Communications in Mathematics **31** (2023), no. 2, 35–59
DOI: https://doi.org/10.46298/cm.11155

Minkowski's successive minima in convex and discrete geometry

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Abstract. In this short survey we want to present some of the impact of Minkowski's successive minima within Convex and Discrete Geometry. Originally related to the volume of an o-symmetric convex body, we point out relations of the successive minima to other functionals, as e.g., the lattice point enumerator or the intrinsic volumes and we present some old and new conjectures about them. Additionally, we discuss an application of successive minima to a version of Siegel's lemma.

1 Introduction

One of the basic questions in Geometry of Numbers, as well as in other areas of mathematics like number theory or integer linear programming, is to decide when a set S in the n-dimensional Euclidean space $\mathbb{R}^n$ contains an integral point, i.e., a point of the lattice $\mathbb{Z}^n$, possibly $\neq \mathbf{0}$, and if necessary to determine such a point. Here "when" usually refers to certain sizes/properties of the set S, such as volume, thickness, covering or packing properties etc.

With respect to the class $\mathcal{K}^n_{os}$ of o-symmetric convex bodies, i.e., non-empty convex and compact sets $K \subset \mathbb{R}^n$ satisfying $K = -K$, and the volume $\mathrm{vol}(\cdot)$, i.e., the n-dimensional Lebesgue measure, Minkowski's classical, so called *convex body theorem* gives a beautiful answer.

MSC 2020: 52C07, 11H06, 52A40, 52A39
Keywords: successive minima, convex body, Siegel's lemma, centered convex bodies, intrinsic volumes, lattice point enumerator
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Theorem 1.1 (Minkowski's convex body theorem, 1893, [62]). *Let* $K \in \mathcal{K}^n_{os}$ *with* $\mathrm{vol}(K) \geq 2^n$. *Then* K *contains a non-trivial lattice point, i.e.,* $K \cap \mathbb{Z}^n \setminus \{\mathbf{0}\} \neq \emptyset$.

The lower bound on the volume is best possible, as, e.g., the cube $C_n := [-1,1]^n$ shows and Minkowski called any convex body $K \in \mathcal{K}^n_{os}$ with $\mathrm{vol}(K) = 2^n$ and $\mathrm{int}(K) \cap \mathbb{Z}^n = \{\mathbf{0}\}$, where $\mathrm{int}(\cdot)$ denotes the interior, an *extremal convex body*. These are – up to a factor of 2 – exactly those convex bodies, actually polytopes, which tile the space via $\mathbb{Z}^n$, i.e., $\mathbb{Z}^n$ is a covering lattice as well as a packing lattice of $\frac{1}{2}K$ (see Section 2 for precise definitions).

In his 1896 published book "Geometrie der Zahlen", Minkowski describes this result as "…ein Satz, der nach meinem Dafürhalten zu den fruchtbarsten in der Zahlenlehre zu rechnen ist." ([63], p. 75), and indeed this theorem has numerous applications in different areas for which we refer to [16, 40, 59, 70, 71].

Actually, in [62] Minkowski proved Theorem 1.1 as an inequality relating the volume of K and the minimal norm of a non-trivial lattice point, measured with respect to the gauge body K. In order to state this inequality – and later to generalize it – we will use his 1896 introduced "kleinstes System von unabhängig gerichteten Strahlendistanzen im Zahlengitter" ([63], p. 178), the so called *successive minima*.

For $K \in \mathcal{K}^n_{os}$, $\dim(K) = n$, and $1 \leq i \leq n$, the i-th successive minimum $\lambda_i(K)$ is the smallest dilation factor λ such that λK contains at least i linearly independent lattice points, i.e.,

$$\lambda_i(K) := \min\{\lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i\}.$$

For instance, if $B = B(a_1, \ldots, a_n) := [-a_1, a_1] \times \cdots \times [-a_n, a_n]$ is a box with

$$a_1 \geq a_2 \geq \cdots \geq a_n > 0,$$

then $\lambda_i(B) = 1/a_i$, $1 \leq i \leq n$.

The successive minima form a non-decreasing sequence and, in particular, $\lambda_1(K)$ is the smallest number λ such that λK contains a non-trivial lattice point. Hence, Theorem 1.1 is equivalent to

Theorem 1.2. *Let* $K \in \mathcal{K}^n_{os}$, $\dim K = n$. *Then*

$$\lambda_1(K)^n \ \mathrm{vol}(K) \leq 2^n.$$

The proof is based on the observation that due to the symmetry of K and the definition of $\lambda_1(K)$, $\mathbb{Z}^n$ is a packing lattice of $\frac{1}{2}\lambda_1(K)\,K$ and so its volume is at most 1. Minkowski's so-called theorem on successive minima is a far reaching extension of Theorem 1.2 in which $\lambda_1(K)^n$ is replaced by the product of all successive minima. In addition, in this way a lower bound is also possible which does not exist in Theorem 1.2.

Theorem 1.3 (Minkowski's theorem on successive minima, 1896, [63, Kapitel 5]). *Let* $K \in \mathcal{K}^n_{os}$, $\dim K = n$. *Then*

$$\frac{2^n}{n!} \leq \lambda_1(K)\lambda_2(K)\cdots\lambda_n(K) \ \mathrm{vol}(K) \leq 2^n. \tag{1}$$

The lower bound follows by an inclusion argument and it is attained, e.g., for the regular cross-polytope $C_n^\star := \mathrm{conv}(\{\pm \boldsymbol{e}_i : 1 \le i \le n\})$, where $\mathrm{conv}(\cdot)$ denotes the convex hull of a set, and $\boldsymbol{e}_i$ are the canonical unit vectors. The crucial part is the upper bound, which is considered as a deep and important result in Geometry of Numbers. There are many alternative proofs available for this result (see, e.g., [41, Section 9 and Section ii], [40, Section 23], [76, Section 3.5] and the references within), but, maybe, the most geometric one is still Minkowski's original proof (see, e.g., [40, Theorem 23.1]).

The applications of the upper bound in Theorem 1.3 are probably not as numerous as those of its precursor Theorem 1.2, but, in general, they are also less "elementary". For instance, the bound is used in order to prove Minkowski's finiteness theorems in reduction theory (see, e.g., [80]), it appears in the proof of W. M. Schmidt of his (strong) subspace theorem [70, pp. 162] or it is also used in proofs of Freiman's theorem in additive combinatorics (see, e.g., [25]). Another prominent application is the Bombieri-Vaaler extension of Siegel's lemma which we will discuss in more detail in Section 6.

Since Minkowski's introduction of the successive minima they have become an important tool/measure in different areas of mathematics. In this short survey we want to present some of their impact within convex and discrete geometry by showing relations of the successive minima to other functionals (e.g., lattice point enumerator, intrinsic volumes) as well as presenting some old and new conjectures related to them. For their immense impact on Diophantine Geometry we refer to [16, 70] and for algorithmic questions related to them see [59, 65].

2 Preliminaries

Here we briefly introduce some more basic notation, for a thorough treatment we refer to [22, 41, 40, 72]. Let $\mathcal{K}^n$ be the family of all non-empty convex bodies in $\mathbb{R}^n$, i.e., compact convex sets $K \subset \mathbb{R}^n$, and let $\mathcal{K}^n_o \subset \mathcal{K}^n$ be the set of those convex bodies with $\mathbf{0} \in \mathrm{int}(K)$. The subfamily of centered convex bodies, i.e., those $K \in \mathcal{K}^n_o$ whose centroid

$$\frac{1}{\mathrm{vol}(K)} \int_K \boldsymbol{x} \, \mathrm{d}^n \boldsymbol{x}$$

is at the origin is denoted by $\mathcal{K}^n_{oc}$. The n-dimensional Euclidean unit ball is denoted by B_n and its volume by ω_n. According to a result of Steiner the volume of $K + \rho B_n$, $\rho \ge 0$, is a polynomial of degree n in ρ which we can write as [72, Section 4.2]

$$\mathrm{vol}(K + \rho B_n) = \sum_{i=0}^{n} \omega_i \, \mathrm{V}_{n-i}(K) \rho^i.$$

The coefficient $\mathrm{V}_i(K)$ is called the i-th intrinsic volume of K, $i = 0, \ldots, n$; in particular,

$$\mathrm{V}_n(K) = \mathrm{vol}(K), \quad \mathrm{V}_{n-1}(K) = \frac{1}{2}\,\mathrm{F}(K), \quad \mathrm{V}_0(K) = 1, \tag{2}$$

where $\mathrm{F}(K)$ is the surface area of K. We remark that intrinsic volumes are, up to some normalization, are mixed volumes for which we refer to [20, 72].

For $K \in \mathcal{K}_o^n$ the polar set $K^\star$, defined as

$$K^\star := \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \text{ for all } \boldsymbol{x} \in K\},$$

is again a convex body. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. The set of all m-dimensional lattices $\Lambda \subset \mathbb{R}^n$ will be denoted by $\mathcal{L}_m^n$, i.e.,

$$\mathcal{L}_m^n := \{B\mathbb{Z}^m : B \in \mathbb{R}^{n\times m},\ \mathrm{rank}(B) = m\}.$$

In the case $m = n$ we just write $\mathcal{L}^n$. As usual, for $\Lambda = B\mathbb{Z}^m \in \mathcal{L}_m^n$, $\det(\Lambda) := \sqrt{\det(B^\intercal B)}$ is called the determinant of Λ. The polar lattice of Λ is given by

$$\Lambda^\star := \{\boldsymbol{y} \in \mathrm{lin}(\Lambda) : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{Z} \text{ for all } \boldsymbol{x} \in \Lambda\},$$

and it is $\det(\Lambda^\star) = 1/\det(\Lambda)$. Here $\mathrm{lin}(\cdot)$ denotes the linear hull.

With respect to a convex body $K \in \mathcal{K}_o^n$ and a general lattice $\Lambda \in \mathcal{L}^n$, the i-th successive minimum $\lambda_i(K, \Lambda)$ is given by

$$\lambda_i(K, \Lambda) := \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\},$$

where the dimension of a set S is always meant with respect to its affine hull $\mathrm{aff}(S)$. In case $\Lambda = \mathbb{Z}^n$ we write $\lambda_i(K)$ and if $K = B_n$, we abbreviate $\lambda_i(B_n, \Lambda)$ by $\lambda_i(\Lambda)$, which is also called the i-th successive minimum of the lattice Λ. Since for $\Lambda = B\mathbb{Z}^n \in \mathcal{L}^n$ we have $\lambda_i(B^{-1}K) = \lambda_i(K, \Lambda)$ and $\mathrm{vol}(B^{-1}K) = \mathrm{vol}(K)/\det(\Lambda)$, Minkowski's Theorem 1.3 can be equivalently stated for arbitrary lattices and $K \in \mathcal{K}_{os}^n$ as

$$\det(\Lambda)\frac{2^n}{n!} \leq \lambda_1(K, \Lambda)\lambda_2(K, \Lambda)\cdots\lambda_n(K, \Lambda)\ \mathrm{vol}(K) \leq 2^n \det(\Lambda). \tag{3}$$

A lattice $\Lambda \in \mathcal{L}^n$ will be called a covering lattice of $K \in \mathcal{K}^n$ if $\mathbb{R}^n = \Lambda + K$ and a packing lattice if $\mathrm{int}(\boldsymbol{a} + K) \cap \mathrm{int}(\boldsymbol{b} + K) = \emptyset$ for all $\boldsymbol{a} \neq \boldsymbol{b} \in \Lambda$. Given a $K \in \mathcal{K}^n$ and a lattice $\Lambda \in \mathcal{L}^n$, then

$$\lambda_1(K - K, \Lambda) = \max\{\rho : \Lambda \text{ packing lattice of } \rho\, K\}$$

and

$$\delta(K) := \max\left\{\frac{\mathrm{vol}\left(\lambda_1(K - K, \Lambda)\, K\right)}{\det(\Lambda)} : \Lambda \in \mathcal{L}^n\right\} \tag{4}$$

is called the density of a densest lattice packing of K. Here the ratio

$$\mathrm{vol}(\lambda_1(K - K, \Lambda)\, K)/\det(\Lambda)$$

describes the proportion of space which is occupied by the packing $\Lambda + \lambda_1(K - K)\, K$. Observe, that $0 < \delta(K) \leq 1$.

The covering counterpart to $\lambda_1(K-K,\Lambda)$ is the so called covering radius

$$\mu(K,\Lambda)=\min\{\mu>0:\mu\,K+\Lambda=\mathbb{R}^n\},$$

i.e., the smallest $\mu>0$ such that Λ is a covering lattice of $\mu\,K$. In analogy to (4), the minimum of $\mathrm{vol}(\mu(K,\Lambda)\,K)/\det(\Lambda)$ with respect to all lattices leads to the density of a thinnest lattice covering, but we do not need this quantity here.

By definition we have $\lambda_1(K-K,\Lambda)\leq\mu(K,\Lambda)$ and more generally,

$$\lambda_n(K-K,\Lambda)\leq\mu(K,\Lambda)\leq\lambda_1(K-K,\Lambda)+\cdots+\lambda_n(K-K,\Lambda).$$

This was shown in the symmetric case by Jarník (see [40, Theorem 23.4]). The general case was treated by Kannan and Lovász in [49, Lemma 2.4], where they also introduced the so called covering minima $\mu_i(K,\Lambda)$, $1\leq i\leq n$, which can be regarded as covering counterparts to the successive minima. For more information on these functionals see,e.g., [49, 39].

3 Possible Tightenings and Generalizations of Minkowski's theorem

First we state a straightforward extension of Minkowski's Theorem 1.3 to arbitrary, not necessarily o-symmetric convex bodies. To this end we consider for $K\in\mathcal{K}^n$ its central symmetrical

$$K_s:=\frac{1}{2}(K-K)\in\mathcal{K}^n_{os}.$$

Obviously, $K_s=K$ for $K\in\mathcal{K}^n_{os}$ and by the classical Brunn-Minkowski inequality [72, Section 7] we know that

$$\mathrm{vol}(K)\leq\mathrm{vol}(K_s)$$

with equality if and only if K and K_s are translates of each other. Thus, the upper bound in (3) applied to K_s gives

$$\lambda_1(K_s,\Lambda)\lambda_2(K_s,\Lambda)\cdots\lambda_n(K_s,\Lambda)\ \mathrm{vol}(K)\leq 2^n\det(\Lambda),$$

and it is also easy to see that the corresponding lower bound in (3) holds true (see, e.g., the discussion in [43]). Hence, Minkowski's Theorem 1.3 can be stated in a bit more general form.

Theorem 3.1. *Let $K\in\mathcal{K}^n$, $\dim K=n$, and let $\Lambda\in\mathcal{L}^n$. Then*

$$\det(\Lambda)\frac{2^n}{n!}\leq\lambda_1(K_s,\Lambda)\lambda_2(K_s,\Lambda)\cdots\lambda_n(K_s,\Lambda)\ \mathrm{vol}(K)\leq 2^n\det(\Lambda). \tag{5}$$

In fact, the body K_s is a rather natural candidate for such an extension. As Λ is a packing lattice of $\lambda_1(K-K,\Lambda)\,K$ we get by the definition of the density of a densest lattice packing (4) the inequality

$$\lambda_1(K_s,\Lambda)^n \operatorname{vol}(K) = 2^n \lambda_1(K-K,\Lambda)^n \operatorname{vol}(K) \le \delta(K)\, 2^n \det(\Lambda)$$

which is a tightening of Minkowski's Theorem 1.2 (for general $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n$). If such an improvement is also possible for the upper bound in Minkowski's Theorem 1.2 is the content of a famous problem posed by Davenport.

Problem 3.2 (Davenport, 1946, [26]). Let $K \in \mathcal{K}^n$. Is it true that

$$\lambda_1(K_s,\Lambda)\cdots\lambda_n(K_s,\Lambda)\operatorname{vol}(K) \le \delta(K)\, 2^n \det(\Lambda)\,? \tag{6}$$

Actually, Davenport formulated it for $K \in \mathcal{K}^n_{os}$ but this is equivalent to the above statement. So far it has only been verified for $n = 2$ and for ellipsoids by Minkowski (see [63, pp. 196], [41, pp. 195]), the case $n = 3$ has been settled by Woods [83]. For more information see [41, Section 18].

The inequalities in Theorem 5 have the nice properties that they generalize the symmetric setting and they are invariant with respect to translations of K. A different way to extend the class $\mathcal{K}^n_{os}$ is to consider centered convex bodies $K \in \mathcal{K}^n_{oc}$. For those bodies Ehrhart posed in 1964 the following conjecture as an analog to Minkowski's convex body Theorem 1.1.

Conjecture 3.3 (Ehrhart, 1964, [30]). *Let $K \in \mathcal{K}^n_{oc}$ with $\operatorname{vol}(K) \ge (n+1)^n/n!$. Then K contains a non-trivial lattice point, i.e., $K \cap \mathbb{Z}^n \setminus \{\mathbf{0}\} \ne \emptyset$.*

Moreover, he conjectured that the bound is best possible only (up to $\mathbb{Z}^n$ preserving linear transformations) for the simplex

$$T_n := -\sum_{i=1}^{n} \boldsymbol{e}_i + (n+1)\operatorname{conv}\{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}. \tag{7}$$

Ehrhart proved his conjecture in various special cases, e.g., in the plane [28] and for simplices [31]. Berman and Berndtsson [12] proved it for a special class of n-dimensional lattice polytopes including so called reflexive polytopes (see also [64]). The best known bound for which the conclusion of Conjecture 3.3 holds true is based on a lower bound for the ratio

$$\alpha(K) := \frac{\operatorname{vol}(K \cap (-K))}{\operatorname{vol}(K)}, \quad K \in \mathcal{K}^n_{oc},$$

since then $\operatorname{vol}(K) \ge \alpha(K)^{-1} 2^n$ implies by Minkowski's Theorem 1.1 that $K \cap (-K)$ and thus K contains a non-trivial lattice point.

Milman and Pajor [60] proved $\alpha(K) \geq 2^{-n}$, this was improved by Huang et al. [47] to $2^{-n}\mathrm{e}^{c\sqrt{n}}$ where c is an absolute constant, and recently it was shown by Campos et al. [21, Theorem 4.1] that

$$\alpha(K) \geq 2^{-n}\,\mathrm{e}^{c\,n/L_n^2},$$

where L_n is the so called isotropic constant (see, e.g., [19]). Together with the very recently announced upper bound of $O(\sqrt{\log(n)})$ onto L_n by Klartag [50], the result of Campos et al. shows that $K \in \mathcal{K}^n_{oc}$ contains a non-trivial lattice point if $\mathrm{vol}(K) \geq 4^n \mathrm{e}^{-c\,n/\log(n)^2}$ (cf. [21, Theorem 1.4]); observe that $(n+1)^n/n! \sim \mathrm{e}^n$ and it is believed that $\alpha(K)$ is minimized for a simplex which would give "almost" a bound of e^n.

In view of Minkowski's successive minima it is also tempting to consider the following generalization of Ehrhart's conjecture (see [43]).

Problem 3.4. Let $K \in \mathcal{K}^n_{oc}$ and $\Lambda \in \mathcal{L}^n$. Is it true that

$$\lambda_1(K,\Lambda)\cdots\lambda_n(K,\Lambda)\,\mathrm{vol}(K) \leq \frac{(n+1)^n}{n!}\,\det(\Lambda)\,?$$

By the same reasoning as above and the monotonicity of the successive minima such an inequality exists with $4^n\mathrm{e}^{-c\sqrt{n}}$ instead of $(n+1)^n/n!$, and in [43] it is also verified in some special cases, e.g., in the plane. Moreover, in analogy to Minkowski's lower bound in Theorem 1.3, the authors of [43] prove that for $K \in \mathcal{K}^n_{oc}$,

$$\frac{n+1}{n!}\,\det(\Lambda) \leq \lambda_1(K,\Lambda)\cdots\lambda_n(K,\Lambda)\,\mathrm{vol}(K)$$

along with a characterization of the equality case. In particular, if $\Lambda = \mathbb{Z}^n$, equality is attained for

$$T_n{}^\star = \mathrm{conv}\{-(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n), \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}. \tag{8}$$

As the volume is a particular intrinsic volume $\mathrm{V}_i(K)$ (see (2)) one may also ask for inequalities relating these functionals to the successive minima. The $\mathrm{V}_i(K)$, however, are not invariant under linear transformations of determinant 1 and so results for $\mathbb{Z}^n$ cannot be equivalently formulated for arbitrary lattices as in the case of the volume inequalities presented so far.

As the i-th intrinsic volume is not smaller than the volume of any intersection of K with an i-dimensional plane, the lower bound in Theorem 3.1 implies for $K \in \mathcal{K}^n$, $\dim K = n$, and $\Lambda = \mathbb{Z}^n$ (see [82])

$$\frac{2^i}{i!} \leq \lambda_1(K_s)\cdots\lambda_i(K_s)\,\mathrm{V}_i(K), \quad 1 \leq i \leq n. \tag{9}$$

An o-symmetric convex body without non-trivial lattice points, but with a large i-dimensional section shows there is no upper bound on the right hand side for $i \leq n-1$. On the other hand, it was shown in [42] that for $K \in \mathcal{K}^n$, $\dim K = n$,

$$\lambda_{i+1}(K_s)\cdots\lambda_n(K_s) < 2^{n-i}\frac{\mathrm{V}_i(K)}{\mathrm{vol}(K)}, \quad 1 \leq i \leq n-1, \tag{10}$$

and that this inequality is best possible. Actually, both inequalities (9) and (10) were originally proved for o-symmetric convex bodies, but the proofs also work in this slightly more general setting. We also remark, that in general the ratio $\mathrm{V}_i(K)/\mathrm{vol}(K)$ in (10) is not bounded from above in terms of the successive minima. To generalize (10) to the setting of arbitrary lattices is still an open problem.

Conjecture 3.5 (Schnell, 1995, [73]). *Let* $K \in \mathcal{K}^n$, $\dim K = n$, *and let* $\Lambda \in \mathcal{L}^n$. *Then*

$$\lambda_{i+1}(K_s,\Lambda)\cdots\lambda_n(K_s,\Lambda) < 2^{n-i}\frac{\mathrm{V}_i(K)}{\mathrm{vol}(K)}\frac{\det(\Lambda)}{\det_i(\Lambda)}, \quad 1 \leq i \leq n-1,$$

where $\det_i(\Lambda)$ *is the smallest determinant of an* i*-dimensional sublattice of* Λ.

Schnell made this conjecture only for $K \in \mathcal{K}^n_{os}$, but in view of (10) it is quite plausible that it holds true for any $K \in \mathcal{K}^n$.

The case $i = n-1$ in (10) gives the particularly nice inequality

$$\lambda_n(K_s) < \frac{\mathrm{F}(K)}{\mathrm{vol}(K)}, \tag{11}$$

where $\mathrm{F}(K)$ is the surface area of K (cf. (2)). In (26) we will also see a kind of discrete counterpart to (11).

For $K \in \mathcal{K}^n_{os}$ and $i = n-1$, (9) was improved in [43] to the tight inequality

$$\frac{2^n}{(n-1)!} \leq \left(\sum_{i=1}^{n}\prod_{j=1,\, j\neq i}^{n}\lambda_j(K)^2\right)^{\frac{1}{2}}\mathrm{F}(K).$$

A corresponding best possible inequality for $K \in \mathcal{K}^n$, or with respect to arbitrary lattices or for other intrinsic volumes is not known (see the discussion in [43]).

4 Successive Minima and polarity

In the context of so-called transference theorems in number theory the goal is (roughly speaking) to establish relations between integral solutions of different linear Diophantine approximation problems (see, e.g., [41, Section 45], [40, Section 23.2]). From a geometric point of view, this means to relate functionals such as volume or successive minima of K and $K^\star$. The study of this fruitful interplay goes back to Kurt Mahler in 1930s (see, e.g, [32]), and in [53] he studied for $K \in \mathcal{K}^n_{os}$ the linearly invariant volume product $\mathrm{vol}(K)\,\mathrm{vol}(K^\star)$, today also known as Mahler volume.

Conjecture 4.1 (Mahler, 1939, [53]). *Let* $K \in \mathcal{K}^n_{os}$. *Then*

$$\frac{4^n}{n!} = \mathrm{vol}(C_n)\,\mathrm{vol}(C_n{}^\star) \leq \mathrm{vol}(K)\,\mathrm{vol}(K^\star). \tag{12}$$

Mahler [52] verified the conjecture in dimension 2, and it was also recently proved in dimension 3 [48]. He also proved a lower bound of $4^n/(n!)^2$ and an upper bound on the product of 4^n. The best known general bounds are

$$\frac{\pi^n}{n!} \leq \mathrm{vol}(K)\,\mathrm{vol}(K^\star) \leq \omega_n^2. \tag{13}$$

The apparently best possible upper bound, which was also conjectured by Mahler [53], is known as the Blaschke-Santaló Theorem [68]. The lower bound is due to Kuperberg [51]. In the general case, i.e., for $K \in \mathcal{K}^n$ it is conjectured, and also attributed to Mahler, that

$$\frac{(n+1)^{n+1}}{(n!)^2} = \mathrm{vol}(S_n)\,\mathrm{vol}({S_n}^\star) \leq \mathrm{vol}(K)\,\mathrm{vol}(K^\star), \tag{14}$$

where S_n is any simplex with the centroid at the origin, e.g., T_n from (7). This is only known to be true in the plane [52]. For more information on the Mahler volume and its central role within Convex Geometry we refer to [34].

Combining the upper bound in Minkowski's inequality (5) for $K^\star$ with the conjectured lower bound in Mahler's Conjecture (4.1) leads for $K \in \mathcal{K}^n_{os}$ to the following conjectural inequality, which is also due to Mahler.

Conjecture 4.2 (Mahler, 1974, [54]). *Let $K \in \mathcal{K}^n_{os}$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{2^n}{n!}\det(\Lambda)\ \lambda_1(K^\star,\Lambda^\star)\cdots\lambda_n(K^\star,\Lambda^\star) \leq \mathrm{vol}(K). \tag{15}$$

The inequality would be best possible, for instance, for the cross-polytope ${C_n}^\star$, and the previously mentioned results on the volume product imply that it is true for $n = 2, 3$. Even the weaker inequality (as an analogue to Theorem 1.2),

$$\frac{2^n}{n!}\det(\Lambda)\ \lambda_1(K^\star,\Lambda^\star)^n \leq \mathrm{vol}(K), \tag{16}$$

which has also been studied by Mahler, is open for $n \geq 4$. For general convex bodies the same problem was studied by Makai Jr.

Conjecture 4.3 (Makai, Jr., 1978, [55]). *Let $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{n+1}{n!}\det(\Lambda)\ \lambda_1({K_s}^\star,\Lambda^\star)^n \leq \mathrm{vol}(K). \tag{17}$$

It was shown to be true for $n = 2$ by L. Fejes Tóth and Makai, Jr. [33] (see also [39] for applications). In view of (15) one might even conjecture the stronger inequality

$$\frac{n+1}{n!}\det(\Lambda)\lambda_1({K_s}^\star,\Lambda^\star)\cdots\lambda_n({K_s}^\star,\Lambda^\star) \leq \mathrm{vol}(K), \tag{18}$$

which would be also best possible as the simplex ${T_n}^\star$ from (8) shows. For $n = 2$, (18) is an immediate consequence of the upper bound in (5) and Eggleston's [27] inequality

$$6 \le \mathrm{vol}(K)\,\mathrm{vol}({K_s}^\star) \tag{19}$$

for planar convex bodies K. Actually, taking into account all successive minima, there seems to be stronger lower bounds possible than the one in (18). For the planar case see [45]. In contrast to Minkowski's Theorem 3.1, here upper bounds on $\mathrm{vol}(K)$ in terms of $\lambda_i(K^\star, \Lambda^\star)$ are the easy part. In [45] it was shown that for $K \in \mathcal{K}^n$

$$\mathrm{vol}(K) \le 2^n \det(\Lambda)\lambda_1({K_s}^\star, \Lambda^\star)\cdots\lambda_n({K_s}^\star, \Lambda^\star),$$

and if $K \in \mathcal{K}^n_{oc}$ then

$$\mathrm{vol}(K) \le \frac{(n+1)^n}{n!}\det(\Lambda)\lambda_1({K_s}^\star, \Lambda^\star)\cdots\lambda_n({K_s}^\star, \Lambda^\star).$$

Both inequalities are best possible as the usual suspects show.

The inequalities (16) and (17) have a nice geometric interpretation in terms of the so called lattice width. For $K \in \mathcal{K}^n$ we have by the definition of the polar set that

$$\lambda_1((K-K)^\star, \Lambda^\star) = \min_{\boldsymbol{a}^\star \in \Lambda^\star \setminus \{\boldsymbol{0}\}} \max_{\boldsymbol{y} \in K-K} \langle \boldsymbol{a}^\star, \boldsymbol{y} \rangle =: \mathrm{w}(K, \Lambda), \tag{20}$$

and $\mathrm{w}(K, \Lambda)$ is the lattice width of K with respect to Λ. It describes, roughly speaking, the minimal number of parallel lattice hyperplanes of the lattice Λ intersecting K; a lattice hyperplane is a hyperplane containing n affinely independent lattice points of Λ. Hence, (16) and (17) claim that the volume of a convex body of lattice width 2 is at least the volume of ${C_n}^\star$ if K is symmetric and, otherwise, at least the volume of ${T_n}^\star$.

An interesting weaker inequality than (17) was conjectured in the context of isosystolic inequalities for optical hypersurfaces.

Conjecture 4.4 (Álvarez Paiva et al., 2016, [5])**.** *Let $K \in \mathcal{K}^n_o$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{n+1}{n!}\det(\Lambda)\ \lambda_1(K^\star, \Lambda^\star)^n \le \mathrm{vol}(K), \tag{21}$$

with equality if and only if K is a simplex whose vertices are its only non-trivial lattice points.

It was pointed out in [45] that in general there is no upper bound on $\mathrm{vol}(K)$ in this setting. For possible extensions of Makai's conjecture (17) via covering minima (instead of the successive minima) we refer to González Merino and Schymura [39].

Applying Minkowski's upper bound in (5) to K, Λ and $K^\star, \Lambda^\star$ gives with (13)

$$\lambda_1(K, \Lambda)\lambda_1(K^\star, \Lambda^\star)\cdots\lambda_n(K, \Lambda)\lambda_n(K^\star, \Lambda^\star) \le \frac{4^n}{\pi^n}n!$$

and, in particular,

$$\lambda_1(K,\Lambda)\lambda_1(K^\star,\Lambda^\star)\leq c\,n,$$

where, in the following c denotes an absolute constant which may vary from line to line. This easily derived inequality is essentially best possible, as Banaszczyk [8, Lemma 2] proved that in every dimension n there exist a lattice $\Lambda\in\mathcal{L}^n$ with

$$c\,n\leq\lambda_1(K,\Lambda)\lambda_1(K^\star,\Lambda^\star). \tag{22}$$

In the case $K=B_n$ this was shown earlier by Conway and Thompson [61, Chapter II, Theorem 9.5].

For products of the type $\lambda_i(K,\Lambda)\lambda_j(K^\star,\Lambda^\star)$ one can only expect a non-trivial lower bound if $j\geq n+1-i$ and an upper bound if $j\leq n+1-i$.

In the already mentioned paper [53], Mahler was the first who studied the products $\lambda_i(K,\Lambda)\lambda_{n+1-i}(K^\star,\Lambda^\star)$ and proved for $K\in\mathcal{K}^n_{os}$ the bounds

$$1\leq\lambda_i(K,\Lambda)\lambda_{n+1-i}(K^\star,\Lambda^\star)\leq n!.$$

The lower bound is clearly optimal, but the upper bound has been improved considerably in the last decades. The currently best bounds are due to Banaszczyk and are based on his groundbreaking Gaussian-like measures on lattices introduced in [7]. In [8] he proves for $K\in\mathcal{K}^n_{os}$

$$\lambda_i(K,\Lambda)\lambda_{n+1-i}(K^\star,\Lambda^\star)\leq c\,n(1+\log(n)),$$

which, in view of (22) is close to optimal. Moreover, he also shows that the $(1+\log(n))$ term can be improved for various classes of symmetric convex bodies. In particular, for $K=B_n$ it was already shown in [7] that

$$\lambda_i(\Lambda)\lambda_{n+1-i}(\Lambda^\star)\leq n.$$

If $K\in\mathcal{K}^n$ is lattice point free with respect to a lattice Λ. i.e., $\mathrm{int}(K)\cap\Lambda=\emptyset$, then the covering radius is at least 1, i.e., $\mu(K,\Lambda)\geq 1$. Hence, any upper bound on $\mu(K,\Lambda)\lambda_1({K_s}^\star,\Lambda^\star)$, $K\in\mathcal{K}^n$, is a bound on the so-called flatness constant $\mathrm{flt}(n)$, the maximal lattice width of a lattice point free convex body in $\mathbb{R}^n$.

That this quantity can be indeed bounded by a constant only depending on the dimension was first shown by Khinchin [46].

For $K\in\mathcal{K}^n_{os}$, Banaszczyk [8] proved

$$\mu(K,\Lambda)\lambda_1(K^\star,\Lambda^\star)\leq c\,n\,\log(n) \tag{23}$$

and so $\mathrm{flt}(n)\leq c\,n\log(n)$ for all convex bodies having a center of symmetry. For general $K\in\mathcal{K}^n$, the following bound was very recently announced by Reis and Rothvoss [66] (see also [67], [9] for the former best known bounds)

$$\mathrm{flt}(n)\leq c\,n\,\log(n)^8.$$

Actually, their main result implies the astonishing relation

$$\mu(K,\Lambda)\leq c\,\log(n)^7\mu(K-K,\Lambda),$$

which gives the bound on the flatness constant via the symmetric case (23). The best lower bounds on $\mathrm{flt}(n)$ are of order n (see [9, 58] and the references within).

5 Successive Minima and lattice point enumerator

Since the successive minima measure or reflect lattice point properties of a convex body one may also ask for direct relations between the successive minima and the lattice point enumerator $\mathrm{G}(K,\Lambda) := \#(K\cap\Lambda)$; in the case $\Lambda=\mathbb{Z}^n$ we just write $\mathrm{G}(K)$. A result in this spirit is again due to Minkowski who proved in analogy to his convex body Theorem 1.1.

Theorem 5.1 (Minkowski, 1896, [63, p. 79]). *Let $K\in\mathcal{K}^n_{os}$, $\dim K = n$ and $\mathrm{G}(K)\geq 3^n+1$. Then K contains a non-trivial lattice point in its interior.*

The cube C_n shows that the bound cannot be improved in general. Minkowski also proved a sharper bound of $2^{n+1}-1$ for the class of strictly o-symmetric convex bodies, but for simplification we will deal only with the general case and we refer to [41, p. 63] for details, and to [38] for an interesting generalization of these statements of Minkowski.

Betke et al. [13] embedded the above result of Minkowski in an inequality for o-symmetric convex bodies in the sense of Theorem 1.2 which was later extended to $K\in\mathcal{K}^n$ by Malikiosis [56]. Let $K\in\mathcal{K}^n$ and $\Lambda\in\mathcal{L}^n$. Then

$$\mathrm{G}(K,\Lambda)\leq\left\lfloor\frac{2}{\lambda_1(K_s,\Lambda)}+1\right\rfloor^n.$$

And, obviously, the conjecture is that this can also be improved via the product of all successive minima as in Minkowski's Theorem 1.3 on successive minima.

Conjecture 5.2 (Betke et al.[13]; Malikiosis [56]). *Let $K\in\mathcal{K}^n$ and $\Lambda\in\mathcal{L}^n$. Then*

$$\mathrm{G}(K,\Lambda)\leq\left\lfloor\frac{2}{\lambda_1(K_s,\Lambda)}+1\right\rfloor\cdots\left\lfloor\frac{2}{\lambda_n(K_s,\Lambda)}+1\right\rfloor.$$

The cube, or more generally, a box $[-l_1,l_1]\times\cdots\times[-l_n,l_n]$, $l_i\in\mathbb{N}$, shows that the bound would be tight. Again Betke et al. just considered the symmetric case, in which they also proved a best possible lower bound [13, Corollary 2.1] if $\lambda_n(K,\Lambda)\leq 2$:

$$\frac{1}{n!}\left(\frac{2}{\lambda_1(K,\Lambda)}-1\right)\cdots\left(\frac{2}{\lambda_n(K,\Lambda)}-1\right)\leq\mathrm{G}(K,\Lambda).$$

In [56], Malikiosis proved the Conjecture 5.2 for $n=3$, and in general he showed that

$$\mathrm{G}(K,\Lambda)\leq\frac{4}{\mathrm{e}}\sqrt{3}^{\,n-1}\left\lfloor\frac{2}{\lambda_1(K_s,\Lambda)}+1\right\rfloor\cdots\left\lfloor\frac{2}{\lambda_n(K_s,\Lambda)}+1\right\rfloor, \tag{24}$$

where $\sqrt{3}$ can be replaced by $\sqrt[3]{40/9}$ if $K=-K$. Moreover, in [57] he verified it for ellipsoids in every dimension.

Recently, Tointon [78] presented a different type of upper bound on $\mathrm{G}(K,\Lambda)$ in terms of the successive minima:

$$\mathrm{G}(K,\Lambda)\leq\left(1+\frac{\lambda_k(K,\Lambda)}{2}\right)\frac{2}{\lambda_1(K,\Lambda)}\cdots\frac{2}{\lambda_k(K,\Lambda)},$$

where the index k is chosen such that $k = \max\{j : \lambda_j(K,\Lambda) \leq 2\}$ which can be replaced in the symmetric setting by $k = \max\{j : \lambda_j(K,\Lambda) \leq 1\}$. Tointon's inequality improves on (24) in the symmetric case, as well as in various cases for general $K \in \mathcal{K}^n$. Moreover, it has, as well as Conjecture 5.2, the nice and important feature that it implies the continuous case, i.e., the upper bound in Minkowski's Theorem 3.1. The reason is that, roughly speaking, for "fat" convex bodies there is almost no difference between $\mathrm{vol}(K)$ and $\mathrm{G}(K)$, or, more precisely, the Jordan measurability of convex bodies gives

$$\lim_{\rho\to\infty} \frac{\mathrm{vol}(\rho\, K)}{\det(\Lambda)\, \mathrm{G}(\rho\, K,\Lambda)} = 1. \tag{25}$$

In order to control the gap between $\mathrm{vol}(K)$ and $\mathrm{G}(K)$ for "thin" convex bodies, Betke et al. also started to study bounds on $\mathrm{G}(K)/\mathrm{vol}(K)$ in terms of the successive minima. And here the following inequalities could be true.

Conjecture 5.3 (Betke et al., 1993). *Let $K \in \mathcal{K}^n$, $\dim(K) = n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\prod_{i=1}^{n}\left(1 - i\frac{\lambda_i(K_s,\Lambda)}{2}\right) \leq \frac{\mathrm{G}(K,\Lambda)}{\mathrm{vol}(K)}\det(\Lambda) \leq \prod_{i=1}^{n}\left(1 + i\frac{\lambda_i(K_s,\Lambda)}{2}\right),$$

where for the lower bound $n\,\lambda_n(K_s,\Lambda) \geq 2$ is assumed and $\mathrm{G}(K)$ might be replaced by $\mathrm{G}(\mathrm{int}(K))$.

Actually, in [13, Conjecture 2.2] Betke et al. state only a conjecture about a corresponding lower bound for symmetric convex bodies in which the is in the factors of the product are replaced by 1, and they pose the problem to consider also upper bounds.

The bounds in the Conjecture 5.3 would be tight as, e.g., positive integral multiples of the standard simplex $\mathrm{conv}\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ show (see [35]). Freyer and Lucas verified in [35] the upper bound in Conjecture 5.3 in the plane and the lower bound for planar lattice polytopes. In arbitrary dimensions they proved the following weaker inequalities

$$\prod_{i=1}^{n}\left(1 - n\frac{\lambda_i(K_s,\Lambda)}{2}\right) \leq \frac{\mathrm{G}(K,\Lambda)}{\mathrm{vol}(K)}\det(\Lambda) \leq \prod_{i=1}^{n}\left(1 + n\frac{\lambda_i(K_s,\Lambda)}{2}\right).$$

Observe that the upper bound together with Minkowski's upper bound (5) give

$$\mathrm{G}(K,\Lambda) \leq \left(\frac{2}{\lambda_1(K,\Lambda)} + n\right)\cdots\left(\frac{2}{\lambda_n(K,\Lambda)} + n\right),$$

which in turn via (25) implies Minkowski's upper bound (5) [35].

A different point of view on Conjecture 5.2 is given by Ehrhart theory from Discrete Geometry. By the monotonicity of the successive minima it suffices to prove the conjecture instead of for $K \in \mathcal{K}^n$ for the associated lattice polytope $P := \mathrm{conv}(K \cap \Lambda)$. A polytope is called a lattice polytope (with respect to a lattice Λ) if all its vertices are lattice points of Λ. According to a result due to Ehrhart [29] we have for $k \in \mathbb{N}$

$$\mathrm{G}(k\,P,\Lambda) = \sum_{i=0}^{n} \mathrm{G}_i(P,\Lambda)k^i,$$

which is also known as the Ehrhart-polynomial. The coefficients $\mathrm{G}_i(P,\Lambda)$ have been the subject of intensive investigations over the last decades (see [11]) and (at least) three of them have a very clear geometric meaning

$$\mathrm{G}_n(P,\Lambda)=\frac{\mathrm{vol}(P)}{\det(\Lambda)},\quad \mathrm{G}_{n-1}(P,\Lambda)=\frac{1}{2}\sum_{i=1}^{m}\frac{\mathrm{vol}_{n-1}(F_i)}{\det(\mathrm{aff}(F_i)\cap\Lambda)},\quad \mathrm{G}_0(P,\Lambda)=1,$$

where $F_1,\dots,F_m$ are the facets of P, and $\mathrm{G}_0(P,\Lambda)$ corresponds to the Euler-characteristic of P. Hence, one may also ask for relations of the other coefficients (apart from the volume) to the successive minima. In [44] it was shown for o-symmetric lattice polytopes

$$\frac{\mathrm{G}_{n-1}(P,\Lambda)}{\mathrm{G}_n(P,\Lambda)}\leq\frac{1}{2}\sum_{i=1}^{n}\lambda_i(P,\Lambda). \tag{26}$$

The inequality is best possible, e.g., for C_n and $C_n{}^{\star}$, and maybe regarded as a discrete counterpart to (11) for o-symmetric lattice polytopes. For generalizations to not necessarily symmetric polytopes or lattice polytopes having their centroid at the origin we refer to [43]. Here we want to point out one nice feature of the above inequality for symmetric polytopes; together with Minkowski's upper bound it gives the best possible inequality

$$\mathrm{G}_{n-1}(P,\Lambda)\leq\sum_{j=1}^{n}\prod_{i\neq j}\frac{2}{\lambda_i(P,\Lambda)}.$$

Hence, in view of Minkowski's upper bound on the volume it is tempting to conjecture that $\mathrm{G}_i(P,\Lambda)$ is bounded by the i-th elementary symmetric function of the successive minima. This, however, fails already for $i=n-2$ as shown in [14, Proposition 1.1], but, on the positive side it is true for special lattice polytopes, as parallelepipeds and symmetric lattice-face polytopes (see [14]).

The problem of counting lattice points inside a convex body may also be considered from the more general point of view of covering lattice points by a minimum number of k-dimensional affine subspaces. For those types of covering problems we refer to [6, 10, 15, 18, 37] and here, as an appetizer we only state the following result by Balko et al. [6, Theorem 2.5]: Let $K\in\mathcal{K}^n_{os}$ containing n linearly independent lattice points of Λ and let $1\leq k\leq n-1$. Then up to constants depending on k and n the lattice points of $K\cap\Lambda$ can be covered by

$$(\lambda_{k+1}(K,\Lambda)\cdots\lambda_n(K,\Lambda))^{-1}$$

k-dimensional affine subspaces and the bound is optimal – up to constants depending on k and n.

6 An application: Bombieri-Vaaler extension of Siegel's lemma

Let $A=(a_{ij})\in\mathbb{Z}^{m\times n}$, $m<n$, be an integral matrix of rank m. Consider the system of homogeneous linear equations

$$A\boldsymbol{x}=\boldsymbol{0}\,. \tag{27}$$

Since $m < n$, the system (27) has a non–trivial solution in integers. If the entries of A are relatively small integers, then it is reasonable to expect that there will be a solution in relatively small integers. This principle was applied by Thue in [77] to a problem from Diophantine approximations. Siegel [74, Bd. I, p. 213, Hilfssatz] was the first to state this idea formally.

Let us denote by $\|A\|_\infty$ the maximum absolute value of an entry of A, that is $\|A\|_\infty = \max_{ij} |a_{ij}|$. Following Siegel's work, one can obtain the following result (included with proof in Schmidt [71]).

Theorem 6.1 (Siegel's Lemma)**.** *The system* (27) *has a solution* $\boldsymbol{x} \in \mathbb{Z}^n$ *with*

$$0 < \|\boldsymbol{x}\|_\infty < 1 + (n\|A\|_\infty)^{m/(n-m)} . \tag{28}$$

Notably, the exponent $m/(n-m)$ on the right hand side of (28) is optimal. Siegel's lemma type results have been motivated by their numerous applications in number theory (see e.g. [17, 71, 36]). In more recent years, new applications have been developed, in particular in mathematical optimization [3, 4].

To establish a link between Siegel's lemma and successive minima, we follow the work of Bombieri and Vaaler [17]. They have proved, by using geometry of numbers, the following advanced version of Siegel's lemma.

Theorem 6.2. *The system* (27) *has* $n-m$ *linearly independent solutions* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-m}$ *in* $\mathbb{Z}^n$*, with*

$$\prod_{i=1}^{n-m} \|\boldsymbol{x}_i\|_\infty \leq \frac{\sqrt{\det(AA^T)}}{\gcd(A)},$$

where $\gcd(A)$ *is the greatest common divisor of all* $m \times m$ *subdeterminants of* A*.*

Recall that $C_n = [-1,1]^n$ and let $\ker(A) = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{0}\}$. Consider the section $S(A) = C_n \cap \ker(A)$ of the cube C_n and the lattice $\Lambda(A) = \mathbb{Z}^n \cap \ker(A)$. The lattice $\Lambda(A)$ has determinant $\det(\Lambda(A)) = \sqrt{\det(AA^T)}/\gcd(A)$. The $(n-m)$-dimensional subspace $\ker(A)$ can be considered as a usual Euclidean $(n-m)$-dimensional space. This immediately extends the definition of successive minima to o-symmetric bounded convex sets with nonempty relative interior in $\ker(A)$ and $(n-m)$-dimensional lattices in $\ker(A)$. Theorem 6.2 is an immediate corollary of the following result.

Theorem 6.3. *Let* $A \in \mathbb{Z}^{m\times n}$*,* $m < n$*, be an integral matrix of rank* m*. Then the inequality*

$$\prod_{i=1}^{n-m} \lambda_i(S(A), \Lambda(A)) \leq \det(\Lambda(A)) \tag{29}$$

holds.

Proof. By a result of Vaaler [79], we have $\mathrm{vol}_{n-m}(S(A)) \geq 2^{n-m}$. Hence, Minkowski's theorem on successive minima in the form (3) gives

$$\prod_{i=1}^{n-m} \lambda_i(S(A), \Lambda(A)) \leq \frac{2^{n-m} \det(\Lambda(A))}{\mathrm{vol}_{n-m}(S(A))} \leq \det(\Lambda(A)) \,. \qquad \square$$

In what follows, we will focus on the special case $m = 1$, that is when A is just an n-dimensional nonzero row vector. Theorem 6.2 implies that for every nonzero vector $\boldsymbol{a}$ in $\mathbb{Z}^n$, $n \geq 2$, there exists a vector $\boldsymbol{x}$ in $\mathbb{Z}^n$, such that

$$\langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0 \,, \quad 0 < \|\boldsymbol{x}\|_\infty^{n-1} \leq \sqrt{n} \|\boldsymbol{a}\|_\infty \,. \tag{30}$$

The exponent $n - 1$ in the latter bound is optimal. Let us define

$$c(n) = \sup_{\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}} \inf_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \setminus \{0\} \\ \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0}} \frac{\|\boldsymbol{x}\|_\infty^{n-1}}{\|\boldsymbol{a}\|_\infty} \,.$$

That is $c(n)$ is the optimal constant in the bound (30).

It is easy to see that $c(2) = 1$. Further, the equality $c(3) = 4/3$ is implicit in [24]. Namely, the inequality $c(3) \leq 4/3$ is contained in [24, Lemma 4], while the inequality $c(3) \geq 4/3$ is a consequence of [24, Lemma 7]. We have also $c(4) = 27/19$. The inequality $c(4) \geq 27/19$ was proved by Chaladus in [23] and its counterpart $c(4) \leq 27/19$ was obtained in [1] (see also [69]). For $n > 4$, the exact values of the constants $c(n)$ remain unknown.

In this vein, Schinzel [69] proved the following general result that gives a geometric interpretation for the constant $c(n)$. Given $K \in \mathcal{K}_{os}^n$ we denote by $\Delta(K)$ its critical determinant, defined as

$$\Delta(K) = \min\{\det(\Lambda) : 2\Lambda \text{ is a packing lattice for } K\} \,.$$

In terms of the density $\delta(K)$ (see (4)), we have

$$\Delta(K) = \frac{\mathrm{vol}(K)}{2^n \delta(K)} \,.$$

Theorem 6.4. *For $n \geq 3$*

$$c(n) = \sup \Delta(H_{\alpha_1,\ldots,\alpha_{n-3}})^{-1} \,,$$

where $H_{\alpha_1,\ldots,\alpha_{n-3}}$ is a generalised hexagon given by

$$H_{\alpha_1,\ldots,\alpha_{n-3}} = \left\{ \boldsymbol{x} \in \mathbb{R}^{n-1} : \|\boldsymbol{x}\|_\infty \leq 1, \left| \sum_{i=1}^{n-3} \alpha_i x_i + x_{n-2} + x_{n-1} \right| \leq 1 \right\}$$

and the supremum is taken over all rational numbers $\alpha_1, \ldots, \alpha_{n-3}$ in the interval $(0, 1]$.

Based on the values of $c(n)$ for $n \leq 4$, the following conjecture was proposed in [2].

Conjecture 6.5. *The equality*

$$c(n) = \Delta(H_{1,\dots,1})^{-1}$$

holds. Here $H_{1,\dots,1}$ *is a generalised hexagon in* $\mathbb{R}^{n-1}$.

From the perspective of Theorems 6.2–6.3 and the successive minima, it is natural to consider for $n \ge 2$ the constant

$$s(n) = \sup_{\boldsymbol{a}\in\mathbb{Z}^n\setminus\{0\}} \inf_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{n-1}} \frac{\prod_{i=1}^{n-1}\|\boldsymbol{x}_i\|_\infty}{\|\boldsymbol{a}\|_\infty}\,, \tag{31}$$

where the infimum is taken over all linearly independent integer vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_{n-1}\in\mathbb{Z}^n$ such that $\langle \boldsymbol{a},\boldsymbol{x}_1\rangle = \cdots = \langle \boldsymbol{a},\boldsymbol{x}_{n-1}\rangle = 0$.

The bound in Theorem (6.2) immediately implies

$$s(n) \le \sqrt{n}\,. \tag{32}$$

In [2], the constant $s(n)$ was estimated as

$$s(n) \le \sigma_n^{-1}\,, \tag{33}$$

where σ_n is the sinc integral

$$\sigma_n = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt\,.$$

The bound (33) asymptotically improves on (32) with factor $\sqrt{\pi/6}$. The numbers σ_n are rational, the sequences of numerators and denominators of $\sigma_n/2$ can be found in [75] (sequences A049330 and A049331).

Clearly, $s(2) = c(2) = 1$. In this section we prove the following result, mentioned without proof in [2, Remark 1 (ii)].

Theorem 6.6. *For* $n\in\{3,4\}$ *we have* $s(n) = c(n)$. *That is*

$$s(3) = \frac{4}{3} \text{ and } s(4) = \frac{27}{19}\,.$$

Proof. Observe that for any nonzero $\boldsymbol{a}\in\mathbb{Z}^n$

$$\inf_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\setminus\{0\}\\ \langle\boldsymbol{a},\boldsymbol{x}\rangle=0}} \frac{\|\boldsymbol{x}\|_\infty^{n-1}}{\|\boldsymbol{a}\|_\infty} \le \inf_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{n-1}} \frac{\prod_{i=1}^{n-1}\|\boldsymbol{x}_i\|_\infty}{\|\boldsymbol{a}\|_\infty}\,,$$

where the latter infimum is taken over the same set as in (31). Hence for all $n\ge 2$ we have

$$c(n) \le s(n)\,. \tag{34}$$

In view of (34), it is sufficient to show that $s(3) \le 4/3$ and $s(4) \le 27/19$. To achieve this goal, we will first express $s(n)$ in terms of successive minima. Given a nonzero $\boldsymbol{a} \in \mathbb{Z}^n$, let $\ker(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$, and consider the lattice

$$\Lambda(\boldsymbol{a}) = \mathbb{Z}^n \cap \ker(\boldsymbol{a}) \,,$$

and the $(n-1)$-dimensional section

$$S(\boldsymbol{a}) = C_n \cap \ker(\boldsymbol{a})$$

of the cube C_n. Then, by the definition of successive minima, we have

$$s(n) = \sup_{\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}} \frac{\prod_{i=1}^{n-1} \lambda_i(S(\boldsymbol{a}), \Lambda(\boldsymbol{a}))}{\|\boldsymbol{a}\|_\infty} \,. \tag{35}$$

Given $K \in \mathcal{K}^n_{os}$, its anomaly $a(K)$ is defined as

$$a(K) = \sup_{\Lambda \in \mathcal{L}^n} \frac{\Delta(K) \prod_{i=1}^{n} \lambda_i(K, \Lambda)}{\det(\Lambda)} \,.$$

The Problem 3.2 of Davenport in terms of the anomaly is asking whether $a(K) = 1$. Woods [83] proved that $a(K) = 1$ holds in dimension up to three. As above, notice that the hyperplane $\ker(\boldsymbol{a})$ can be considered as a usual Euclidean $(n-1)$-dimensional space. This immediately extends the definition of the critical determinant to o-symmetric convex sets with nonempty relative interior in $\ker(\boldsymbol{a})$. Hence, for $n \le 4$ we have

$$\prod_{i=1}^{n-1} \lambda_i(S(\boldsymbol{a}), \Lambda(\boldsymbol{a})) \le \frac{\det(\Lambda(\boldsymbol{a}))}{\Delta(S(\boldsymbol{a}))} = \frac{\|\boldsymbol{a}\|_2}{\gcd(\boldsymbol{a}) \Delta(S(\boldsymbol{a}))} \,. \tag{36}$$

We may assume without loss of generality that $\boldsymbol{a}$ does not have zero entries. Otherwise, we can replace n with $n-1$. Hence, without loss of generality, we may assume that $0 < a_1 \le \cdots \le a_n$. Consider the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that forgets the last coordinate. Since all entries of $\boldsymbol{a}$ are positive, the mapping π restricted to $\ker(\boldsymbol{a})$ is bijective. Let $K(\boldsymbol{a}) = \pi(S(\boldsymbol{a}))$. One can write $K(\boldsymbol{a})$ as follows. Given a sequence of rational numbers $0 < \alpha_1 \le \cdots \le \alpha_{n-1} \le 1$, let

$$K_{\alpha_1,\ldots,\alpha_{n-1}} = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : \|\boldsymbol{x}\|_\infty \le 1, |\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}| \le 1\} \,.$$

Then $K(\boldsymbol{a}) = K_{\alpha_1,\ldots,\alpha_{n-1}}$ with

$$\alpha_1 = \frac{a_1}{a_n}, \ldots, \alpha_{n-1} = \frac{a_{n-1}}{a_n} \,. \tag{37}$$

For any $(n-1)$-dimensional lattice $\Lambda \subset \ker(\boldsymbol{a})$, the lattice 2Λ is a packing lattice for $S(\boldsymbol{a})$ if and only if $2\pi(\Lambda)$ is a packing lattice for $K(\boldsymbol{a})$. Hence

$$\Delta(S(\boldsymbol{a})) = \Delta(K(\boldsymbol{a})) \frac{\|\boldsymbol{a}\|_2}{\|\boldsymbol{a}\|_\infty}$$

and, by (36),

$$\prod_{i=1}^{n-1} \lambda_i(S(\boldsymbol{a}), \Lambda(\boldsymbol{a})) \leq \frac{\|\boldsymbol{a}\|_\infty}{\gcd(\boldsymbol{a})\Delta(K(\boldsymbol{a}))}\,.$$

Consequently, by (35), we obtain the inequality

$$s(n) \leq \sup_{\boldsymbol{a}\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{\gcd(\boldsymbol{a})\Delta(K(\boldsymbol{a}))}\,. \tag{38}$$

The main tool of the proof of Theorem 6.6 is the following lemma.

Lemma 6.7. *For any $n \geq 3$ and any rational numbers $0 < \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq 1$, the following statements hold:*

(i) If $n = 3$ then

$$K_{1,1} \subset K_{\alpha_1,\alpha_2}\,.$$

(ii) If $n > 3$ then there exists rational numbers $0 < \beta_1 \leq \cdots \leq \beta_{n-3} \leq 1$ such that

$$K_{\beta_1,\dots,\beta_{n-3},1,1} \subset K_{\alpha_1,\dots,\alpha_{n-1}}\,.$$

This result was originally proved for $n \leq 4$ in [1], and, subsequently, for all n in [69]. We include a proof for completeness.

Proof. We start with part (i). For $\boldsymbol{x} \in K_{1,1}$, we have

$$|x_1| \leq 1, |x_2| \leq 1, |x_1 + x_2| \leq 1\,. \tag{39}$$

Multiplying the inequality $|x_2| \leq 1$ by $\alpha_2/\alpha_1 - 1$ and adding it to the last inequality in (39) we obtain the inequality $|\alpha_1 x_1 + \alpha_2 x_2| \leq \alpha_2 \leq 1$. Hence $\boldsymbol{x} \in K_{\alpha_1,\alpha_2}$.

For part (ii) take

$$\beta_1 = \frac{\alpha_1}{\alpha_{n-2}}, \dots, \beta_{n-3} = \frac{\alpha_{n-3}}{\alpha_{n-2}}\,.$$

Then if $\boldsymbol{x} \in K_{\beta_1,\dots,\beta_{n-3},1,1}$, we have

$$\begin{gathered} |x_i| \leq 1,\ i \in \{1,\dots,n-1\}\,, \\ \left|\sum_{i=1}^{n-3} \frac{\alpha_i}{\alpha_{n-2}} x_i + x_{n-2} + x_{n-1}\right| \leq 1\,. \end{gathered} \tag{40}$$

Multiplying the inequality $|x_{n-1}| \leq 1$ by $\alpha_{n-1}/\alpha_{n-2} - 1$ and adding it to the last inequality in (40), we obtain the inequality

$$\frac{\alpha_{n-1}}{\alpha_{n-2}} \geq \left|\sum_{i=1}^{n-3} \frac{\alpha_i}{\alpha_{n-2}} x_i + x_{n-2} + x_{n-1}\right| + \left(\frac{\alpha_{n-1}}{\alpha_{n-2}} - 1\right)|x_{n-1}|\,.$$

Multiplying both sides by α_{n-2} we obtain

$$1 \geq \alpha_{n-1} \geq \left| \sum_{i=1}^{n-3} \alpha_i x_i + \alpha_{n-2} x_{n-2} + \alpha_{n-2} x_{n-1} \right| + (\alpha_{n-1} - \alpha_{n-2}) |x_{n-1}|$$
$$\geq |\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}| .$$

Hence $\boldsymbol{x} \in K_{\alpha_1,\dots,\alpha_{n-1}}$. □

For the rest of the proof we choose the numbers α_i as in (37). Suppose first that $n = 3$. Since $K_{1,1} \subset K_{\alpha_1,\alpha_2}$ we have by Lemma 6.7 part (i)

$$\Delta(K(\boldsymbol{a})) = \Delta(K_{\alpha_1,\alpha_2}) \geq \Delta(K_{1,1}) .$$

Further, since the hexagon $\Delta(K_{1,1})$ is a space filling convex body (we refer the reader to [41, Section 20.4] for details), we have $\delta(K_{1,1}) = 1$ and, consequently,

$$\Delta(K_{1,1}) = \frac{\mathrm{vol}(K_{1,1})}{4} = \frac{3}{4} .$$

The bound (38) completes the proof in this case.

It remains to consider the case $n = 4$. By Lemma 6.7 part (ii) there exists a rational $0 < \beta \leq 1$ such that $K_{\beta,1,1} \subset K_{\alpha_1,\alpha_2,\alpha_3}$. A result of Whitworth [81] implies that $\Delta(K_{\beta,1,1})$ equals

$$\begin{cases} 3/4 \, , & 0 \leq \beta < 1/2 \, , \\ -(\beta^2 + 3\beta - 24 + 1/\beta)/27 \, , & 1/2 \leq \beta \leq 1 \, . \end{cases} \tag{41}$$

Hence $\Delta(K_{\beta,1,1})$ takes the minimum in the interval $[0, 1]$ at $\beta = 1$ and

$$\Delta(K(\boldsymbol{a})) = \Delta(K_{\alpha_1,\alpha_2,\alpha_3}) \geq \Delta(K_{1,1,1}) = 19/27.$$

The bound (38) completes the proof of Theorem 6.6. □

Acknowledgements

The authors thank Matthias Schymura for many valuable comments on an earlier version, and the anonymous referee for detailed and constructive comments.

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Received: April 4, 2023
Accepted for publication: April 27, 2023
Communicated by: Camilla Hollanti and Lenny Fukshansky