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Note on geodesics of cotangent bundle with Berger-type deformed Sasaki metric over standard Kähler manifold

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Abstract. In this paper, first, we introduce the Berger-type deformed Sasaki metric on the cotangent bundle T^*M over a standard Kähler manifold (M^{2m}, J, g) and investigate the Levi-Civita connection of this metric. Secondly, we present the unit cotangent bundle equipped with Berger-type deformed Sasaki metric, and we investigate the Levi-Civita connection. Finally, we study the geodesics on the cotangent bundle and the unit cotangent bundle concerning the Berger-type deformed Sasaki metric.

1 Introduction

One can define natural Riemannian metrics on the cotangent bundle of a Riemannian manifold. Their construction makes use of the Levi-Civita connection. Among them, the so-called Sasaki metric is of particular interest. That is why A.A. Salimov and F. Agca have studied the geometry of a cotangent bundle equipped with the Sasaki metric [7, 8], A.A. Salimov and F. Ocak [9]. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on the cotangent bundle. This is the reason why some authors have attempted to search for different metrics on the cotangent bundle, which are different deformations of the Sasaki metric. In this direction, some authors defined and studied some metrics, which are called Cheeger-Gromoll metric [2] or g-Natural Metrics [1, 19] or new metric in the cotangent bundle [6, 5] or a new class of metrics on the cotangent bundle [12, 20]. In another direction, A. Zagane has introduced the notion of Berger-type deformed Sasaki metric on the cotangent bundle over anti-paraKähler manifold [13, 14, 17]. For deformations of the Sasaki metric or Cheeger-Gromoll metric, we also refer to [3, 15, 16, 18].

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The main idea in this paper, firstly, we introduce the Berger-type deformed Sasaki metric on the cotangent bundle T^*M over a standard Kähler manifolds manifold (M^{2m}, J, g) and we investigate the formulas relating to its Levi-Civita connection (Theorem 3.5). Secondly, we present the unit cotangent bundle equipped with the Berger-type deformed Sasaki metric, and we establish the formulas relating to the Levi-Civita connection of this metric (Theorem 4.1). In the last section, we study the geodesics on the cotangent bundle (Theorem 5.2, Corollary 5.3, Corollary 5.4 and Theorem 5.8) and on unit cotangent bundle (Theorem 5.10, Theorem 5.11, Theorem 5.12 and Theorem 5.14).

2 Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{\bar{i}=\overline{m+1,2m}}$ on T^*M, where p_i is the component of covector p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, denote by $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M \tag{1}$$

of the tangent bundle to T^*M at any $(x, p) \in T^*M$ into vertical subspace

$$V_{(x,p)}T^*M = \ker(d\pi_{(x,p)}) = \{\omega_i \partial_{\bar{i}}|_{(x,p)},\ \omega_i \in \mathbb{R}\}, \tag{2}$$

and the horizontal subspace

$$H_{(x,p)}T^*M = \{X^i \partial_i|_{(x,p)} + X^i p_a \Gamma^a_{hi} \partial_{\bar{h}}|_{(x,p)},\ X^i \in \mathbb{R}\}. \tag{3}$$

Note that the map $X \to {}^H X = X^i \partial_i|_{(x,p)} + X^i p_a \Gamma^a_{hi} \partial_{\bar{h}}|_{(x,p)}$ is an isomorphism between the vector spaces $T_x M$ and $H_{(x,p)}T^*M$.

Similarly, the map $\omega \to {}^V \omega = \omega_i \partial_{\bar{i}}|_{(x,p)}$ is an isomorphism between the vector spaces $T^*_x M$ and $V_{(x,p)}T^*M$. Obviously, each tangent vector $Z \in T_{(x,p)}T^*M$ can be written in the form $Z = {}^H X + {}^V \omega$, where $X \in T_x M$ and $\omega \in T^*_x M$ are uniquely determined.

Let $X = X^i \partial_i$ and $\omega = \omega_i dx^i$ be local expressions in $(U, x^i)_{i=\overline{1,m}}$, of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal lift ${}^H X \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift ${}^V \omega \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$\begin{aligned} {}^H X &= X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{i}}, && (4) \\ {}^V \omega &= \omega_i \partial_{\bar{i}}, && (5) \end{aligned}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, (see [11] for more details).

From (4) and (5) we see that ${}^H(\partial_i)$ and ${}^V(dx^i)$ have respectively local expressions of the form

$$
\begin{aligned}
{}^H(\partial_i) &= \partial_i + p_a \Gamma^a_{hi} \partial_{\bar{h}}, & (6)\\
{}^V(dx^i) &= \partial_{\bar{i}}. & (7)
\end{aligned}
$$

The set of vector fields $\{{}^H(\partial_i)\}$ on $\pi^{-1}(U)$ define a local frame for HT^*M over $\pi^{-1}(U)$ and the set of vector fields $\{{}^V(dx^i)\}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$. The set $\{{}^H(\partial_i), {}^V(dx^i)\}$ define a local frame on T^*M, adapted to the direct sum decomposition (1).

In particular, we have the vertical spray ${}^Vp$ on T^*M defined by

$$
{}^Vp = p_i {}^V(dx^i) = p_i \partial_{\bar{i}}, \tag{8}
$$

${}^Vp$ is also called the canonical or Liouville vector field on T^*M.

Lemma 2.1 ([11]). *Let (M^m, g) be a Riemannian manifold, ∇ be the Levi-Civita connection, and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

(1) $[{}^V\omega, {}^V\theta] = 0$,

(2) $[{}^HX, {}^V\theta] = {}^V(\nabla_X \theta)$,

(3) $[{}^HX, {}^HY] = {}^H[X, Y] + {}^V(pR(X, Y))$,

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, such that $pR(X, Y) = p_a R^a_{ijk} X^i Y^j \, dx^k$, where R^a_{ijk} are local components of R on (M^m, g).

Let (M^m, g) be a Riemannian manifold, we define the map

$$
\begin{array}{ccc} \Im^0_1(M) & \to & \Im^1_0(M) \\ \omega & \mapsto & \widetilde{\omega} \end{array} \quad by \quad g(\widetilde{\omega}, X) = \omega(X),
$$

for all $X \in \Im^1_0(M)$. Locally if $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have

$$
\widetilde{\omega} = g^{ij} \omega_i \partial_j, \tag{9}
$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

The scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T^*_xM by

$$
g^{-1}(\omega, \theta) = g(\widetilde{\omega}, \tilde{\theta}) = g^{ij} \omega_i \theta_j,
$$

for all $x \in M$ and $\omega, \theta \in \Im^0_1(M)$. In this case we have $\widetilde{\omega} = g^{-1} \circ \omega$.

We also define the map

$$
\begin{array}{ccc} \Im^1_0(M) & \to & \Im^0_1(M) \\ X & \mapsto & \widetilde{X} \end{array} \quad by \quad \widetilde{X}(Y) = g(X, Y),
$$

for all $Y \in \Im_0^1(M)$. Locally if $X = X^i\partial_i \in \Im_0^1(M)$, we have

$$\widetilde{X} = g_{ij}X^i dx^j, \tag{10}$$

we also write $\widetilde{X} = g \circ X$.

Lemma 2.2 ([14]). *Let (M, g) be a Riemannian manifold, then we have the following:*

$$\widetilde{\widetilde{\omega}} = \omega \quad , \quad \widetilde{\widetilde{X}} = X, \tag{11}$$

$$g^{-1}(\omega, \theta J) = g(J\widetilde{\omega}, \tilde{\theta}), \tag{12}$$

$$\nabla_X \widetilde{\omega} = \widetilde{\nabla_X \omega}, \tag{13}$$

$$X g^{-1}(\omega, \theta) = g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta), \tag{14}$$

$$\widetilde{\omega R(X, Y)} = R(Y, X)\widetilde{\omega}, \tag{15}$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$ and $J \in \Im_1^1(M)$, where ∇ is the Levi-Civita connection of (M, g).

3 Berger-type deformed Sasaki metric

Let M^r be an r-dimensional differentiable manifold. An almost complex structure J on M is a $(1,1)$-tensor field on M such that $J^2 = -I$, (I is the $(1,1)$-identity tensor field on M. The pair (M^r, J) is called an almost complex manifold. Since every almost complex manifold is even dimensional, We will take $r = 2m$. Also, note that every complex manifold (Topological space endowed with a holomorphic atlas) carries a natural almost complex structure [4]. The integrability of a complex structure J on M is equivalent to the vanishing of the Nijenhuis tensor N_J:

$$N_J(X, Y) = [JX, JY] - J\,[JX, Y] - J\,[X, JY] - [X, Y] \tag{16}$$

for all vector fields X, Y on M.

On an almost complex manifold (M^{2m}, J), a Hermitian metric is a Riemannian metric g on M such that

$$g(JX, Y) = -g(X, JY) \Leftrightarrow g(JX, JY) = g(X, Y), \tag{17}$$

or from (12) equivalently

$$g^{-1}(\omega J, \theta) = -g^{-1}(\omega, \theta J) \Leftrightarrow g^{-1}(\omega J, \theta J) = g^{-1}(\omega, \theta), \tag{18}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

The almost complex manifold (M^{2m}, J) having the Hermitian metric g is called an almost Hermitian manifold. Let (M^{2m}, J, g) be an almost Hermitian manifold. We define the fundamental or Kähler 2-form Ω on M by

$$\Omega(X, Y) = g(X, JY), \tag{19}$$

for any vector fields X and Y on M. A Hermitian metric g on an almost Hermitian manifold M^{2m} is called a standard Kähler metric if the fundamental 2-form Ω is closed, i.e., $d\Omega = 0$. In this case, the triple (M^{2m}, J, g) is called an almost standard Kähler manifold. If the almost complex structure is integrable, then the triple (M^{2m}, J, g) is called a standard Kähler manifold. Moreover, the following conditions are equivalent

1. $\nabla J = 0$,
2. $\nabla \Omega = 0$,
3. $N_J = 0$ and $d\Omega = 0$,

where ∇ is the Levi-Civita connection of g [4].

As a result, the almost Hermitian manifold (M^{2m}, J, g) is a standard Kähler manifold if and only if $\nabla J = 0$. Using the formula

$$\omega(\nabla_X J) = \nabla_X(\omega J) - (\nabla_X \omega)J. \tag{20}$$

Also, the almost Hermitian manifold (M^{2m}, J, g) is a standard Kähler manifold if and only if

$$\nabla_X(\omega J) = (\nabla_X \omega)J. \tag{21}$$

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$. The Riemannian curvature tensor R of a standard Kähler manifold possess the following properties:

$$\begin{cases} R(Y,Z)J &= JR(Y,Z), \\ R(JY,JZ) &= R(Y,Z), \\ R(JY,Z) &= -R(Y,JZ), \end{cases} \tag{22}$$

for all $Y, Z \in \Im^1_0(M)$.

Lemma 3.1. *Let (M^{2m}, J, g) be an almost Hermitian manifold. We have the following:*

$$\widetilde{\omega J} \;=\; -J\widetilde{\omega}, \tag{23}$$

for any $\omega \in \Im^0_1(M)$.

Definition 3.2. Let (M^{2m}, J, g) be an almost Hermitian manifold and T^*M be its cotangent bundle. A fiber-wise Berger-type deformation of the Sasaki metric noted ${}^{BS}g$ is defined on T^*M by

$$\begin{aligned} {}^{BS}g({}^{H}X, {}^{H}Y) &= g(X,Y), \\ {}^{BS}g({}^{H}X, {}^{V}\theta) &= 0, \\ {}^{BS}g({}^{V}\omega, {}^{V}\theta) &= g^{-1}(\omega,\theta) + \delta^2 g^{-1}(\omega, pJ) g^{-1}(\theta, pJ), \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, where δ is some constant [13],[14],[17], for version tangent bundle see [10].

In the following, we put $\lambda = 1+\delta^2 r^2$ and $r^2 = g^{-1}(p,p) = |p|^2$. where $|.|$ denote the norm with respect to g^{-1}.

Lemma 3.3. *Let (M^{2m}, J, g) be a standard Kähler manifold and $f : \mathbb{R} \to \mathbb{R}$ a smooth function, we have the following:*

1. ${}^{H}X(f(r^2)) = 0$,
2. ${}^{V}\theta(f(r^2)) = 2f'(r^2)g^{-1}(\theta, p)$,
3. ${}^{H}Xg^{-1}(\omega, p) = g^{-1}(\nabla_X\omega, p)$,
4. ${}^{V}\theta g^{-1}(\omega, p) = g^{-1}(\omega, \theta)$,
5. ${}^{H}Xg^{-1}(\omega, pJ) = g^{-1}(\nabla_X\omega, pJ)$,
6. ${}^{V}\theta g^{-1}(\omega, pJ) = g^{-1}(\omega, \theta J)$,

for all $X \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where $r^2 = g^{-1}(p,p)$, see [12],[14].

Lemma 3.4. *Let (M^{2m}, J, g) be a standard Kähler manifold and T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$, we have the following:*

$$\begin{aligned}
(1)\ \ {}^{H}X\,{}^{BS}g({}^{V}\omega, {}^{V}\theta) &= {}^{BS}g({}^{V}(\nabla_X\omega), {}^{V}\theta) + {}^{BS}g({}^{V}\omega, {}^{V}(\nabla_X\theta)),\\
(2)\ \ {}^{V}\eta\,{}^{BS}g({}^{V}\omega, {}^{V}\theta) &= \delta^2 g^{-1}(\omega, \eta J)g^{-1}(\theta, pJ) + \delta^2 g^{-1}(\omega, pJ)g^{-1}(\theta, \eta J),
\end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, see as well [14]

We shall calculate the Levi-Civita connection ${}^{BS}\nabla$ of T^*M with Berger-type deformed Sasaki metric ${}^{BS}g$. The Koszul formula characterizes this connection:

$$\begin{aligned}
2\,{}^{BS}g({}^{BS}\nabla_U V, W) = {} & U\,{}^{BS}g(V,W) + V\,{}^{BS}g(W,U) - W\,{}^{BS}g(U,V)\\
& + {}^{BS}g(W,[U,V]) + {}^{BS}g(V,[W,U]) - {}^{BS}g(U,[V,W]),
\end{aligned} \tag{24}$$

for all $U, V, W \in \Im^1_0(T^*M)$.

Theorem 3.5. *Let (M^{2m}, J, g) be a standard Kähler manifold and T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$, then we have the following formulas:*

$$\begin{aligned}
(i)\ \ {}^{BS}\nabla_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) + \frac{1}{2}{}^{V}(pR(X,Y)),\\
(ii)\ \ {}^{BS}\nabla_{{}^{H}X}{}^{V}\theta &= {}^{V}(\nabla_X\theta) + \frac{1}{2}\big({}^{H}(R(\widetilde{p},\widetilde{\theta})X) - \delta^2 g^{-1}(\theta, pJ){}^{H}(R(\widetilde{p}, J\widetilde{p})X)\big),\\
(iii)\ \ {}^{BS}\nabla_{{}^{V}\omega}{}^{H}Y &= \frac{1}{2}\big({}^{H}(R(\widetilde{p},\widetilde{\omega})Y) - \delta^2 g^{-1}(\omega, pJ){}^{H}(R(\widetilde{p}, J\widetilde{p})Y)\big),\\
(iv)\ \ {}^{BS}\nabla_{{}^{V}\omega}{}^{V}\theta &= \delta^2\big(g^{-1}(\omega, pJ){}^{V}(\theta J) + g^{-1}(\theta, pJ){}^{V}(\omega J)\big)\\
&\quad - \frac{\delta^4}{\lambda}\big(g^{-1}(\omega, pJ)g^{-1}(\theta, p) + g^{-1}(\omega, p)g^{-1}(\theta, pJ)\big){}^{V}(pJ),
\end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$*, where* ∇ *is the Levi-Civita connection of* (M^{2m}, J, g) *and* R *is its curvature tensor, (for anti-paraKähler manifold, see [14]).*

Proof. The proof of Theorem 3.5 follows directly from Kozul formula (24), Lemma 2.1, Definition 3.2 and Lemma 3.4.

(1) Direct calculations give

$$\begin{aligned} 2\,{}^{BS}g({}^{BS}\nabla_{{}^HX}{}^HY, {}^HZ) &= {}^HX\,{}^{BS}g({}^HY, {}^HZ) + {}^HY\,{}^{BS}g({}^HZ, {}^HX) - {}^HZ\,{}^{BS}g({}^HX, {}^HY) \\ &\quad + {}^{BS}g({}^HZ, [{}^HX, {}^HY]) + {}^{BS}g({}^HY, [{}^HZ, {}^HX]) - {}^{BS}g({}^HX, [{}^HY, {}^HZ]) \\ &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) - g(X,[Y,Z]) \\ &= 2g(\nabla_X Y, Z) \\ &= 2\,{}^{BS}g({}^H(\nabla_X Y), {}^HZ), \\ {}^{BS}g({}^{BS}\nabla_{{}^HX}{}^HY, {}^V\eta) &= {}^HX\,{}^{BS}g({}^HY, {}^V\eta) + {}^HY\,{}^{BS}g({}^V\eta, {}^HX) - {}^V\eta\,{}^{BS}g({}^HX, {}^HY) \\ &\quad + {}^{BS}g({}^V\eta, [{}^HX, {}^HY]) + {}^{BS}g({}^HY, [{}^V\eta, {}^HX]) - {}^{BS}g({}^HX, [{}^HY, {}^V\eta]) \\ &= {}^{BS}g({}^V\eta, [{}^HX, {}^HY]) \\ &= {}^{BS}g({}^V(pR(X,Y)), {}^V\eta), \end{aligned}$$

Thus, we find

$${}^{BS}\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y)).$$

(2) In a similar way,

$$\begin{aligned} 2\,{}^{BS}g({}^{BS}\nabla_{{}^HX}{}^V\theta, {}^HZ) &= {}^HX\,{}^{BS}g({}^V\theta, {}^HZ) + {}^V\theta\,{}^{BS}g({}^HZ, {}^HX) - {}^HZ\,{}^{BS}g({}^HX, {}^V\theta) \\ &\quad + {}^{BS}g({}^HZ, [{}^HX, {}^V\theta]) + {}^{BS}g({}^V\theta, [{}^HZ, {}^HX]) - {}^{BS}g({}^HX, [{}^V\theta, {}^HZ]) \\ &= {}^{BS}g({}^V\theta, [{}^HZ, {}^HX]) \\ &= {}^{BS}g({}^V(pR(Z,X)), {}^V\theta) \\ &= g^{-1}(pR(Z,X), \theta) + \delta^2 g^{-1}(pR(Z,X), pJ)g^{-1}(\theta, pJ), \end{aligned}$$

From (15), we have

$$\begin{aligned} g^{-1}(pR(Z,X), \theta) &= g(\widetilde{\mathrm{pR(Z,X)}}, \tilde{\theta}) = g(R(X,Z)\tilde{p}, \tilde{\theta}) = g(R(\tilde{p}, \tilde{\theta})X, Z) \\ &= {}^{BS}g({}^H(R(\tilde{p}, \tilde{\theta})X), {}^HZ). \end{aligned}$$

On the other hand, using (12), (15) and (22), we have

$$\begin{aligned} g^{-1}(pR(Z,X), pJ) &= g(J(\widetilde{\mathrm{pR(Z,X)}}), \tilde{p}) = g(JR(X,Z)\tilde{p}, \tilde{p}) \\ &= g(R(X,Z)J\tilde{p}, \tilde{p}) = g(R(J\tilde{p}, \tilde{p})X, Z) \\ &= {}^{BS}g({}^H(R(J\tilde{p}, \tilde{p})X), {}^HZ), \end{aligned}$$

then,

$$2\,{}^{BS}g({}^{BS}\nabla_{{}^HX}{}^V\theta, {}^HZ) = {}^{BS}g({}^H(R(\tilde{p},\tilde{\theta})X) - \delta^2 g^{-1}(\theta, pJ)^H(R(\tilde{p}, J\tilde{p})X), {}^HZ),$$

and also with direct calculations, we obtain

$$\begin{aligned} 2\,{}^{BS}g({}^{BS}\nabla_{{}^HX}{}^V\theta, {}^V\eta) &= {}^HX\,{}^{BS}g({}^V\theta, {}^V\eta) + {}^V\theta\,{}^{BS}g({}^V\eta, {}^HX) - {}^V\eta\,{}^{BS}g({}^HX, {}^V\theta) \\ &\quad + {}^{BS}g({}^V\eta, [{}^HX, {}^V\theta]) + {}^{BS}g({}^V\theta, [{}^V\eta, {}^HX]) - {}^{BS}g({}^HX, [{}^V\theta, {}^V\eta]) \\ &= {}^HX\,{}^{BS}g({}^V\theta, {}^V\eta) + {}^{BS}g({}^V\eta, [{}^HX, {}^V\theta]) + {}^{BS}g({}^V\theta, [{}^V\eta, {}^HX]). \end{aligned}$$

Using the first formula of Lemma 3.4 we have:

$$\begin{aligned} 2\,{}^{BS}g({}^{BS}\nabla_{{}^HX}{}^V\theta, {}^V\eta) &= {}^{BS}g({}^V(\nabla_X\theta), {}^V\eta) + {}^{BS}g({}^V\theta, {}^V(\nabla_X\eta)) \\ &\quad + {}^{BS}g({}^V\eta, {}^V(\nabla_X\theta)) - {}^{BS}g({}^V\theta, {}^V(\nabla_X\eta)) \\ &= 2\,{}^{BS}g({}^V(\nabla_X\theta), {}^V\eta). \end{aligned}$$

Which gives the formula

$${}^{BS}\nabla_{{}^HX}{}^V\theta = {}^V(\nabla_X\theta) + \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\theta})X) - g^{-1}(\theta, pJ)^H(R(\tilde{p}, J\tilde{p})X)\big).$$

Similar calculations obtain the other formulas. □

As a consequence of Theorem 3.5, we get the following Lemma.

Lemma 3.6. *Let (M^n, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric, then*

$$\begin{aligned} {}^{BS}\nabla_{{}^HX}{}^Vp &= 0, \\ {}^{BS}\nabla_{{}^Vp}{}^HX &= 0, \\ {}^{BS}\nabla_{{}^V\omega}{}^Vp &= {}^V\omega + \frac{\delta^2}{\lambda}g^{-1}(\omega, pJ)^V(pJ), \\ {}^{BS}\nabla_{{}^Vp}{}^V\omega &= \frac{\delta^2}{\lambda}g^{-1}(\omega, pJ)^V(pJ), \\ {}^{BS}\nabla_{{}^Vp}{}^Vp &= {}^Vp, \end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$.

4 Unit cotangent bundle with Berger-type deformed Sasaki metric

The unit cotangent (sphere) bundle over a standard Kähler manifold (M^n, J, g), is the hyper-surface

$$T^*_1M = \big\{(x,p) \in T^*M,\ g^{-1}(p,p) = 1\big\}. \qquad (25)$$

The unit normal vector field to T_1^*M is given by

$$\begin{array}{rcl}\mathcal{N}: T^*M & \to & T(T^*M)\\ (x,p) & \mapsto & \mathcal{N}_{(x,p)} = {}^V p.\end{array} \tag{26}$$

The tangential lift ${}^T\omega$ with respect to ${}^{BS}g$ of a covector $\omega \in T_x^*M$ to $(x,p) \in T_1^*M$ as the tangential projection of the vertical lift of ω to (x,p) with respect to $\mathcal{N}$, that is

$${}^T\omega = {}^V\omega - {}^{BS}g_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} = {}^V\omega - g_x^{-1}(\omega,p){}^V p_{(x,p)}.$$

For the sake of notational clarity, we will use $\overline{\omega} = \omega - g^{-1}(\omega,p)p$, then ${}^T\omega = {}^V\overline{\omega}$.

From the above, we get the direct sum decomposition

$$T_{(x,p)}T^*M = T_{(x,p)}T_1^*M \oplus span\{\mathcal{N}_{(x,p)}\} = T_{(x,p)}T_1^*M \oplus span\{{}^V p_{(x,p)}\}, \tag{27}$$

where $(x,p) \in T_1^*M$.

Indeed, if $W \in T_{(x,p)}T^*M$, then they exist $X \in T_xM$ and $\omega \in T_x^*M$, such that

$$\begin{array}{rcl}W & = & {}^HX + {}^V\omega\\ & = & {}^HX + {}^T\omega + {}^{BS}g_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)}\\ & = & {}^HX + {}^T\omega + g_x^{-1}(\omega,p){}^V p_{(x,p)}.\end{array} \tag{28}$$

From the (28) we can say that the tangent space $T_{(x,p)}T_1^*M$ of T_1^*M at (x,p) is given by

$$T_{(x,p)}T_1^*M = \{{}^HX + {}^T\omega \,/\, X \in T_xM, \omega \in \{p\}^\perp \subset T_x^*M\},$$

where $\{p\}^\perp = \big\{\omega \in T_x^*M,\ g^{-1}(\omega,p) = 0\big\}$. Hence $T_{(x,p)}T_1^*M$ is spanned by vectors of the form ${}^HX$ and ${}^T\omega$.

Given a covector field ω on M, the tangential lift ${}^T\omega$ of ω is given by

$${}^T\omega_{(x,p)} = \big({}^V\omega - {}^{BS}g({}^V\omega, \mathcal{N})\mathcal{N}\big)_{(x,p)} = {}^V\omega_{(x,p)} - g_x^{-1}(\omega_x,p){}^V p_{(x,p)}. \tag{29}$$

If ${}^{BS}\hat{g}$ is the Riemannian metric on T_1^*M induced by ${}^{BS}g$, then the Levi-Civita connection ${}^{BS}\widehat{\nabla}$ of $(T_1^*M, {}^{BS}\hat{g})$ is characterized by the formula:

$${}^{BS}\widehat{\nabla}_U V = {}^{BS}\nabla_U V - {}^{BS}g({}^{BS}\nabla_U V, \mathcal{N})\mathcal{N}, \tag{30}$$

for all $U, V \in \Im_0^1(T^*M)$.

Theorem 4.1. *Let (M^n, J, g) be a standard Kähler manifold and $(T_1^*M, {}^{BS}\hat{g})$ its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric, then we have the following formulas:*

$$\begin{array}{rcl}{}^{BS}\widehat{\nabla}_{{}^HX}{}^HY & = & {}^H(\nabla_X Y) + \dfrac{1}{2}{}^T(pR(X,Y)),\\ {}^{BS}\widehat{\nabla}_{{}^HX}{}^T\theta & = & {}^T(\nabla_X\theta) + \dfrac{1}{2}\big({}^H(R(\tilde{p},\tilde{\theta})X) - \delta^2 g^{-1}(\theta,pJ){}^H(R(\tilde{p},J\tilde{p})X)\big),\\ {}^{BS}\widehat{\nabla}_{{}^T\omega}{}^HY & = & \dfrac{1}{2}\big({}^H(R(\tilde{p},\tilde{\omega})Y) - \delta^2 g^{-1}(\omega,pJ){}^H(R(\tilde{p},J\tilde{p})Y)\big),\\ {}^{BS}\widehat{\nabla}_{{}^T\omega}{}^T\theta & = & -g^{-1}(\theta,p){}^T\omega + \delta^2\big(g^{-1}(\omega,pJ){}^T(\theta J) + g^{-1}(\theta,pJ){}^T(\omega J)\big)\\ & & -\delta^2\big(g^{-1}(\omega,pJ)g^{-1}(\theta,p) + g^{-1}(\omega,p)g^{-1}(\theta,pJ)\big){}^T(pJ),\end{array}$$

for all $X,Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$, where ∇ is the Levi-Civita connection and R is its curvature tensor.

Proof. In the proof, we will use the Theorem 3.5, Lemma 3.6 and the formula (30).

1. By direct calculation, we have

$$\begin{aligned} {}^{BS}\widehat{\nabla}_{{}^HX}{}^HY &= {}^{BS}\nabla_{{}^HX}{}^HY - {}^{BS}g({}^{BS}\nabla_{{}^HX}{}^HY,\mathcal{N})\mathcal{N} \\ &= {}^H(\nabla_XY) + \frac{1}{2}{}^V(pR(X,Y)) - {}^{BS}g(\frac{1}{2}{}^V(pR(X,Y)),\mathcal{N})\mathcal{N} \\ &= {}^H(\nabla_XY) + \frac{1}{2}{}^T(pR(X,Y)). \end{aligned}$$

2. We have ${}^{BS}\widehat{\nabla}_{{}^HX}{}^T\theta = {}^{BS}\nabla_{{}^HX}{}^T\theta - {}^{BS}g({}^{BS}\nabla_{{}^HX}{}^T\theta,\mathcal{N})\mathcal{N}$, by direct calculation, we get

$${}^{BS}\nabla_{{}^HX}{}^T\theta = {}^T(\nabla_X\theta) + \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\theta})X) + \delta^2 g^{-1}(\theta,pJ){}^H(R(J\tilde{p},\tilde{p})X)\big)$$

and

$${}^{BS}g({}^{BS}\nabla_{{}^HX}{}^T\theta,\mathcal{N})\mathcal{N} = 0.$$

Hence

$${}^{BS}\widehat{\nabla}_{{}^HX}{}^T\theta = {}^T(\nabla_X\theta) + \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\theta})X) + \delta^2 g^{-1}(\theta,pJ){}^H(R(J\tilde{p},\tilde{p})X)\big).$$

3. Also, we have ${}^{BS}\widehat{\nabla}_{{}^T\omega}{}^HY = {}^{BS}\nabla_{{}^T\omega}{}^HY - {}^{BS}g({}^{BS}\nabla_{{}^T\omega}{}^HY,\mathcal{N})\mathcal{N}$, by direct calculation, we get

$${}^{BS}\nabla_{{}^T\omega}{}^HY = \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\omega})Y) + \delta^2 g^{-1}(\omega,pJ){}^H(R(J\tilde{p},\tilde{p})Y)\big)$$

and

$${}^{BS}g({}^{BS}\nabla_{{}^T\omega}{}^HY,\mathcal{N})\mathcal{N} = 0.$$

Hence

$${}^{BS}\widehat{\nabla}_{{}^T\omega}{}^HX = \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\omega})Y) + \delta^2 g^{-1}(\omega,pJ){}^H(R(J\tilde{p},\tilde{p})Y)\big).$$

4. In the same way above, we have ${}^{BS}\widehat{\nabla}_{{}^T\omega}{}^T\theta = {}^{BS}\nabla_{{}^T\omega}{}^T\theta - {}^{BS}g({}^{BS}\nabla_{{}^T\omega}{}^T\theta,\mathcal{N})\mathcal{N}$,

$$\begin{aligned} {}^{BS}\nabla_{{}^T\omega}{}^T\theta &= \delta^2\big(g^{-1}(\omega,pJ){}^V(\theta J) + g^{-1}(\theta,pJ){}^V(\omega J)\big) - \delta^2\big(g^{-1}(\omega,pJ)g^{-1}(\theta,p) \\ &\quad + g^{-1}(\omega,p)g^{-1}(\theta,pJ)\big){}^V(pJ) - g^{-1}(\theta,p){}^V\omega - g^{-1}(\omega,\theta){}^Vp \\ &\quad + 2g^{-1}(\omega,p)g^{-1}(\theta,p){}^Vp, \end{aligned}$$

and

$$\begin{aligned} {}^{BS}g({}^{BS}\nabla_{T_\omega}{}^T\theta, \mathcal{N})\mathcal{N} &= \delta^2\big(g^{-1}(\omega, Jp)g^{-1}(\theta J, p) + g^{-1}(\theta, Jp)g^{-1}(\omega J, p)\big)^V p \\ &\quad - g^{-1}(\omega, \theta)^V p + g^{-1}(\omega, p)g^{-1}(\theta, p)^V p. \end{aligned}$$

Hence

$$\begin{aligned} {}^{BS}\widehat{\nabla}_{T_\omega}{}^T\theta &= -g^{-1}(\theta, p)^T\omega + \delta^2\big(g^{-1}(\omega, pJ)^T(\theta J) + g^{-1}(\theta, pJ)^T(\omega J)\big) \\ &\quad -\delta^2\big(g^{-1}(\omega, pJ)g^{-1}(\theta, p) + g^{-1}(\omega, p)g^{-1}(\theta, pJ)\big)^T(pJ). \end{aligned}$$

□

5 Geodesics of the Berger-type deformed Sasaki metric

Let $\gamma : I \to M$ be a curve on M, I is an open interval of $\mathbb{R}$ and C be a curve on T^*M expressed by $C = (\gamma(t), \vartheta(t))$, for all $t \in I$, where $\vartheta(t) \in T^*M$ i.e. $\vartheta(t)$ is a covector field along γ.

Lemma 5.1 ([12]). *Let (M, g) be a Riemannian manifold, and ∇ denote the Levi-Civita connection of (M, g). If $C = (\gamma(t), \vartheta(t))$ is a curve on T^*M, then*

$$\dot{C} = \dot{\gamma}^H + (\nabla_{\dot{\gamma}}\vartheta)^V,$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$ and $\dot{C} = \frac{dC}{dt}$.

Subsequently we denote $\gamma' = \frac{dx}{dt}$, $\gamma'' = \nabla_{\gamma'}\gamma'$, $\vartheta' = \nabla_{\gamma'}\vartheta$, $\vartheta'' = \nabla_{\gamma'}\vartheta'$ and $C' = \frac{dC}{dt}$. Then

$$C' = {}^H\gamma' + {}^V\vartheta'. \tag{31}$$

Theorem 5.2. *Let (M^{2m}, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. The curve $C = (\gamma(t), \vartheta(t))$ is a geodesic on T^*M if and only if*

$$\begin{cases} \gamma'' = \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' \\ \vartheta'' = 2\delta^2 g^{-1}(\vartheta', \vartheta J)(\dfrac{\delta^2}{\lambda} g^{-1}(\vartheta', \vartheta)\vartheta J - \vartheta' J), \end{cases} \tag{32}$$

where $\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta}) = R(\widetilde{\vartheta'}, \widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta', \vartheta J)R(\widetilde{\vartheta}, J\widetilde{\vartheta})$ and R is the curvature tensor of the manifold (M^{2m}, J, g).

Proof. From formula (31) and Theorem 3.5, we obtain

$$\begin{aligned}
{}^{BS}\nabla_{C'}C' &= {}^{BS}\nabla_{({}^H\gamma' + {}^V\vartheta')}({}^H\gamma' + {}^V\vartheta') \\
&= {}^{BS}\nabla_{{}^H\gamma'}{}^H\gamma' + {}^{BS}\nabla_{{}^H\gamma'}{}^V\vartheta' + {}^{BS}\nabla_{{}^V\vartheta'}{}^H\gamma' + {}^{BS}\nabla_{{}^V\vartheta'}{}^V\vartheta' \\
&= {}^H\gamma'' + {}^H(R(\widetilde{\vartheta},\widetilde{\vartheta'})\gamma' - \delta^2 g^{-1}(\vartheta',\vartheta J)R(\widetilde{\vartheta},J\widetilde{\vartheta})\gamma')) + {}^V\vartheta'' \\
&\quad +2\delta^2 g^{-1}(\vartheta',\vartheta J){}^V(\vartheta' J) - \frac{2\delta^4}{\lambda}g^{-1}(\vartheta',\vartheta)g^{-1}(\vartheta',J\vartheta){}^V(\vartheta J) \\
&= {}^H\big(\gamma'' + R(\widetilde{\vartheta},\widetilde{\vartheta'})\gamma' - \delta^2 g^{-1}(\vartheta',\vartheta J)R(\widetilde{\vartheta},J\widetilde{\vartheta})\gamma'\big) \\
&\quad +{}^V\big(\vartheta'' + 2\delta^2 g^{-1}(\vartheta',\vartheta J)(\vartheta' J - \frac{\delta^2}{\lambda}g^{-1}(\vartheta',\vartheta)\vartheta J)\big) \\
&= {}^H\big(\gamma'' - (R(\widetilde{\vartheta'},\widetilde{\vartheta})\gamma' + \delta^2 g^{-1}(\vartheta',\vartheta J)R(\widetilde{\vartheta},J\widetilde{\vartheta})\gamma')\big) \\
&\quad +{}^V\big(\vartheta'' - 2\delta^2 g^{-1}(\vartheta',\vartheta J)(\frac{\delta^2}{\lambda}g^{-1}(\vartheta',\vartheta)\vartheta J - \vartheta' J)\big).
\end{aligned}$$

If we put ${}^{BS}\nabla_{C'}C'$ equal to zero, we find (32). □

A curve $C = (\gamma(t), \vartheta(t))$ on T^*M is said to be a horizontal lift of the curve γ on M if and only if $\vartheta' = 0$ [11]. Thus, we have

Corollary 5.3. *Let (M^{2m}, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. The horizontal lift of any geodesic on (M^{2m}, J, g) is a geodesic on $(T^*M, {}^{BS}g)$.*

Corollary 5.4. *Let (M^{2m}, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. The curve $C = (\gamma(t), \widetilde{\gamma'(t)})$ is a geodesic on T^*M if and only if γ is a geodesic on (M^{2m}, J, g).*

Proof. We have, $\gamma'(t) \in TM$, then $\vartheta(t) = \widetilde{\gamma'(t)} \in T^*M$. From (11) and (13), we get $\vartheta' = \nabla_{\gamma'}\vartheta = \widetilde{\widetilde{\nabla_{\gamma'}\vartheta}} = \widetilde{\nabla_{\gamma'}\widetilde{\vartheta}} = \widetilde{\nabla_{\gamma'}\gamma'} = \widetilde{\gamma''}$, then γ is a geodesic on M equivalent to C is a horizontal lift of the curve γ on M. Using Corollary 5.3, we deduce the result. □

Remark 5.5. If γ is a geodesic on M locally we have:

$$\gamma'' = 0 \quad \Leftrightarrow \quad \gamma''_h + \sum_{i,j=1}^{2m} \Gamma^h_{ij}(\gamma')^i(\gamma')^j = 0, \quad h = \overline{1, 2m}.$$

If C such that $C(t) = (\gamma(t), \vartheta(t))$ is a horizontal lift of the curve γ, locally we have:

$$\vartheta' = 0 \quad \Leftrightarrow \quad \vartheta'_h - \sum_{i,j=1}^{2m} \Gamma^i_{jh}\vartheta_i(\gamma')^j = 0, \quad h = \overline{1, 2m}.$$

Example 5.6. Let $\mathbb{R}^2$ be endowed with the structure standard Kähler (J, g) defined by

$$g = x^2 dx^2 + y^2 dy^2.$$

and

$$J\partial_x = -\frac{x}{y}\partial_y \quad , \quad J\partial_y = \frac{y}{x}\partial_x.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^1_{11} = \frac{1}{x},\ \Gamma^2_{22} = \frac{1}{y}.$$

The geodesics γ such that $\gamma(t) = (x(t), y(t))$, $\gamma(0) = (a, b)$ and $\gamma'(0) = (\alpha, \beta) \in \mathbb{R}^2$ satisfy the system of equations,

$$\gamma''_h + \sum_{i,j=1}^{2} \Gamma^h_{ij}(\gamma')^i(\gamma')^j = 0 \Leftrightarrow \begin{cases} x'' + \dfrac{(x')^2}{x} = 0 \\ \\ y'' + \dfrac{(y')^2}{y} = 0 \end{cases} \Leftrightarrow \begin{cases} x(t) = \sqrt{2a\alpha t + a^2} \\ \\ y(t) = \sqrt{2b\beta t + b^2} \end{cases}$$

Hence $\gamma'(t) = \dfrac{a\alpha}{\sqrt{2a\alpha t + a^2}}\partial_x + \dfrac{b\beta}{\sqrt{2b\beta t + a^2}}\partial_y$, $\gamma(t) = (\sqrt{2a\alpha t + a^2}, \sqrt{2b\beta t + b^2})$.

1) Let $C_1 = (\gamma(t), \vartheta(t))$ be a horizontal lift of the geodesic γ then,

$$\vartheta'_h - \sum_{i,j=1}^{2} \Gamma^i_{jh}\vartheta_i(\gamma')^j = 0 \Leftrightarrow \begin{cases} \vartheta'_1 - \dfrac{x'}{x}\vartheta_1 = 0 \\ \\ \vartheta'_2 - \dfrac{y'}{y}\vartheta_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \vartheta_1(t) = k_1\sqrt{2a\alpha t + a^2} \\ \\ \vartheta_2(t) = k_2\sqrt{2b\beta t + b^2} \end{cases}$$

Hence $\vartheta(t) = k_1\sqrt{2a\alpha t + a^2}dx + k_2\sqrt{2b\beta t + b^2}dy$, where $k_1, k_2 \in \mathbb{R}$. From Corollary 5.3, the curve C_1 is a geodesic on $T^*\mathbb{R}^2$.

2) Let $C_2 = (\gamma(t), \widetilde{\gamma'(t)})$ be a curve on $T^*\mathbb{R}^2$, from (10), we have

$$\widetilde{\gamma'(t)} = \sum_{i,j=1}^{2} g_{ij}(\gamma')^j(t)dx_i = a\alpha\sqrt{2a\alpha t + a^2}dx + b\beta\sqrt{2b\beta t + b^2}dy.$$

From Corollary 5.4, the curve C_2 is a geodesic on $T^*\mathbb{R}^2$.

Corollary 5.7. *Let (M^{2m}, J, g) be a flat standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. Then the curve $C = (\gamma(t), \vartheta(t))$ is a geodesic on T^*M if and only if γ is a geodesic on (M^{2m}, J, g) and*

$$\vartheta'' = 2\delta^2 g^{-1}(\vartheta', \vartheta J)(\frac{\delta^2}{\lambda}g^{-1}(\vartheta', \vartheta)\vartheta J - \vartheta' J).$$

Let C be a curve on T^*M, the cure $\gamma = \pi \circ C$ is called the projection (projected curve) of the curve C on M.

Theorem 5.8. *Let (M^{2m}, φ, g) be a standard Kähler locally symmetric manifold, $(T^*M, {}^{BS}g)$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric, and C be a geodesic on T^*M. Then $\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})$ is parallel along the projected curve $\gamma = \pi \circ C$.*

Proof. Using (14), (21) and (22) we have

$$\begin{aligned}
(\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta}))' &= (R(\widetilde{\vartheta'}, \widetilde{\vartheta}))' + \delta^2(g^{-1}(\vartheta', \vartheta J))' R(\widetilde{\vartheta}, J\widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta', \vartheta J)(R(\widetilde{\vartheta}, J\widetilde{\vartheta}))' \\
&= R'(\widetilde{\vartheta'}, \widetilde{\vartheta}) + R(\widetilde{\vartheta''}, \widetilde{\vartheta}) + R(\widetilde{\vartheta'}, \widetilde{\vartheta'}) + \delta^2 g^{-1}(\vartheta'', \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&\quad + \delta^2 g^{-1}(\vartheta', \vartheta' J) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta', \vartheta J) R'(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&\quad + \delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta'}, J\widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta'}) \\
&= R(\widetilde{\vartheta''}, \widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta'', \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) + \delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta'}, J\widetilde{\vartheta}) \\
&\quad + \delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta'}),
\end{aligned}$$

from second equation of (32) and (23) we get

$$\begin{aligned}
(\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta}))' &= \frac{2\delta^4}{\lambda} g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) R(\widetilde{\vartheta J}, \widetilde{\vartheta}) - 2\delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta' J}, \widetilde{\vartheta}) \\
&\quad + \frac{2\delta^6}{\lambda} g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) g^{-1}(\vartheta J, \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&\quad - 2\delta^4 g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta' J, \vartheta J) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) + 2\delta^2 g^{-1}(\vartheta', \vartheta J) R(\widetilde{\vartheta'}, J\widetilde{\vartheta}) \\
&= \frac{2\delta^4}{\lambda} g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&\quad + \frac{2\delta^4(\lambda - 1)}{\lambda} g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&\quad - 2\delta^4 g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&= \left(\frac{2\delta^4}{\lambda} + \frac{2\delta^4(\lambda - 1)}{\lambda} - 2\delta^4\right) g^{-1}(\vartheta', \vartheta J) g^{-1}(\vartheta', \vartheta) R(\widetilde{\vartheta}, J\widetilde{\vartheta}) \\
&= 0.
\end{aligned}$$

□

We now study the geodesics on the unit cotangent bundle with respect to the Berger-type deformed Sasaki metric.

Lemma 5.9. *Let (M^{2m}, φ, g) be a standard Kähler manifold, $(T_1^*M, {}^{BS}\hat{g})$ its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric and $C = (\gamma(t), \vartheta(t))$ be a curve on T_1^*M. Then we have*

$$C' = {}^H\gamma' + {}^T\vartheta'. \tag{33}$$

Proof. Using (31), we have

$$C' = {}^H\gamma' + {}^V\vartheta' = {}^H\gamma' + {}^T\vartheta' + g^{-1}(\vartheta', \vartheta)^V\vartheta.$$

Since $C(t) = (\gamma(t), \vartheta(t)) \in T_1^*M$ then $g^{-1}(\vartheta, \vartheta) = 1$, on the other hand

$$0 = (g^{-1}(\vartheta, \vartheta))' = 2g^{-1}(\vartheta', \vartheta),$$

hence

$$g^{-1}(\vartheta', \vartheta) = 0. \tag{34}$$

The proof of the lemma is completed. □

Subsequently, let t be an arc length parameter on C, From 33, we have

$$1 = |\gamma'|^2 + |\vartheta'|^2 + \delta^2 g^{-1}(\vartheta', \vartheta J)^2. \tag{35}$$

Theorem 5.10. *Let (M^{2m}, φ, g) be a standard Kähler manifold, $(T_1^*M, {}^{BS}\hat{g})$ its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric and $C = (\gamma(t), \vartheta(t))$ be a curve on T_1^*M. Let $\kappa = |\vartheta'|$ and $\mu = g^{-1}(\vartheta', \vartheta J)$. Then C is a geodesic on T_1^*M if and only if*

$$\begin{cases} \gamma'' = \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' \\ \vartheta'' = -2\delta^2\mu\, \vartheta' J, \end{cases} \tag{36}$$

where $\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta}) = R(\widetilde{\vartheta'}, \widetilde{\vartheta}) + \delta^2\mu R(\widetilde{\vartheta}, J\widetilde{\vartheta})$. Moreover,

$$\begin{cases} |\vartheta'| = \kappa \\ |\gamma'| = \sqrt{1-K} \end{cases} \tag{37}$$

where $K = \kappa^2 + \delta^2\mu^2 = const$, $0 \leq K \leq 1$, $\kappa = const$ and $\mu = const$.

Proof. Using formula (33) and Theorem 4.1, we compute the derivative $\widehat{\nabla}_{C'}C'$.

$$\begin{aligned} \widehat{\nabla}_{C'}C' &= \widehat{\nabla}_{({}^H\gamma' + {}^T\vartheta')}({}^H\gamma' + {}^T\vartheta') \\ &= \widehat{\nabla}_{{}^H\gamma'}{}^H\gamma' + \widehat{\nabla}_{{}^H\gamma'}{}^T\vartheta' + \widehat{\nabla}_{{}^T\vartheta'}{}^H\gamma' + \widehat{\nabla}_{{}^T\vartheta'}{}^T\vartheta' \\ &= {}^H\gamma'' + {}^T\vartheta'' + {}^H(R(\widetilde{\vartheta}, \widetilde{\vartheta'})\gamma' - \delta^2 g^{-1}(\vartheta', \vartheta J)R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma')) \\ &\quad + 2\delta^2 g^{-1}(\vartheta', \vartheta J)^T(\vartheta' J) \\ &= {}^H\gamma'' - {}^H(R(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' + \delta^2 g^{-1}(\vartheta', \vartheta J)R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma')) \\ &\quad + {}^T\vartheta'' + 2\delta^2 g^{-1}(\vartheta', \vartheta J)^T(\vartheta' J) \\ &= {}^H\big(\gamma'' - \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'\big) + {}^T\big(\vartheta'' + 2\delta^2\mu\, \vartheta' J\big). \end{aligned}$$

If we put $\widehat{\nabla}_{C'}C'$ equal to zero, we find (36). Moreover, we have $\kappa = |\vartheta'|$, then

$$(\kappa^2)' = 2g^{-1}(\vartheta'', \vartheta'),$$

from second equation of (36), we have $g^{-1}(\vartheta'', \vartheta') + 2\delta^2\mu g^{-1}(\vartheta' J, \vartheta') = 0$, on the other hand, from (18), we find $g^{-1}(\vartheta' J, \vartheta') = 0$, then $g^{-1}(\vartheta'', \vartheta') = 0$, hence $\kappa = const$. We have, $\mu = g^{-1}(\vartheta', \vartheta J)$, then $\mu' = g^{-1}(\vartheta'', \vartheta J) + g^{-1}(\vartheta', \vartheta' J) = g^{-1}(\vartheta'', \vartheta J)$, from second equation of (36), we have $\mu' = g^{-1}(\vartheta'', \vartheta J) = 2\delta^2\mu g^{-1}(\vartheta' J, \vartheta J)$, from (18), we find

$$g^{-1}(\vartheta' J, \vartheta J) = g^{-1}(\vartheta', \vartheta) = 0,$$

hence $\mu = const$. Using (35), we get $1 = |\gamma'|^2 + \kappa^2 + \delta^2\mu^2$, then

$$|\gamma'| = \sqrt{1 - (\kappa^2 + \delta^2\mu^2)} = \sqrt{1 - K}.$$

where, $K = \kappa^2 + \delta^2\mu^2 = const$. □

Theorem 5.11. *Let (M^{2m}, J, g) denote a standard Kähler locally symmetric manifold, $(T_1^*M, {}^{BS}\hat{g})$ be its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric, and C be a geodesic on T_1^*M. Then $\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})$ is parallel along the projected curve $\gamma = \pi \circ C$.*

Proof. Similarly, proving Theorem 5.8, using $\mu = g^{-1}(\vartheta', \vartheta J)$ and Theorem 5.10, we get the result. □

Theorem 5.12. *Let (M^{2m}, J, g) denote a standard Kähler locally symmetric manifold, $(T_1^*M, {}^{BS}\hat{g})$ be its unit tangent bundle equipped with Berger-type deformed Sasaki metric, and C be a geodesic on T_1^*M, then all Frenet curvatures of the projected curve $\gamma = \pi \circ C$ are constants.*

Proof. Using the first equation of (36), we have

$$\gamma'' = \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' = R(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' + \delta^2\mu R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma'.$$

Since $(g(\gamma', \gamma'))' = 2g(\gamma'', \gamma') = 2g(\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma', \gamma') = 0$, hence $|\gamma'| = const$.

$$\begin{aligned}
\gamma''' &= (\mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma')' \\
&= (R(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma')' + \delta^2\mu(R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma')' \\
&= R'(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta'}, \widetilde{\vartheta'})\gamma' + R(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'' \\
&\quad + \delta^2\mu(R'(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta'}, J\widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta}, J\widetilde{\vartheta'})\gamma' + R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma'') \\
&= R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'' + \delta^2\mu(R(\widetilde{\vartheta'}, J\widetilde{\vartheta})\gamma' + R(\widetilde{\vartheta}, J\widetilde{\vartheta'})\gamma' + R(\widetilde{\vartheta}, J\widetilde{\vartheta})\gamma'') \\
&= R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' - 2\delta^2\mu\, R(J\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma' + \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'' \\
&= R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' + 2\delta^2\mu\, R(\widetilde{\vartheta' J}, \widetilde{\vartheta})\gamma' + \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'' \\
&= R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' - R(\widetilde{\vartheta''}, \widetilde{\vartheta})\gamma' + \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma'' \\
&= \mathcal{R}(\widetilde{\vartheta'}, \widetilde{\vartheta})\gamma''.
\end{aligned}$$

Since $(g(\gamma'',\gamma''))' = 2g(x''',\gamma'') = 2g(\mathcal{R}(\widetilde{\vartheta}',\widetilde{\vartheta})\gamma'',\gamma'') = 0$, hence $|\gamma''| = const$.

Continuing the process by recurrence, we get

$$\gamma^{(p+1)} = \mathcal{R}(\widetilde{\vartheta}',\widetilde{\vartheta})\gamma^{(p)}, \quad p \geq 1$$

and

$$(g(\gamma^{(p)},\gamma^{(p)}))' = 2g(\gamma^{(p+1)},\gamma^{(p)}) = 2g(\mathcal{R}(\widetilde{\vartheta}',\widetilde{\vartheta})\gamma^{(p)},\gamma^{(p)}) = 0.$$

Thus, we get

$$|\gamma^{(p)}| = const, \quad p \geq 1. \tag{38}$$

Denote by s an arc length parameter on γ, i.e. ($|\gamma'_s| = 1$). Then $\gamma' = \gamma'_s \frac{ds}{dt}$, and using (37), we get

$$\frac{ds}{dt} = \sqrt{1-K} = const. \tag{39}$$

Let $\nu_1 = \gamma'_s$ be the first vector in the Frenet frame $\nu_1,\ldots,\nu_{2m-1}$ along γ and let $k_1,\ldots,k_{2m-1}$ the Frenet curvatures of γ. Then the Frenet formulas verify

$$\begin{cases} (\nu_1)'_s &= k_1\nu_2 \\ (\nu_i)'_s &= -k_{i-1}\nu_{i-1} + k_i\nu_{i+1}, \quad 2 \leq i \leq 2m-2 \\ (\nu_{2m-1})'_s &= -k_{2m-2}\nu_{2m-2} \end{cases} \tag{40}$$

Using (39) and the Frenet formulas (40), we obtain

$$\gamma' = \gamma'_s \frac{ds}{dt} = \sqrt{1-K}\,\nu_1.$$

$$\gamma'' = \sqrt{1-K}(\nu_1)'_t = \sqrt{1-K}(\nu_1)'_s\frac{ds}{dt} = (1-K)k_1\nu_2.$$

Now (38) implies $k_1 = const$. Next, in a similar way, we have

$$\gamma''' = (1-K)k_1(\nu_2)'_t = (1-K)k_1(\nu_2)'_s\frac{ds}{dt} = (1-K)\sqrt{1-K}k_1(-k_1\nu_1 + k_2\nu_3).$$

and again (38) implies $k_2 = const$. By continuing the process, we finish the proof. □

Lemma 5.13. *Let (M^{2m},J,g) be a standard Kähler manifold, $(T_1^*M, {}^{BS}\hat{g})$ its unit cotangent bundle equipped with Berger-type deformed Sasaki metric and $C = (\gamma(t),\vartheta(t))$ be a curve on T_1^*M, we put $\xi = \vartheta J$, then we have*

1. *$\Gamma = (\gamma(t),\xi(t))$ is a curve on T_1^*M.*
2. *Γ is a geodesic on T_1^*M if and only if C is a geodesic on T_1^*M.*

Proof. 1. Since we have $\xi(t) = \vartheta J(t)$, then $g^{-1}(\xi,\xi) = g^{-1}(\vartheta J,\vartheta J) = g^{-1}(\vartheta,\vartheta)$. Since $C = (\gamma(t),\vartheta(t)) \in T_1^*M$ we get $g(\vartheta,\vartheta) = 1$. Hence, $g(\xi,\xi) = 1$, which means that $\Gamma = (\gamma(t),\xi(t)) \in T_1^*M$.

2. In a similar way proof of (36), we have

$$\widehat{\nabla}_{\Gamma'}\Gamma' \;=\; {}^{H}\!\big(\gamma'' - \mathcal{R}(\widetilde{\xi'},\widetilde{\xi})\gamma'\big) + {}^{T}\!\big(\xi'' + 2\delta^2\mu\,\xi' J\big),$$

since $\xi' = \vartheta' J$, $\xi'' = \vartheta'' J$ and $\mathcal{R}(\widetilde{\xi'},\widetilde{\xi}) = \mathcal{R}(\widetilde{\vartheta'},\widetilde{\vartheta})$, we have

$$\widehat{\nabla}_{\Gamma'}\Gamma' \;=\; {}^{H}\!\big(\gamma'' - \mathcal{R}(\widetilde{\vartheta'},\widetilde{\vartheta})\gamma'\big) + {}^{T}\!\big((\vartheta'' + 2\delta^2\mu\,\vartheta' J)J\big).$$

$$\widehat{\nabla}_{\Gamma'}\Gamma' = 0 \Leftrightarrow \begin{cases} \gamma'' - \mathcal{R}(\widetilde{\vartheta'},\widetilde{\vartheta})\gamma' = 0 \\ \vartheta'' + 2\delta^2\mu\,\vartheta' J = 0 \end{cases} \Leftrightarrow \begin{cases} \gamma'' = \mathcal{R}(\widetilde{\vartheta'},\widetilde{\vartheta})\gamma' \\ \vartheta'' = -2\delta^2\mu\,\vartheta' J \end{cases} \Leftrightarrow \widehat{\nabla}_{C'}C' = 0. \qquad \square$$

From Theorem 5.12 and Lemma 5.13, we have the following theorem:

Theorem 5.14. *Let (M^{2m}, J, g) denote a standard Kähler locally symmetric manifold, $(T_1^*M, {}^{BS}\hat{g})$ be its unit cotangent bundle equipped with Berger-type deformed Sasaki metric, and $C = (\gamma(t), \vartheta(t))$ be a geodesic on T_1^*M, we put $\xi = \vartheta J$, then all Frenet curvatures of the projected curve $\gamma = \pi \circ \Gamma$ are constants, where $\Gamma = (\gamma(t), \xi(t))$.*

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