrm{Cl}_2(\Bbbk)\simeq(2^m,2^n)$, $m\geq 1$, $n\geq 2$, and $d_\Bbbk=d_1d_2r_1r_2$ or $r_1r_2r_3r_4$ be its discriminant, where $d_1$ and $d_2$ are positive prime discriminants and $r_1$, $r_2$, $r_3$, $r_4$ are negative prime discriminants. Denote by $\mathbb{K}_i$ $(i=1,2,3)$, the three unramified quadratic extensions of $k$. If $h_2(\Bbbk_2^{(1)})=2$ then $Q_{\mathbb{K}_i}=Q_{\mathbb{K}_j}=2$, and $Q_{\mathbb{K}_s}=1$ or $2$ for some $\{i,j,s\}=\{1,2,3\}$, where $Q_\mathbb{K}$ denotes the unit index of $\mathbb{K}$.*

## 3 The 4-rank of the 2-class group of $\Bbbk=\mathbb{Q}(\sqrt{2p_1p_2q})$.

Let $p_1\equiv p_2\equiv -q\equiv 1 \pmod 4$ be different positive prime integers and $\Bbbk=\mathbb{Q}(\sqrt{2p_1p_2q})$. It is well known, by genus theory, that the 2-rank of the class group of $\Bbbk$ is 2. The purpose of this section is to determine the 4-rank of the 2-class group of $\Bbbk$.

E. Benjamin and C. Snyder characterized real quadratic fields whose 2-class group is of type $(2,2)$ in [10]. In particular, they proved the following theorem.

**Theorem 3.1.** *Let* $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ *be different prime integers. Then the* 2*-class group of* $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$ *is of type* $(2,2)$ *(i.e.* $4\text{-rank}(\mathrm{Cl}_2(\Bbbk)) = 0$*) if and only if one of the following conditions is satisfied.*

1. 
   - $\left(\frac{p_1}{p_2}\right) = 1$, *and*
   - *either* $\left(\frac{2}{p_1}\right) = -1$ *or* $\left(\frac{q}{p_1}\right) = -1$, *and*
   - *either* $\left(\frac{2}{p_2}\right) = -1$ *or* $\left(\frac{q}{p_2}\right) = -1$, *and*
   - $\left(\frac{2}{p_1}\right), \left(\frac{2}{p_2}\right), \left(\frac{q}{p_1}\right), \left(\frac{q}{p_2}\right)$ *are not all equal.*
2. $\left(\frac{p_1}{p_2}\right) = -1$ *and* $\left(\frac{2}{p_1}\right), \left(\frac{2}{p_2}\right), \left(\frac{q}{p_1}\right), \left(\frac{q}{p_2}\right)$ *are not all equal.*

In the following theorem, we give necessary and sufficient conditions for the 2-class group of $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$ to be of type $(2, 2^n)$ or $(2^m, 2^n)$, where $n \geq 2$ and $m \geq 2$.

**Theorem 3.2.** *Let* $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ *be different prime integers. Then the* 2*-class group of* $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$ *is of type* $(2, 2^n)$*, where* $n \geq 2$*,(i.e.* $4\text{-rank}(\mathrm{Cl}_2(\Bbbk)) = 1$*) if and only if one of the following conditions is satisfied.*

1. $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = 1$ *and* $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right) = -1.$
2. $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$ *and* $\left(\frac{p_1}{p_2}\right) = -1.$
3. $\left(\frac{2}{p_1}\right) = -\left(\frac{2}{p_2}\right) = 1$ *and* $\left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = 1.$
4. $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -1.$

*Moreover,* $4\text{-rank}(\mathrm{Cl}_2(\Bbbk)) = 2$ *if and only if*

$$\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1.$$

*Proof.* Proceeding as in [3], the results are deduced by applying Theorem 2.4 and Remark 2.5. □

## 4 The FSUs of certain biquadratic number fields

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different prime integers. Put $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$. Consider the following three unramified quadratic extensions of $\Bbbk$:

$$\mathbb{K}_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2q}), \quad \mathbb{K}_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1q}) \text{ and } \mathbb{K}_3 = \mathbb{Q}(\sqrt{2q}, \sqrt{p_1p_2}).$$

Let $\varepsilon_{2p_1p_2q}=x+y\sqrt{2p_1p_2q}$, $\varepsilon_{2p_1q}=z+t\sqrt{2p_1q}$ and $\varepsilon_{2p_2q}=a+b\sqrt{2p_2q}$ be the fundamental units of $\mathbb{Q}(\sqrt{2p_1p_2q})$, $\mathbb{Q}(\sqrt{2p_2q})$ and $\mathbb{Q}(\sqrt{2p_1q})$ respectively. The goal of this section is to determine a Fundamental System of Units (FSU) of $\mathbb{K}_i$ basing on the conditions cited in Theorem 3.2.

Using similar arguments as in the proof of Lemma 4.1 of [3] (see also [5]), we get the following lemmas.

**Lemma 4.1.** *Suppose that* $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=\left(\frac{p_1}{p_2}\right)=\left(\frac{q}{p_1}\right)=-\left(\frac{q}{p_2}\right)=1.$

1. *If* $x\pm1$ *is a square in* $\mathbb{N}$, *then*
   - i. $\{\varepsilon_{p_1},\varepsilon_{2p_2q},\sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.
   - ii. $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $z\pm1$ *is or not a square in* $\mathbb{N}$.
   - iii. $\{\varepsilon_{2q},\varepsilon_{p_1p_2},\sqrt{\varepsilon_{2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_3$.
2. *If* $p_1(x\pm1)$ *is a square in* $\mathbb{N}$, *then*
   - i. $\{\varepsilon_{p_1},\varepsilon_{2p_2q},\sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.
   - ii. $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $p_1(z\pm1)$ *is or not a square in* $\mathbb{N}$.
   - iii. $\{\varepsilon_{2q},\ \varepsilon_{p_1p_2},\ \sqrt{\varepsilon_{2q}\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{2q},\ \varepsilon_{p_1p_2},\ \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* 1 *or* $-1$.
3. *If* $2p_1(x\pm1)$ *is a square in* $\mathbb{N}$, *then*
   - i. $\{\varepsilon_{p_1},\varepsilon_{2p_2q},\sqrt{\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.
   - ii. $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $2p_1(z+1)$ *is or not a square in* $\mathbb{N}$.
   - iii. $\{\varepsilon_{2q},\varepsilon_{p_1p_2},\sqrt{\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{2q},\varepsilon_{p_1p_2},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* 1 *or* $-1$.

**Lemma 4.2.** *Suppose that* $\left(\frac{2}{p_1}\right)=-\left(\frac{2}{p_2}\right)=\left(\frac{p_1}{p_2}\right)=\left(\frac{q}{p_1}\right)=-\left(\frac{q}{p_2}\right)=1.$

1. *If* $2p_1(x+1)$ *is a square in* $\mathbb{N}$, *then*
   - i. $\{\varepsilon_{p_1},\varepsilon_{2p_2q},\sqrt{\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.
   - ii. $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2},\varepsilon_{2p_1q},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $2p_1(z+1)$ *is or not a square in* $\mathbb{N}$.
   - iii. $\{\varepsilon_{2q},\varepsilon_{p_1p_2},\sqrt{\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{2q},\varepsilon_{p_1p_2},\varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* 1 *or* $-1$.
2. *If* $2p_2(x\pm1)$ *is a square in* $\mathbb{N}$, *then*

i. $\{\varepsilon_{p_1}, \varepsilon_{2p_2q}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.

ii. $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \sqrt{\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_2$.

iii. $\{\varepsilon_{2q}, \varepsilon_{p_1p_2}, \sqrt{\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{q}, \varepsilon_{p_1p_2}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* $1$ *or* $-1$.

3. *If* $q(x\pm 1)$ *is a square in* $\mathbb{N}$, *then*

i. $\{\varepsilon_{p_1}, \varepsilon_{2p_2q}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.

ii. $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according to whether* $q(z-1)$ *is or not a square in* $\mathbb{N}$.

iii. $\{\varepsilon_{2q}, \varepsilon_{p_1p_2}, \sqrt{\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_3$.

**Lemma 4.3.** *Suppose that* $\left(\frac{2}{p_1}\right) = -\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$.

1. *If* $2p_1(x\pm 1)$ *is a square in* $\mathbb{N}$, *then*

i. $\{\varepsilon_{p_1}, \varepsilon_{2p_2q}, \sqrt{\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.

ii. $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $2p_1(z\pm1)$ *is or not a square in* $\mathbb{N}$.

iii. $\{\varepsilon_{2q}, \varepsilon_{p_1p_2}, \sqrt{\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{2q}, \varepsilon_{p_1p_2}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* $1$ *or* $-1$.

2. *If* $p_2(x\pm 1)$ *is a square in* $\mathbb{N}$, *then*

i. $\{\varepsilon_{p_1}, \varepsilon_{2p_2q}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.

ii. $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $(z\pm 1)$ *is or not a square in* $\mathbb{N}$.

iii. $\{\varepsilon_{2q},\ \varepsilon_{p_1p_2},\ \sqrt{\varepsilon_{2q}\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{2q},\ \varepsilon_{p_1p_2},\ \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_3$ *according to whether* $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})$ *equals* $1$ *or* $-1$.

3. *If* $2q(x\pm 1)$ *is a square in* $\mathbb{N}$, *then*

i. $\{\varepsilon_{p_1}, \varepsilon_{2p_2q}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_1$.

ii. $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \sqrt{\varepsilon_{2p_1q}\varepsilon_{2p_1p_2q}}\}$ *or* $\{\varepsilon_{p_2}, \varepsilon_{2p_1q}, \varepsilon_{2p_1p_2q}\}$ *is a* FSU *of* $\mathbb{K}_2$ *according as* $2q(z\pm1)$ *is or not a square in* $\mathbb{N}$.

iii. $\{\varepsilon_{2q}, \varepsilon_{p_1p_2}, \sqrt{\varepsilon_{2q}\varepsilon_{2p_1p_2q}}\}$ *is a* FSU *of* $\mathbb{K}_3$.

## 5 The structure of the group $\mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk)$.

In this section we consider the field $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$, where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$, and the three unramified quadratic extensions

$$\mathbb{K}_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2q}),\ \mathbb{K}_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1q}) \text{ and } \mathbb{K}_3 = \mathbb{Q}(\sqrt{2q}, \sqrt{p_1p_2}).$$

Let $\mathrm{Cl}_2(\mathbb{K}_i)$ denote the 2-class group of $\mathbb{K}_i$ $(i = 1, 2, 3)$.

### 5.1 The metacyclic case

**Theorem 5.1.** *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different prime integers, and $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2q})$. Assume $\mathrm{Gal}(\Bbbk_2^{(1)}/\Bbbk)$ is a non-elementary 2-group. Then $\mathrm{G} = \mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk)$ is metacyclic if and only if*

$$\left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = \left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1.$$

*More precisely,*

i. *if $\left(\frac{p_1}{p_2}\right) = 1$, then G is a metacyclic non-abelian non-modular 2-group,*

ii. *if $\left(\frac{p_1}{p_2}\right) = -1$, then G is a modular or abelian 2-group according as $2p_1p_2(x+1)$ is a square or not in $\mathbb{N}$.*

*Proof.* According to Theorems 3.1 and 3.2, there are five cases to distinguish. By [1] and [2] we have:

1. If $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 3.$$

2. If $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{q}{p_1}\right) = -\left(\frac{q}{p_2}\right) = 1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = 3 \text{ and } \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 2.$$

3. If $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$ and $\left(\frac{p_1}{p_2}\right) = -1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = 2 \text{ and } \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 3.$$

4. If $\left(\frac{2}{p_1}\right) = -\left(\frac{2}{p_2}\right) = 1$ and $\left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = 1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = 3 \text{ and } \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 2.$$

5. If $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -1$, then

   i. If $\left(\frac{p_1}{p_2}\right) = 1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 2.$$

   ii. If $\left(\frac{p_1}{p_2}\right) = -1$, then
$$\mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_1)) = \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_2)) = 1 \text{ and } \mathrm{rank}(\mathrm{Cl}_2(\mathbb{K}_3)) = 2.$$

Hence the results are deduced from Theorem 2.3, Lemma 2.1 and [8, Theorem 2]. □

### 5.2 The non-metacyclic case

Assuming $\mathrm{Cl}_2(\Bbbk) \simeq (2, 2^n)$, $n \geq 2$, for the non-metacyclic case we have four cases to distinguish, according to Theorems 3.2 and 5.1. For simplicity, we will denote by $q_i$ the unit index of the field $\mathbb{K}_i$ $(i = 1, 2, 3)$. In all that follows, we use the notations of [19, page 336]. Put $p_1 = 2e^2 + (-1)^\gamma d^2$, $q = 2r^2 + (-1)^\gamma s^2$ and $A = sd + 2er + 2\gamma(es + dr)$ according as $\left(\frac{2}{q}\right) = (-1)^{\gamma+1}$, where $\gamma \in \{0, 1\}$.

#### 5.2.1 Case 1: $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = -\left(\frac{q}{p_2}\right) = 1$

**Theorem 5.2.** *Let $\delta \in \{1, p_1, 2p_1\}$ be such that $\delta(x \pm 1)$ is a square in $\mathbb{N}$. The group $\mathrm{Cl}_2(\Bbbk_2^{(1)}) \simeq \mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk_2^{(1)})$ is non-elementary cyclic if and only if one of the two following assertions holds:*

I. i. *$\delta(z \pm 1)$ is not a square in $\mathbb{N}$,*

ii. *at least one of the elements $\left\{\left(\frac{A}{p_1}\right), \left(\frac{2q}{p_1}\right)_4\right\}$ equals $-1$, and*

iii. *either*

a. *$\delta = 1$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4$, or*

b. *$\delta \neq 1$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$.*

II. i. *$\delta(z \pm 1)$ is a square in $\mathbb{N}$ or $\left(\frac{A}{p_1}\right) = \left(\frac{2q}{p_1}\right)_4 = 1$, and*

ii. *either*

a. *$\delta = 1$ and $\left(\frac{p_1}{p_2}\right)_4 \neq \left(\frac{p_2}{p_1}\right)_4$, or*

b. *$\delta \neq 1$ and one of $\left(\frac{p_1}{p_2}\right)_4, \left(\frac{p_2}{p_1}\right)_4$ is equal to $-1$.*

*Proof.* Form Theorem 2.7, we must calculate the 2-class numbers of $\mathbb{K}_i$.

- By [19], if $\left(\frac{q}{p_2}\right) = -1$, then $h_2(2p_2q) = 2$, and, according to Lemma 4.1, $q_1 = 2$. In this case, the 2-class number of $\mathbb{K}_1$ is given by [27]:

$$h_2(\mathbb{K}_1) = \frac{1}{4} q_1 h_2(p_1) h_2(2p_2q) h_2(2p_1p_2q) = h_2(2p_1p_2q).$$

- If $\left(\frac{2}{p_1}\right) = \left(\frac{q}{p_1}\right) = 1$, then, by [19], $h_2(2p_1q) \geq 4$. More precisely, $h_2(2p_1q) = 4$ if and only if at least one of the elements $\left(\frac{A}{p_1}\right), \left(\frac{2q}{p_1}\right)_4$ equals $-1$. The 2-class number of $\mathbb{K}_2$ is given by:

$$h_2(\mathbb{K}_2) = \frac{1}{4} q_2 h_2(p_2) h_2(2p_1q) h_2(2p_1p_2q) = \frac{1}{4} q_2 h_2(2p_1q) h_2(2p_1p_2q),$$

so $h_2(\mathbb{K}_2) = h_2(2p_1p_2q)$ if and only if $q_2 = 1$ and $h_2(2p_1q) = 4$. On the other hand, by Lemma 4.1, $q_2 = 1$ if and only if $\delta(z \pm 1)$ is not a square in $\mathbb{N}$.

- Similarly, the 2-class number of $\mathbb{K}_3$ is given by:

$$h_2(\mathbb{K}_3)=\frac{1}{4}q_3h_2(2q)h_2(p_1p_2)h_2(2p_1p_2q)=\frac{1}{4}q_3h_2(p_1p_2)h_2(2p_1p_2q),$$

so $h_2(\mathbb{K}_3)=h_2(2p_1p_2q)$ if and only if either $q_3=1$ and $h_2(p_1p_2)=4$, or $q_3=2$ and $h_2(p_1p_2)=2$. In this case, according to [21] (see also [25]) and Lemma 4.1, $h_2(\mathbb{K}_3)=h_2(2p_1p_2q)$ if and only if one of the two following conditions is satisfied:

1. $\delta=1$ and $\left(\frac{p_1}{p_2}\right)_4\neq\left(\frac{p_2}{p_1}\right)_4$.
2. $\delta\neq1$ and either $\left(\frac{p_1}{p_2}\right)_4=-1$ or $\left(\frac{p_2}{p_1}\right)_4=-1$.

Using Theorem 2.7, we get the results. □

**Example 5.3.** Put $\alpha=\left(\frac{A}{p_1}\right)$, $s=\left(\frac{2q}{p_1}\right)_4$, $t_1=\left(\frac{p_1}{p_2}\right)_4$, $t_2=\left(\frac{p_2}{p_1}\right)_4$, $c=\mathrm{Cl}(\Bbbk_2^{(1)})$, $n=h_2(\Bbbk)$, $n_i=h_2(\mathbb{K}_i)$ $(i=1,2,3)$ and $q_0=q(\mathbb{K}_0/\Bbbk)$, and by using PARI/GP [26], we get the following examples for the case: $x\pm1$ is a square, $z+1$ and $z-1$ are not squares, ($\alpha=-1$ or $s=-1$) and $t_1=t_2$.

| $d=2p_1p_2q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $38982=2\cdot73\cdot89\cdot3$ | 2 | 1 | 2 | $-1$ | $-1$ | 1 | 1 | 8 | 8 | 8 | 16 | 16 | [4] |
| $60006=2\cdot73\cdot137\cdot3$ | 2 | 1 | 2 | $-1$ | $-1$ | 1 | 1 | 8 | 8 | 8 | 64 | 16 | [16] |
| $298862=2\cdot73\cdot89\cdot23$ | 2 | 1 | 2 | $-1$ | $-1$ | 1 | 1 | 8 | 8 | 8 | 16 | 16 | [12] |

**Theorem 5.4.** *Let $\delta\in\{1,p_1,2p_1\}$ such that $\delta(x\pm1)$ is a square in $\mathbb{N}$. Then $\#\mathrm{Cl}_2(\Bbbk_2^{(1)})=2$ if and only if the following conditions are satisfied:*

i. *$\delta(z\pm1)$ is not a square in $\mathbb{N}$,*

ii. *at least one of the elements $\left\{\left(\frac{A}{p_1}\right),\left(\frac{2q}{p_1}\right)_4\right\}$ equals $-1$.*

iii. $\left(\frac{p_1}{p_2}\right)_4\neq\left(\frac{p_2}{p_1}\right)_4$.

*Proof.* Suppose that $\#\mathrm{Cl}_2(\Bbbk_2^{(1)})=2$. Then, according to Corollary 2.9, $h_2(\mathbb{K}_i)=h_2(\Bbbk)$ for all $i=1,2,3$. By the proof of Theorem 5.2, the equality $h_2(\mathbb{K}_2)=h_2(\Bbbk)$ implies the two first conditions and $q_2=1$. On the other hand, as $q_1=2$, from Theorem 2.10 we infer $q_3=2$. Accordingly, $h_2(p_1p_2)=2$ and $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})=1$, which is equivalent to $\left(\frac{p_1}{p_2}\right)_4\neq\left(\frac{p_2}{p_1}\right)_4$ (see [25]).

Reciprocally, suppose the three conditions (i), (ii) and (iii) are satisfied. Applying results of the proof of Theorem 5.2, we get $h_2(\mathbb{K}_i)=h_2(\Bbbk)$ for all $i=1,2,3$. Let $\mathbb{K}_0=\mathbb{K}_1\mathbb{K}_2\mathbb{K}_3=\mathbb{Q}(\sqrt{p_1},\sqrt{p_2},\sqrt{2q})$, and denote by $E_{\mathbb{K}_i}$ the unit group of $\mathbb{K}_i$ and by $q(\mathbb{K}_0/\Bbbk)=[E_{\mathbb{K}_0}:E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}]$ the unit index of $\mathbb{K}_0/\Bbbk$. Hence, by Corollary 2.9, it remains to prove only $q(\mathbb{K}_0/\Bbbk)=4$. According to Lemma 4.1, we have:

1. $E_{\mathbb{K}_1} = \langle -1, \varepsilon_{p_1}, \varepsilon_{2p_2q}, \varepsilon\rangle$, where $\varepsilon = \sqrt{\varepsilon_{2p_1p_2q}}$ or $\sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}}$ according to $2p_1(x\pm 1)$ is or not a square in $\mathbb{N}$,

2. $E_{\mathbb{K}_2} = \langle -1, \varepsilon_{p_2}, \varepsilon_{2p_1q}, \varepsilon_{2p_1p_2q}\rangle$,

3. $E_{\mathbb{K}_3} = \langle -1, \varepsilon_{2q}, \varepsilon_{p_1p_2}, \varepsilon'\rangle$, where $\varepsilon' = \sqrt{\varepsilon_{2q}\varepsilon_{2p_1p_2q}}$, $\sqrt{\varepsilon_{2q}\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}$ or $\sqrt{\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}$ according to $(x\pm 1)$, $p_1(x\pm 1)$ or $2p_1(x\pm 1)$ is a square in $\mathbb{N}$.

Put

$$\begin{aligned} A &= \varepsilon_{p_1}^{a_1}\varepsilon_{2p_2q}^{a_2}\varepsilon^{a_3}, \quad a_1,a_2,a_3\in\{0,1\},\\ B &= \varepsilon_{p_2}^{b_1}\varepsilon_{2p_1q}^{b_2}\varepsilon_{2p_1p_2q}^{b_3}, \quad b_1,b_2,b_3\in\{0,1\},\\ C &= \varepsilon_{2q}^{c_1}\varepsilon_{p_1p_2}^{c_2}\varepsilon'^{c_3}, \quad c_1,c_2,c_3\in\{0,1\},\\ \eta^2 &= \pm A.B.C. \end{aligned}$$

So

$$\begin{aligned} N_{\mathbb{K}_0/\mathbb{K}_1}(\eta^2) &= (-1)^{b_1}(\pm\varepsilon_{2p_1p_2q})^{c_3}(\varepsilon_{2p_1p_2q}^{b_3}A)^2,\\ N_{\mathbb{K}_0/\mathbb{K}_2}(\eta^2) &= (-1)^{a_1}(\pm\varepsilon_{2p_1p_2q})^{a_3}(\pm\varepsilon_{2p_1p_2q})^{c_3}B^2,\\ N_{\mathbb{K}_0/\mathbb{K}_3}(\eta^2) &= (-1)^{a_1}(-1)^{b_1}(\pm\varepsilon_{2p_1p_2q})^{a_3}(\varepsilon_{2p_1p_2q}^{b_3}C)^2. \end{aligned}$$

Assume $\eta\in\mathbb{K}_0$, if $a_3\neq 0$ or $c_3\neq 0$, then $\sqrt{\varepsilon_{2p_1p_2q}}\in\mathbb{K}_2$ or $\sqrt{\varepsilon_{2p_1p_2q}}\in\mathbb{K}_3$, which contradicts Lemma 4.1. On the other hand, if $a_3=c_3=0$ and ($a_1=1$ or $b_1=1$ ), then $N_{\mathbb{K}_0/\mathbb{K}_1}(\eta^2)<0$ or $N_{\mathbb{K}_0/\mathbb{K}_2}(\eta^2)<0$, which contradicts the fact that $N_{\mathbb{K}_0/\mathbb{K}_i}(\eta^2)>0$. Therefore, $a_1=b_1=a_3=c_3=0$ and we get

$$\eta^2 = \pm\varepsilon_{2p_2q}^{a_2}\varepsilon_{2p_1q}^{b_2}\varepsilon_{2p_1p_2q}^{b_3}\varepsilon_{2q}^{c_1}\varepsilon_{p_1p_2}^{c_2}.$$

From the proof of Lemma 4.1, we deduce thet $\sqrt{\varepsilon_{2q}\varepsilon_{2p_iq}}$ or $\sqrt{\varepsilon_{2p_iq}}\in E_{\mathbb{K}_0}$ ($i=1,2$). According to our assumption, $N_{\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}}(\varepsilon_{p_1p_2})=1$, which implies that $\sqrt{\varepsilon_{p_1p_2}}\in E_{\mathbb{K}_0}$. We distinguish the following cases:

i. If $(x\pm 1)$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2p_2q}}, \sqrt{\varepsilon_{2p_1p_2q}}\notin E_{\mathbb{K}_0}$, and Lemma 4.1 implies that

$$\begin{aligned} \sqrt{\varepsilon_{p_1p_2}} &\notin E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3},\\ \sqrt{\varepsilon_{2q}\varepsilon_{2p_1q}}, \sqrt{\varepsilon_{2p_1q}} &\notin E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3},\\ \sqrt{\varepsilon_{2q}\varepsilon_{2p_1p_2q}}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}} &\in E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}. \end{aligned}$$

By a case by case study, we obtain

$$E_{\mathbb{K}_0} = \langle \sqrt{\varepsilon_{2p_1q}} \text{ or } \sqrt{\varepsilon_{2q}\varepsilon_{2p_1q}}, \sqrt{\varepsilon_{p_1p_2}}, E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}\rangle.$$

ii. If $p_1(x\pm 1)$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2p_2q}}, \sqrt{\varepsilon_{2p_1p_2q}} \notin E_{\mathbb{K}_0}$, By Lemma 4.1, we have

$$\begin{aligned} \sqrt{\varepsilon_{p_1p_2}} &\notin E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3},\\ \sqrt{\varepsilon_{2q}\varepsilon_{2p_1q}}, \sqrt{\varepsilon_{2p_1q}} &\notin E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3},\\ \sqrt{\varepsilon_{2q}\varepsilon_{p_1p_2}\varepsilon_{2p_1p_2q}}, \sqrt{\varepsilon_{2p_2q}\varepsilon_{2p_1p_2q}} &\in E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}. \end{aligned}$$

Thus we obtain

$$E_{\mathbb{K}_0} = \langle \sqrt{\varepsilon_{2p_1q}} \text{ or } \sqrt{\varepsilon_{2q}\varepsilon_{2p_1q}}, \sqrt{\varepsilon_{p_1p_2}}, E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}\rangle.$$

iii. If $2p_1(x\pm 1)$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2p_2q}} \notin E_{\mathbb{K}_0}$ and $\sqrt{\varepsilon_{2q}\varepsilon_{2p_2q}} \in E_{\mathbb{K}_0}$, by Lemma 4.1, $\sqrt{\varepsilon_{p_1p_2}}, \sqrt{\varepsilon_{2p_1p_2q}} \in E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}$, from which we deduce that

$$E_{\mathbb{K}_0} = \langle \sqrt{\varepsilon_{2p_1q}} \text{ or } \sqrt{\varepsilon_{2q}\varepsilon_{2p_1q}}, \sqrt{\varepsilon_{2q}\varepsilon_{2p_2q}}, E_{\mathbb{K}_1}E_{\mathbb{K}_2}E_{\mathbb{K}_3}\rangle.$$

In the three cases we get $q(\mathbb{K}_0/\mathbb{k}) = 4$, so it suffices to apply Corollary 2.9 to obtain the results. □

**Example 5.5.** Keep the notation of Example 5.3. For the case $p_1(x\pm 1)$ is a square, $p_1(z\pm 1)$ is not a square, ($\alpha = -1$ or $s = -1$) and $t_1 \neq t_2$, we have

| $d = p_1p_2q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $51798 = 2\cdot 97\cdot 89\cdot 3$ | 2 | 1 | 2 | −1 | 1 | 1 | −1 | 8 | 8 | 8 | 8 | 16 | [2] |
| $64862 = 2\cdot 113\cdot 41\cdot 7$ | 2 | 1 | 2 | −1 | 1 | 1 | −1 | 8 | 8 | 8 | 8 | 16 | [6] |
| $113734 = 2\cdot 73\cdot 41\cdot 19$ | 2 | 1 | 2 | 1 | −1 | −1 | 1 | 8 | 8 | 8 | 8 | 16 | [6] |

### 5.2.2 Case 2: $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -\left(\frac{p_1}{p_2}\right) = 1$

**Lemma 5.6.** *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be three positive prime integers satisfying $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -\left(\frac{p_1}{p_2}\right) = 1$. Then the rank of the 2-class group of $\mathbb{K}_0 = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{2q})$ equals 3.*

*Proof.* As $\left(\frac{p_1}{p_2}\right) = -1$, it is well known (cf. [21]) that the class number of $\mathbb{k}_0 = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$ is odd. Consider the extension $\mathbb{K}_0/\mathbb{k}_0$. Then according to [18], the rank of the 2-class group of $\mathbb{K}_0$ is given by the formula:

$$\operatorname{rank}(\mathrm{Cl}_2(\mathbb{K}_0)) = r - e - 1,$$

where $r$ is the number of finite and infinite primes of $\mathbb{k}_0$ that ramify in $\mathbb{K}_0/\mathbb{k}_0$ and $e$ is defined by $2^e = [E_{\mathbb{k}_0} : E_{\mathbb{k}_0} \cap N_{\mathbb{K}_0/\mathbb{k}_0}(\mathbb{K}_0^*)] \leq 2^4$. As $r = 8$ (4 primes above 2 and 4 above $q$), $\operatorname{rank}(\mathrm{Cl}_2(\mathbb{K}_0)) = r - e - 1 = 8 - e - 1 \geq 3$. On the other hand, the Schreier's inequality implies that $\operatorname{rank}(\mathrm{Cl}_2(\mathbb{K}_0)) \leq 3$, concluding the proof. □

**Theorem 5.7.** *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be three positive prime integers satisfying $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -\left(\frac{p_1}{p_2}\right) = 1$. Then*

$$\operatorname{rank}(\mathrm{Cl}_2(\Bbbk_2^{(1)})) \geq 2.$$

*Proof.* Suppose that $\operatorname{rank}(\mathrm{Cl}_2(\Bbbk_2^{(1)})) = 1$, then, by Lemma 2.2, the proof of Theorem 5.1 and class field theory, the Galois groups $\mathrm{Gal}(\Bbbk_2^{(2)}/\mathbb{K}_1)$ and $\mathrm{Gal}(\Bbbk_2^{(2)}/\mathbb{K}_2)$ are metacyclic, and since $\mathbb{K}_0$ is an unramified quadratic extension of $\mathbb{K}_1$, then Lemma 2.1 implies that $\operatorname{rank}(\mathrm{Cl}_2(\mathbb{K}_0)) \leq 2$, which contradicts Lemma 5.6. □

**Example 5.8.** For $d = 47158 = 2 \cdot 73 \cdot 17 \cdot 19$, we have $\mathrm{Cl}_2(\Bbbk_2^{(1)})$ is of type $(2, 4)$, and for $d = 59942 = 2 \cdot 17 \cdot 41 \cdot 43$, we have $\mathrm{Cl}_2(\Bbbk_2^{(1)})$ is of type $(2, 4)$.

### 5.2.3 Case 3: $\left(\frac{2}{p_1}\right) = -\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{q}{p_1}\right) = -\left(\frac{q}{p_2}\right) = 1$

Using similar arguments as above, we prove the following two theorems.

**Theorem 5.9.** *Let $\delta \in \{2p_1, 2p_2, q\}$ be such that $\delta(x \pm 1)$ is a square in $\mathbb{N}$. The group $\mathrm{Cl}_2(\Bbbk_2^{(1)}) \simeq \mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk_2^{(1)})$ is cyclic non-elementary if and only if one of the following two conditions is satisfied:*

I. i. *$\delta \neq 2p_2$ and $\delta(z \pm 1)$ is not a square in $\mathbb{N}$, and*

ii. *at least one of the elements $\left\{\left(\frac{A}{p_1}\right), \left(\frac{2q}{p_1}\right)_4\right\}$ equals $-1$, and*

iii. *either*

a. *$\delta = q$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4$, or*

b. *$\delta \neq q$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$*

II. i. *$\delta = 2p_2$ or $\delta(z \pm 1)$ is a square in $\mathbb{N}$ or $\left(\frac{A}{p_1}\right) = \left(\frac{2q}{p_1}\right)_4 = 1$, and*

ii. *either*

a. *$\delta = q$ and $\left(\frac{p_1}{p_2}\right)_4 \neq \left(\frac{p_2}{p_1}\right)_4$, or*

b. *$\delta \neq q$ and one of $\left(\frac{p_1}{p_2}\right)_4, \left(\frac{p_2}{p_1}\right)_4$ is equal to $-1$.*

**Example 5.10.** Keep the notation of Example 5.3. For the case $2p_2(x\pm1)$ is a square and $t_1=-1$ or $t_2=-1$, we have

| $d=2p_1p_2q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $17630=2\cdot41\cdot5\cdot43$ | 2 | 2 | 2 | 1 | 1 | 1 | $-1$ | 8 | 8 | 32 | 8 | 32 | [8] |
| $29614=2\cdot17\cdot13\cdot67$ | 2 | 2 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 16 | 8 | 32 | [4] |
| $34238=2\cdot17\cdot53\cdot19$ | 2 | 2 | 1 | 1 | 1 | $-1$ | $-1$ | 8 | 8 | 32 | 8 | 16 | [8] |
| $41830=2\cdot89\cdot5\cdot47$ | 2 | 2 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | 16 | 16 | 32 | 16 | 16 | [4] |
| $59630=2\cdot89\cdot5\cdot67$ | 2 | 2 | 1 | $-1$ | 1 | $-1$ | $-1$ | 8 | 8 | 16 | 8 | 16 | [4] |
| $69782=2\cdot41\cdot37\cdot23$ | 2 | 2 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 16 | 8 | 32 | [4] |
| $91078=2\cdot113\cdot13\cdot31$ | 2 | 2 | 2 | $-1$ | $-1$ | 1 | $-1$ | 16 | 16 | 32 | 16 | 32 | [12] |

**Theorem 5.11.** *Let $\delta\in\{2p_1,2p_2,q\}$ be such that $\delta(x\pm1)$ is a square in $\mathbb{N}$. The order $\#\mathrm{Cl}_2(\Bbbk_2^{(1)})=2$ if and only if the following conditions are satisfied:*

i. *$\delta\neq2p_2$ and $\delta(z\pm1)$ is not a square in $\mathbb{N}$,*

ii. *at least one of the elements $\left\{\left(\frac{A}{p_1}\right),\left(\frac{2q}{p_1}\right)_4\right\}$ equals $-1$,*

iii. $\left(\frac{p_1}{p_2}\right)_4\neq\left(\frac{p_2}{p_1}\right)_4$.

**Example 5.12.** Keep the notation of Example 5.3. For the case $q(x\pm1)$ is a square, $q(z+1)$ and $q(z-1)$ are not squares, ($\alpha=-1$ or $s=-1$) and $t_1\neq t_2$ we have

| $d=2p_1p_2q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $9430=2\cdot41\cdot5\cdot23$ | 2 | 1 | 2 | 1 | $-1$ | 1 | $-1$ | 16 | 16 | 16 | 16 | 16 | [2] |
| $20774=2\cdot17\cdot13\cdot47$ | 2 | 1 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 8 | 8 | 16 | [2] |
| $94054=2\cdot41\cdot37\cdot31$ | 2 | 1 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 8 | 8 | 16 | [2] |
| $102638=2\cdot73\cdot37\cdot19$ | 2 | 1 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 8 | 8 | 16 | [6] |

### 5.2.4 Case 4: $\left(\frac{2}{p_1}\right)=-\left(\frac{2}{p_2}\right)=\left(\frac{p_1}{p_2}\right)=\left(\frac{q}{p_1}\right)=\left(\frac{q}{p_2}\right)=1$

Using similar arguments as above, we prove the following two theorems.

**Theorem 5.13.** *Let $\delta\in\{2p_1,p_2,2q\}$ such that $\delta(x\pm1)$ is a square in $\mathbb{N}$. The group $\mathrm{Cl}_2(\Bbbk_2^{(1)})\simeq\mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk_2^{(1)})$ is cyclic non-elementary if and only if one of the following two conditions is satisfied:*

I. i. *$\delta=p_2$ (resp. $\delta\neq p_2$) and $(z\pm1)$ (resp. $\delta(z\pm1)$) is not a square in $\mathbb{N}$,*

ii. *at least one of the elements $\left\{\left(\frac{A}{p_1}\right),\left(\frac{2q}{p_1}\right)_4\right\}$ equals $-1$,*

iii. *either*

a. *$\delta=2q$ and $\left(\frac{p_1}{p_2}\right)_4=\left(\frac{p_2}{p_1}\right)_4$, or*

b. *$\delta\neq2q$ and $\left(\frac{p_1}{p_2}\right)_4=\left(\frac{p_2}{p_1}\right)_4=1$.*

II. i. $(z \pm 1)$ *(resp.* $\delta(z \pm 1)$*) is a square in* $\mathbb{N}$ *if* $\delta = p_2$ *(resp.* $\delta \neq p_2$*) or* $\left(\frac{A}{p_1}\right) = \left(\frac{2q}{p_1}\right)_4 = 1$,

ii. *either*

a. $\delta = 2q$ *and* $\left(\frac{p_1}{p_2}\right)_4 \neq \left(\frac{p_2}{p_1}\right)_4$, *or*

b. $\delta \neq 2q$ *and one of* $\left(\frac{p_1}{p_2}\right)_4, \left(\frac{p_2}{p_1}\right)_4$ *is equal to* $-1$.

**Example 5.14.** Keep the notation of Example 5.3. For the case $p_2(x \pm 1)$ is a square, ($z \pm 1$ is a square or ($\alpha = s = 1$)) and ($t_1 = -1$ or $t_2 = -1$), we have

| $d = 2p_1p_2q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $84422 = 2 \cdot 17 \cdot 13 \cdot 191$ | 2 | 2 | 2 | 1 | $-1$ | $-1$ | 1 | 8 | 8 | 16 | 8 | 32 | [12] |
| $113102 = 2 \cdot 97 \cdot 53 \cdot 11$ | 2 | 1 | 2 | 1 | 1 | 1 | $-1$ | 8 | 8 | 16 | 8 | 16 | [4] |
| $123710 = 2 \cdot 89 \cdot 5 \cdot 139$ | 2 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | 8 | 8 | 16 | 8 | 16 | [8] |
| $139334 = 2 \cdot 233 \cdot 13 \cdot 23$ | 2 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | 8 | 8 | 16 | 8 | 16 | [8] |
| $159310 = 2 \cdot 89 \cdot 5 \cdot 179$ | 2 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | 8 | 8 | 32 | 8 | 16 | [16] |

**Theorem 5.15.** *Let* $\delta \in \{2p_1, p_2, 2q\}$ *such that* $\delta(x \pm 1)$ *is a square in* $\mathbb{N}$. *The order* $\#\mathrm{Cl}_2(\mathbb{k}_2^{(1)}) = 2$ *if and only if the following conditions are satisfied:*

i. $\delta = p_2$ *(resp.* $\delta \neq p_2$*) and* $(z \pm 1)$ *(resp.* $\delta(z \pm 1)$*) is not a square in* $\mathbb{N}$,

ii. *at least one of the elements* $\left\{\left(\frac{A}{p_1}\right), \left(\frac{2q}{p_1}\right)_4\right\}$ *equals* $-1$,

iii. $\left(\frac{p_1}{p_2}\right)_4 \neq \left(\frac{p_2}{p_1}\right)_4$.

**Example 5.16.** Keep notations of Example 5.3. For the case: $p_2(x \pm 1)$ is a square, $z + 1$ and $z - 1$ are not squares, ($\alpha = -1$ or $s = -1$) and $t_1 \neq t_2$.

| $d = 2p_1.p_2.q$ | $q_1$ | $q_2$ | $q_3$ | $\alpha$ | $s$ | $t_1$ | $t_2$ | $n$ | $n_1$ | $n_2$ | $n_3$ | $q_0$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $45526 = 2 \cdot 17 \cdot 13 \cdot 103$ | 2 | 1 | 2 | $-1$ | $-1$ | $-1$ | 1 | 8 | 8 | 8 | 8 | 16 | [6] |
| $53710 = 2 \cdot 41 \cdot 5 \cdot 131$ | 2 | 1 | 2 | $-1$ | 1 | 1 | $-1$ | 8 | 8 | 8 | 8 | 16 | [2] |
| $56134 = 2 \cdot 17 \cdot 13 \cdot 127$ | 2 | 1 | 2 | $-1$ | 1 | $-1$ | 1 | 16 | 16 | 16 | 16 | 16 | [6] |
| $63438 = 2 \cdot 97 \cdot 109 \cdot 3$ | 2 | 1 | 2 | $-1$ | 1 | 1 | $-1$ | 8 | 8 | 8 | 8 | 16 | [2] |

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*Received:* October 15, 2019
*Accepted for publication:* February 28, 2022
*Communicated by:* Atilla Berczes