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Solvable Leibniz superalgebras whose nilradical has the characteristic sequence $(n-1, 1 \mid m)$ and nilindex $n+m$

Abror Khudoyberdiyev, Khosiyat Muratova.

Abstract. Leibniz superalgebras with nilindex $n+m$ and characteristic sequence $(n-1, 1 \mid m)$ divided into four parametric classes that contain a set of non-isomorphic superalgebras. In this paper, we give a complete classification of solvable Leibniz superalgebras whose nilradical is a nilpotent Leibniz superalgebra with nilindex $n+m$ and characteristic sequence $(n-1, 1 \mid m)$. We obtain a condition for the value of parameters of the classes of such nilpotent superalgebras for which they have a solvable extension. Moreover, the classification of solvable Leibniz superalgebras whose nilradical is a Lie superalgebra with the maximal nilindex is given.

1 Introduction

The theory of supervariety and superalgebras is one of the important direction of modern mathematics which generalizes many objects from differential and algebraic geometry. The interest in superalgebras is explained by their ability to unify bosons and fermions in physics, to integrate the groups of internal and dynamic symmetries into one complex, and to transfer all fundamental powers into a unified field. Lie supergroups and superalgebras are the most widely used supermanifolds in theory of supersymmetry, and the structure theory and classification problems of Lie superalgebras are important problems in non-associative algebras. Lie superalgebras are generalizations of most important object of Lie

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Contact information:
A.Kh.Khudoyberdiyev:
Affiliation: Institute of Mathematics Uzbekistan Academy of Science. National University of Uzbekistan.
Email: khabror@mail.ru
Kh.A. Muratova:
Affiliation: Kimyo International University in Tashkent, Uzbekistan. Institute of Mathematics Uzbekistan Academy of Science.
Email: xalkulova@gmail.com

algebras and for many years they attract the attention of both the mathematicians and physicists [8, 16, 19, 24, 29].

In Lie theory, there are many works devoted to the study of finite-dimensional solvable and nilpotent Lie algebras. First, we recall that in 1945 it was proved by A.I. Malcev that a solvable Lie algebra is fully determined by its nilradical [32]. Further, in 1963, Mubarakzjanov developed a method for constructing solvable algebras using the nilradical and its nil-independent derivations [33]. Using this method, a number of solvable Lie algebras with given nilradicals were constructed, such as: abelian, Heisenberg, filiform, quasi-filiform algebras and others [3, 4, 36, 39, 40, 15].

The notion of Leibniz algebras was introduced in [31] as a non-antisymmetric generalization of Lie algebras. In recent years it has been a common theme to extend various results from Lie algebras to Leibniz algebras [6]. Specifically, variations of Engel's theorem for Leibniz algebras have been proved by different authors and D. Barnes proved Levi's theorem for Leibniz algebras [7]. The analog of Mubarakzjanov's result has been applied to Leibniz algebras case in [14], which shows the importance of the consideration of their nilradicals in the Leibniz algebra case as well. The papers [2, 23, 22, 25, 9, 18, 27, 28] are also devoted to the algebraic and geometric classification of some important classes of finite-dimensional Leibniz algebras.

Leibniz superalgebras are generalizations of Leibniz algebras, and on the other hand, they naturally generalize Lie superalgebras. Leibniz superalgebras first were considered in [30] under the name of graded Leibniz algebras. The concept of Leibniz superalgebra and its cohomology was first introduced by Dzhumadil'daev in [17]. The term Leibniz superalgebras was first used in [1], whose nilpotent Leibniz superalgebras with the maximal index of nilpotency were classified. It should be noted that Lie superalgebras with maximal nilindex were classified in [20]. The distinctive property of Leibniz superalgebra is that the maximal nilindex of $(n+m)$-dimensional Leibniz superalgebra is equal to $n+m+1$. The description of Leibniz superalgebras with dimensions of even and odd parts equal to n and m, respectively, and with nilindex $n+m$ were classified by applying restrictions the invariant called characteristic sequences in [5, 11, 12, 21].

The next natural step in the theory of finite-dimensional Leibniz (Lie) superalgebras is to extend the method of classification of solvable superalgebras with their nilradical. It should be noted that the structures of solvable Lie and Leibniz superalgebras are more complex than the structures of solvable Lie and Leibniz algebras. In particular, an analog of Lie's theorem is not yet known even for Lie superalgebras. However, the corollary to the Lie theorem that the square of a solvable algebra is nilpotent is not true for Lie superalgebras, that is, in [16] an example was constructed of a solvable Lie superalgebra whose square is not nilpotent. Despite all the difficulties, in [10, 13] the solvable extension method for Leibniz superalgebras was established. The papers [26, 38] are also devoted to the description of solvable Lie and Leibniz superalgebras. In particular, in the papers [10, 26], solvable Lie and Leibniz superalgebras are obtained for whose nilradical has the maximal index of nilpotency. Some solvable Leibniz superalgebras for whose nilradical has the nilindex $n+m$ were classified in [34, 35]. In this paper, we give a classification of solvable Leibniz superalgebras whose nilradical has the index of nilpotency $n+m$ and characteristic sequence $(n-1,1|m)$. Specific values are obtained for the parameters of the classes of such nilpotent superalgebras for which they have a solvable extension.

2 Preliminaries

In this section, we give definitions and facts concerning Lie and Leibniz's superalgebras.

Definition 2.1. A $\mathbb{Z}_2$-graded vector space $G = G_0 \oplus G_1$ is called a Lie superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:
1. $[x, y] = -(-1)^{\alpha\beta}[y, x]$, for any $x \in G_\alpha,\ y \in G_\beta$,
2. $(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0$
for any $x \in G_\alpha,\ y \in G_\beta,\ z \in G_\gamma$(Jacobi superidentity).

Definition 2.2. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following condition:

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - (-1)^{\alpha\beta}\big[[x, z], y\big] - \text{Leibniz superidentity}$$

for all $x \in L, y \in L_\alpha, z \in L_\beta$.

The vector spaces L_0 and L_1 are said to be the even and odd parts of the superalgebra L, respectively. It is obvious that L_0 is a Leibniz algebra and L_1 is a representation of L_0. Note that if in Leibniz superalgebra L the identity

$$[x, y] = -(-1)^{\alpha\beta}[y, x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are generalization of Lie superalgebras.

The notions of nilpotency and solvability of Leibniz superalgebras are defined in the same way as for Leibniz algebras. For solvable Leibniz superalgebras we have that a Leibniz superalgebra L is solvable if and only if its Leibniz algebra L_0 is solvable. The concept of derivations of superalgebras differs from the notion of derivations of algebras, and as in a $\mathbb{Z}_2$-graded algebra, the space of derivations consists of even and odd subspaces. Recall, now the definition of superderivations of Leibniz superalgebras [24, 37].

Definition 2.3. A superderivation (or derivation) of a superalgebra L of degree s is a linear map $D : L \to L$ satisfying the following condition:

$$D([x, y]) = [D(x), y] + (-1)^{s\cdot\alpha}[x, D(y)],$$

where $x \in L_\alpha, y \in L$ and $s, \alpha \in \mathbb{Z}_2$

For convenience, let us shorten "derivation of even degree" to just even derivation. Linear operator $R_x : L \to L,\ x \in L$ such that $R_x(y) = (-1)^{\alpha\beta}[y, x],\ x \in L_\alpha, y \in L_\beta$ is called a right multiplication operator. It is known that such an operator is a derivation of the Leibniz superalgebra L of degree s for $x \in L_s$.

Engel's theorem and its direct consequences remain valid for Leibniz superalgebras. In particular, a Leibniz superalgebra L is nilpotent if and only if R_x is nilpotent for every homogeneous element x of L. Here is the definition of nil-independency of the superderivation of degree s imitated from Lie case (see [33]).

Definition 2.4. Let $d_1, d_2, \ldots, d_n$ be derivations of a Leibniz superalgebra L of degree s. The derivations $d_1, d_2, \ldots, d_n$ are said to be a linearly nil-independent if for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and a natural number k

$$(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n)^k = 0 \text{ implies } \ \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Note that the maximal nilpotent ideal N of the Leibniz superalgebra L such that $[L, L] \subset N$ is called a nilradical. In [13] important results regarding the solvable extension method for the finite-dimensional case are given and it is shown that solvable Lie and Leibniz superalgebras can be described using nil-independent even derivations of the nilradical. Additionally, it is proved that the dimension of a solvable Leibniz superalgebra with a given nilradical is bounded by the maximal number of nil-independent even derivations of the nilradical.

Definition 2.5. The set

$$\mathrm{Ann}_r(L) = \{z \in L \mid [L, z] = 0\}$$

is called the right annihilator of the superalgebra L.

Note that, elements of the form $[a, b] + (-1)^{\alpha\beta}[b, a]$, $(a \in L_\alpha,\ b \in L_\beta)$ are contained in $\mathrm{Ann}_r(L)$.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. Operator $R_x, x \in L_0$ is a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Taking into account the property of complex field we can consider the Jordan form of R_x. Denote by $C_i(x)(i \in \{0, 1\})$ the descending sequence of the Jordan blocks with dimensions of R_x. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 2.6. A sequence

$$C(L) = \big(\max_{x \in L_0 \setminus L_0^2} C_0(x) | \max_{\widetilde{x} \in L_0 \setminus L_0^2} C_1(\widetilde{x})\big)$$

is said to be the characteristic sequence of the Leibniz superalgebra L.

3 Solvable Leibniz superalgebras whose nilradical is a Lie superalgebra with maximal nilindex

In this section, we give the description of solvable Leibniz superalgebras whose nilradical is a Lie superalgebra with the maximal index of nilpotency. Note that $(n+m)$-dimensional

Lie superalgebra with nilindex $n+m$ exists only for $n=2$, m is odd and the multiplication table of a superalgebra is as follows:

$$N_{2,m}: \begin{cases} [y_i, e_1] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \le i \le \frac{m+1}{2}. \end{cases}$$

In the next theorems, we present the classification of solvable Leibniz (Lie) superalgebras with the nilradical $N_{2,m}$, which implies using the results of the works [10] and [26]. It should be noted that here we give list of solvable Lie superalgebras after some minor corrections and add the list of solvable non-Lie Leibniz superalgebras.

Theorem 3.1. *Let $L = L_0 \oplus L_1$ be a $(m+3)$-dimensional solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$. Then L is isomorphic to one of the following pairwise non-isomorphic superalgebras:*

$$M_1: \begin{cases} [e_1, x] = -[x, e_1] = e_1, \\ [x, x] = e_2, \\ [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = -[y_i, y_{m+1-i}] = (-1)^{i+1} e_2, & 1 \le i \le \frac{m+1}{2}, \\ [y_i, x] = -[x, y_i] = (i - \frac{m+1}{2}) y_i, & 1 \le i \le m. \end{cases}$$

$$M_2(\alpha): \begin{cases} [e_1, x] = -[x, e_1] = e_1, \\ [e_2, x] = -[x, e_2] = \alpha e_2, \\ [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = -[y_i, y_{m+1-i}] = (-1)^{i+1} e_2, & 1 \le i \le \frac{m+1}{2}, \\ [y_i, x] = -[x, y_i] = (i + \frac{\alpha-m-1}{2}) y_i, & 1 \le i \le m, \end{cases}$$

$$M_3: \begin{cases} [e_1, x] = -[x, e_1] = e_1 + e_2, \\ [e_2, x] = -[x, e_2] = e_2, \\ [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = -[y_i, y_{m+1-i}] = (-1)^{i+1} e_2, & 1 \le i \le \frac{m+1}{2}, \\ [y_i, x] = -[x, y_i] = (i - \frac{m}{2}) y_i, & 1 \le i \le m, \end{cases}$$

$$M_4(b_2, b_4, \dots, b_{m-1}): \begin{cases} [e_2, x] = -[x, e_2] = 2e_2, \\ [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = -[y_i, y_{m+1-i}] = (-1)^{i+1} e_2, & 1 \le i \le \frac{m+1}{2}, \\ [y_i, x] = -[x, y_i] = y_i + \sum\limits_{k=1}^{[\frac{m-i+1}{2}]} b_{2k} y_{i+2k-1}, & 1 \le i \le m, \end{cases}$$

Note that the first nonzero parameter of the algebra $M_4(b_2, b_4, \dots, b_{m-1})$ can be reduced to 1.

Theorem 3.2. *Let $L = L_0 \oplus L_1$ be a $(m+4)$-dimensional solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$. Then L is isomorphic to the following Lie superalgebra:*

$$M_5 : \begin{cases} [e_1, x] = -[x, e_1] = e_1, & \\ [e_2, x] = -[x, e_2] = (m-1)e_2, & \\ [e_2, z] = -[z, e_2] = 2e_2, & \\ [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \le i \le m-1, \\ [y_{m+1-i}, y_i] = -[y_i, y_{m+1-i}] = (-1)^{i+1}e_2, & 1 \le i \le \frac{m+1}{2}, \\ [y_i, x] = -[x, y_i] = (1-i)y_i, & 1 \le i \le m, \\ [y_i, z] = -[z, y_i] = y_i, & 1 \le i \le m. \end{cases}$$

4 Main result

In this section, we give the classification of solvable Leibniz superalgebras whose nilradical has nilindex $n+m$ and characteristic sequence $(n-1, 1|m)$. Note that, in the case of $n = m = 2$ there are two four-dimensional Leibniz superalgebras of nilindex four [5]. Solvable Leibniz superalgebras with these four-dimensional nilradicals are classified in [35]. Thus, we consider the case $n \ge 3$ and in the following theorem we give the list of nilpotent Leibniz superalgebras with nilindex $n+m$ and the characteristic sequence $(n-1, 1|m)$.

Theorem 4.1 ([5]). *Let L be a Leibniz superalgebra of nilindex $n+m$ with characteristic sequence $(n-1, 1|m)$, then $m = n-1$ or $m = n$ and there exists such a basis $\{e_1, e_2, \dots, e_n, y_1, y_2, \dots, y_m\}$ in superalgebra L whose multiplication in this basis is as follows:*

if $m = n-1,\ L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$:

$$\begin{cases} [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [y_j, e_1] = y_{j+1}, & 1 \le j \le n-2, \\ [e_1, y_1] = \frac{1}{2}y_2, \quad [e_i, y_1] = \frac{1}{2}y_i, & 2 \le i \le n-1, \\ [y_1, y_1] = e_1, \quad [y_j, y_1] = e_{j+1}, & 2 \le j \le n-1, \\ [e_1, e_2] = \alpha_4 e_4 + \alpha_5 e_5 + \ldots + \alpha_{n-1}e_{n-1} + \theta e_n, & \\ [e_j, e_2] = \alpha_4 e_{j+2} + \alpha_5 e_{j+3} + \ldots + \alpha_{n+2-j}e_n, & 2 \le j \le n-2, \\ [y_1, e_2] = \alpha_4 y_3 + \alpha_5 y_4 + \ldots + \alpha_{n-1}y_{n-2} + \theta y_{n-1}, & \\ [y_j, e_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \ldots + \alpha_{n+1-j}y_{n-1}, & 2 \le j \le n-3, \end{cases} \tag{4.1}$$

$G(\beta_4,\beta_5,\dots,\beta_n,\gamma)$:

$$\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,e_2]=\beta_4e_4+\beta_5e_5+\ldots+\beta_ne_n, & \\ [e_j,e_2]=\beta_4e_{j+2}+\beta_5e_{j+3}+\ldots+\beta_{n+2-j}e_n, & [e_2,e_2]=\gamma e_n, \quad 3\le j\le n-2,\\ [y_j,e_2]=\beta_4y_{j+2}+\beta_5y_{j+3}+\ldots+\beta_{n+1-j}y_{n-1}, & 1\le j\le n-3, \end{cases} \tag{4.2}$$

if $m=n$, *then* $M(\alpha_4,\alpha_5,\dots,\alpha_n,\theta,\tau)$:

$$\begin{cases} [e_1,e_1]=e_3, \quad [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, \quad [e_i,y_1]=\frac{1}{2}y_i, & 2\le i\le n,\\ [y_1,y_1]=e_1, \quad [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,e_2]=\alpha_4e_4+\alpha_5e_5+\ldots+\alpha_{n-1}e_{n-1}+\theta e_n, & \\ [e_j,e_2]=\alpha_4e_{j+2}+\alpha_5e_{j+3}+\ldots+\alpha_{n+2-j}e_n, & 2\le j\le n-2,\\ [y_1,e_2]=\alpha_4y_3+\ldots+\alpha_{n-1}y_{n-2}+\theta y_{n-1}+\tau y_n, & \\ [y_2,e_2]=\alpha_4y_4+\alpha_5y_4+\ldots+\alpha_{n-1}y_{n-1}+\theta y_n, & \\ [y_j,e_2]=\alpha_4y_{j+2}+\alpha_5y_{j+3}+\ldots+\alpha_{n+2-j}y_n, & 3\le j\le n-2, \end{cases} \tag{4.3}$$

$H(\beta_4,\beta_5,\dots,\beta_n,\delta,\gamma)$:

$$\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,e_2]=\beta_4e_4+\beta_5e_5+\ldots+\beta_ne_n, & \\ [e_j,e_2]=\beta_4e_{j+2}+\beta_5e_{j+3}+\ldots+\beta_{n+2-j}e_n, & [e_2,e_2]=\gamma e_n, \quad 3\le j\le n-2,\\ [y_1,e_2]=\beta_4y_3+\beta_5y_4+\ldots+\beta_ny_{n-1}+\delta y_n, & \\ [y_j,e_2]=\beta_4y_{j+2}+\beta_5y_{j+3}+\ldots+\beta_{n+2-j}y_n, & 2\le j\le n-2. \end{cases} \tag{4.4}$$

First, we describe the even derivations of these nilpotent Leibniz superalgebras.

Proposition 4.2. *An even derivation of* $L(\alpha_4,\alpha_5,\dots,\alpha_n,\theta)$ *has the following form:*

$$\begin{cases} d(e_1)=2a_1e_1+a_2e_3+a_3e_4+\cdots+a_{n-1}e_n,\\ d(e_2)=2a_1e_2+a_2e_3+a_3e_4+\cdots+a_{n-2}e_{n-1}+b_ne_n,\\ d(e_i)=2(i-1)a_1e_i+a_2e_{i+1}+a_3e_{i+2}+\cdots+a_{n-i+1}e_n, & 3\le i\le n,\\ d(y_i)=(2i-1)a_1y_i+a_2y_{i+1}+a_3y_{i+2}+\cdots+a_{n-i}y_{n-1}, & 1\le i\le n-1, \end{cases}$$

where $(n-3)\theta a_1=0,\quad \alpha_ia_1=0,\ 4\le i\le n.$

Proof. Let d be an even derivation of a Leibniz superalgebra which belongs to the class $L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$. Put

$$d(y_1) = a_1y_1 + a_2y_2 + \cdots + a_{n-1}y_{y-1}, \quad d(e_2) = b_1e_1 + b_2e_2 + \cdots + b_ne_n.$$

Using the multiplications of the superalgebra and Definition 2.3, we find the following:

$$d(e_1) = d([y_1, y_1]) = [d(y_1), y_1] + [y_1, d(y_1)] = 2a_1e_1 + a_2e_3 + a_3e_4 + \cdots + a_{n-1}e_n,$$
$$d(e_3) = d([e_2, e_1]) = [d(e_2), e_1] + [e_2, d(e_1)] = (b_1 + b_2 + 2a_1)e_3 + b_3e_4 + \cdots + b_{n-1}e_n,$$

On the other hand,

$$d(e_3) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = 4a_1e_3 + a_2e_4 + a_3e_4 + \cdots + a_{n-2}e_n.$$

Comparing the coefficients at the basic elements, we get that

$$b_1 + b_2 = 2a_1, \ b_i = a_{i-1}, \ 3 \leq i \leq n-1.$$

Since $[e_k, e_1] = e_{k+1}$ for $3 \leq k \leq n$, then from

$$d(e_{k+1}) = d([e_k, e_1]) = [d(e_k), e_1] + [e_k, d(e_1)],$$

we have

$$d(e_i) = 2(i-1)a_1e_i + a_2e_{i+1} + a_3e_{i+2} + \cdots + a_{n-i+1}e_n, \quad 3 \leq i \leq n.$$

Now we consider

$$d(y_2) = d([y_1, e_1]) = [d(y_1), e_1] + [y_1, d(e_1)] = 3a_1y_2 + a_2y_3 + \cdots + a_{n-2}y_{n-1}.$$

From $d(y_i) = d([y_{i-1}, e_1]) = [d(y_{i-1}), e_1] + [y_{i-1}, d(e_1)]$, inductively we get

$$d(y_i) = (2i-1)a_1y_i + a_2y_{i+1} + \cdots + a_{n-i}y_{n-1}, \ 1 \leq i \leq n-1.$$

Consider

$$\begin{aligned} d([y_1, e_2]) &= [d(y_1), e_2] + [y_1, d(e_2)] = \\ &= [a_1y_1 + a_2y_2 + \cdots + a_{n-1}y_{y-1}, e_2] + \\ &+ [y_1, b_1e_1 + b_2e_2 + a_2e_3 + \cdots + a_{n-2}e_{n-1} + b_ne_n] = \\ &= b_1y_2 + (a_1 + b_2)\alpha_4y_3 + (a_1\alpha_5 + a_2\alpha_4 + \alpha_5b_2)y_4 + \\ &+ (a_1\alpha_6 + a_2\alpha_5 + a_3\alpha_4 + \alpha_6b_2)y_5 + \cdots + \\ &+ (a_1\theta + a_2\alpha_{n-1} + a_3\alpha_{n-2} + a_4\alpha_{n-3} + \cdots + a_{n-3}\alpha_4 + \theta b_2)y_{n-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d([y_1, e_2]) &= \alpha_4d(y_3) + \alpha_5d(y_4) + \cdots + \alpha_{n-1}d(y_{n-2}) + \theta d(y_{n-1}) = \\ &= 5\alpha_4a_1y_3 + (\alpha_4a_2 + 7\alpha_5a_1)y_4 + (\alpha_4a_3 + \alpha_5a_2 + 9\alpha_6a_1)y_5 + \cdots + \\ &+ (\alpha_4a_{n-3} + \alpha_5a_{n-4} + \alpha_6a_{n-5} + \cdots + \alpha_{n-1}a_2 + (2n-3)\theta a_1)y_{n-1}. \end{aligned}$$

Comparing the coefficients at the basis elements, we obtain that

$$b_1 = 0, \quad (n-3)\theta a_1 = 0, \quad \alpha_i a_1 = 0, \ 4 \le i \le n-1.$$

From $d([e_2, e_2]) = [d(e_2), e_2] + [e_2, d(e_2)]$, we have $\alpha_n a_1 = 0$. Verification of the property of derivation for the other products give the identity or already obtained restrictions. □

Similarly, to Proposition 4.2, we have the description of the even derivations of the superalgebras for the other classes.

Proposition 4.3. *An even derivation of $M(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta, \tau)$ has the following form:*

$$\begin{cases} d(e_1) = 2a_1e_1 + a_2e_3 + a_3e_4 + \cdots + a_{n-1}e_n, \\ d(e_2) = 2a_1e_2 + a_2e_3 + a_3e_4 + \cdots + a_{n-2}e_{n-1} + b_ne_n, \\ d(e_i) = 2(i-1)a_1e_i + a_2e_{i+1} + a_3e_{i+2} + \cdots + a_{n-i+1}e_n, \quad 3 \le i \le n, \\ d(y_i) = (2i-1)a_1y_i + a_2y_{i+1} + a_3y_{i+2} + \cdots + a_{n-i+1}y_n, \quad 1 \le i \le n, \end{cases}$$

where $\theta a_1 = 0, \quad \tau a_1 = 0, \quad \alpha_i a_1 = 0, \ 4 \le i \le n.$

Proof. The proof is carried out using the property of derivation. □

Proposition 4.4. *An even derivation of $H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma)$ has the following form:*

$$\begin{cases} d(e_1) = 2a_1e_1 + a_2e_3 + a_3e_4 + \cdots + a_{n-1}e_n, \\ d(e_2) = b_2e_2, \\ d(e_i) = 2(i-1)a_1e_i + a_2e_{i+1} + a_3e_{i+2} + \cdots + a_{n-i+1}e_n, \quad 3 \le i \le n, \\ d(y_i) = (2i-1)a_1y_i + a_2y_{i+1} + a_3y_{i+2} + \cdots + a_{n-i+1}y_n, \quad 1 \le i \le n, \end{cases}$$

where

$$\begin{aligned} &\beta_i(2(i-2)a_1 - b_2) = 0, \quad 4 \le i \le n, \\ &\delta(2(n-1)a_1 - b_2) = 0, \quad \gamma((n-1)a_1 - b_2) = 0. \end{aligned} \tag{4.5}$$

Proof. The proof is carried out using the property of derivation. □

Proposition 4.5. *An even derivation of $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$ has the following form:*

$$\begin{cases} d(e_1) = 2a_1e_1 + a_2e_3 + a_3e_4 + \cdots + a_{n-1}e_n, \\ d(e_2) = b_2e_2 + b_ne_n, \\ d(e_i) = 2(i-1)a_1e_i + a_2e_{i+1} + a_3e_{i+2} + \cdots + a_{n-i+1}e_n, \quad 3 \le i \le n, \\ d(y_i) = (2i-1)a_1y_i + a_2y_{i+1} + a_3y_{i+2} + \cdots + a_{n-i}y_{n-1}, \quad 1 \le i \le n-1, \end{cases}$$

where $\gamma((n-1)a_1 - b_2) = 0, \quad \beta_i(2(i-2)a_1 - b_2) = 0, \ 4 \le i \le n.$

Proof. The proof is carried out using the property of derivation. □

From Proposition 4.2, we have the following corollary.

Corollary 4.6. *If R is a non-nilpotent solvable Leibniz superalgebra with the nilradical from the class $L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$, then $\alpha_4 = \alpha_5 = \dots = \alpha_n = \theta = 0$.*

Proof. Suppose $\alpha_i \neq 0$ for some $i (4 \leq i \leq n)$ or $\theta \neq 0$. Then from

$$(n-3)\theta a_1 = 0, \quad \alpha_i a_1 = 0, \ 4 \leq i \leq n,$$

we obtain that that $a_1 = 0$, which implies the nilpotency of any even derivation of $L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$. It is a contradiction to the non-nilpotency of R. Therefore, we have $\alpha_4 = \alpha_5 = \dots = \alpha_n = \theta = 0$. □

Thus, we conclude that solvable Leibniz superalgebra whose nilradical from the class $L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$ exists only under the condition $\alpha_4 = \alpha_5 = \dots = \alpha_n = \theta = 0$ and such solvable Leibniz superalgebras are classified in [34].

Theorem 4.7. *Let R be a solvable Leibniz superalgebra with nilradical $L(0, 0, \dots, 0, 0)$. Then it is isomorphic to the superalgebra*

$$SL: \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-2, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 2 \leq i \leq n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, x] = 2e_1, & [e_i, x] = 2(i-1)e_i, & 2 \leq i \leq n, \\ [y_i, x] = (2i-1)y_i, & & 1 \leq i \leq n-1, \\ [x, e_1] = -2e_1, & [x, y_1] = -y_1. & \end{cases}$$

Analogously to the Corollary 4.6, from Proposition 4.3 for the solvable Leibniz superalgebras with the nilradical from the class $M(\alpha_4, \alpha_5, \dots, \alpha_n, \theta, \tau)$ we get

Corollary 4.8. *If L is a non-nilpotent solvable Leibniz superalgebra with the nilradical from the class $M(\alpha_4, \alpha_5, \dots, \alpha_n, \theta, \tau)$, then $\alpha_4 = \alpha_5 = \dots = \alpha_n = \theta = \tau = 0$.*

The following theorem describes a solvable Leibniz superalgebra whose nilradical is $M(0, 0, \dots, 0, 0, 0)$. It is proved by a similar reason as in Theorem 4.7.

Theorem 4.9. *Let L be a solvable Leibniz superalgebra with nilradical $M(0, 0, \dots, 0, 0, 0)$.*

Then L isomorphic to the superalgebra

$$SM : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \le j \le n-1, \\ [e_1, y_1] = \frac{1}{2}y_2, & [e_i, y_1] = \frac{1}{2}y_i, & 2 \le i \le n, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \le j \le n-1, \\ [e_1, x] = 2e_1, & [e_i, x] = 2(i-1)e_i, & 2 \le i \le n, \\ [y_i, x] = (2i-1)y_i, & & 1 \le i \le n, \\ [x, e_1] = -2e_1, & [x, y_1] = -y_1. & \end{cases}$$

Now consider solvable Leibniz superalgebras whose nilradicals belong to the class $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$. Then from Proposition 4.4, we have the following result.

Corollary 4.10. *If L is a non-nilpotent solvable Leibniz superalgebra with nilradical from the class $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$, then:*

$$(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma) = \begin{cases} (0, 0, \dots, 0, 0, 0), & & \\ (0, 0, \dots, 0, \beta_t, 0, \dots, 0, 0), & 4 \le t \le n, & \beta_t \ne 0, \\ (0, 0, \dots, 0, \delta, 0), & & \delta \ne 0, \\ (0, 0, \dots, 0, \beta_{\frac{n+3}{2}}, 0, \dots, 0, \gamma), & n \text{ is odd}, & \gamma \ne 0. \end{cases}$$

Proof. By the conditions on the parameters of the $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$ from Proposition 4.4, we have the following cases:

- If all parameters are equal to zero, we obtain a split superalgebra $H(0, 0, \dots, 0, 0, 0)$, which has non-nilpotent even derivation.
- If $\beta_i \ne 0, \beta_j \ne 0$ for some $i, j (4 \le i \ne j \le n)$, then from (4.5), we have $(a_1, b_2) = (0, 0)$, which implies that all even derivations of the superalgebra are nilpotent. Therefore, in this case, there is no solvable Leibniz superalgebra with nilradical $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$.
- If $\beta_t \ne 0$ for some t and $\beta_i = 0$ for $i \ne t$, then $b_2 = 2(t-2)a_1$ and

$$\delta(2(n-1)a_1 - b_2) = 0, \quad \gamma((n-1)a_1 - b_2) = 0.$$

  From these equalities we have $\delta a_1(n+1-t) = 0$, $\gamma a_1(n-2t+3) = 0$. If $\delta \ne 0$, then $a_1 = 0$ and the Leibniz superalgebra has only nilpotent even derivations. Thus $\delta = 0$ and

  - if $\gamma = 0$, then we have the superalgebras $H(0, 0, \dots, 0, \beta_t, 0, \dots, 0, 0)$, $4 \le t \le n$, $\beta_t \ne 0$;

- if $\gamma \neq 0$, then in case of $t \neq \frac{n+3}{2}$, we have that $a_1 = 0$ and the Leibniz superalgebra has only nilpotent even derivations which is contradiction with non-nilpotency of the Leibniz superalgebra L. In case of $t = \frac{n+3}{2}$ we have the superalgebras $H(0, 0, \ldots, 0, \beta_{\frac{n+3}{2}}, 0, \ldots, 0, \gamma)$. Note that the case $t = \frac{n+3}{2}$ appears only for n is odd.

- If $\beta_i = 0$ for all $i (4 \leq i \leq n)$ and $\delta \neq 0$, then $\gamma = 0$ and we have the superalgebra $H(0, 0, 0, \ldots, 0, \delta, 0)$.
- If $\beta_i = 0$ for all $i (4 \leq i \leq n)$, $\delta = 0$ and $\gamma \neq 0$, then we have the superalgebra $H(0, 0, 0, \ldots, 0, 0, \gamma)$.

□

Similarly, for the class of superalgebras $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$, we have

Corollary 4.11. *If L is a non-nilpotent solvable Leibniz superalgebra with nilradical from the class $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$, then:*

$$(\beta_4, \beta_5, \ldots, \beta_n, \gamma) = \begin{cases} (0, 0, \ldots, 0, 0), & \\ (0, 0, \ldots, 0, \beta_t, 0, \ldots, 0), & 4 \leq t \leq n, \\ (0, 0, \ldots, 0, \beta_{\frac{n+3}{2}}, 0, \ldots, 0, \gamma), & n \text{ is odd}, \quad \gamma \neq 0. \end{cases}$$

Now using Corollary 4.10, we classify solvable Leibniz superalgebras with nilradical $H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma)$.

First we consider the case when the nilradical of solvable Leibniz superalgebra is the superalgebra $H(0, 0, \ldots, 0, 0)$. From Proposition 4.4, it is easy to conclude that there are two nil-independent even derivations of the superalgebra $H(0, 0, \ldots, 0, 0)$ and other algebras have only one nil-independent even derivations. Moreover, a superalgebra from the class $H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma)$ is split if and only if all parameters are equal to zero, i.e., superalgebra isomorphic to $H(0, 0, \ldots, 0, 0)$.

Theorem 4.12. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to the superalgebra $H(0, 0, \ldots, 0, 0)$. Then L is isomorphic to the following pairwise non-isomorphic superalgebras:*

$$MH_1 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n, \\ [y_i, x_1] = (2i-1) y_i, & & 1 \leq i \leq n, \\ [e_1, x_1] = 2e_1, & [e_i, x_1] = 2(i-1)e_i, & 3 \leq i \leq n, \\ [x_1, e_1] = -2e_1, & [x_1, y_1] = -y_1, & [e_2, x_2] = e_2, \end{cases}$$

$$MH_2:\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n,\\ [y_i,x_1]=(2i-1)y_i, & & 1\le i\le n,\\ [e_1,x_1]=2e_1, & [e_i,x_1]=2(i-1)e_i, & 3\le i\le n,\\ [x_1,e_1]=-2e_1, & [x_1,y_1]=-y_1, & [e_2,x_2]=e_2, & [x_2,e_2]=-e_2,\end{cases}$$

$$H_1(b):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n,\\ [y_i,x]=(2i-1)y_i, & & 1\le i\le n,\\ [e_1,x]=2e_1, & [e_2,x]=be_2, & [e_i,x]=2(i-1)e_i, & 3\le i\le n,\\ [x,e_1]=-2e_1, & [x,e_2]=-be_2, & [x,y_1]=-y_1, & b\neq 0,\end{cases}$$

$$H_2(b):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n,\\ [y_i,x]=(2i-1)y_i, & & 1\le i\le n,\\ [e_1,x]=2e_1, & [e_2,x]=be_2, & [e_i,x]=2(i-1)e_i, & 3\le i\le n,\\ [x,e_1]=-2e_1, & [x,y_1]=-y_1,\end{cases}$$

$$H_3:\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n,\\ [e_1,x]=2e_1, & [e_i,x]=2(i-1)e_i, & 3\le i\le n,\\ [x,e_1]=-2e_1, & [x,y_1]=-y_1, & [x,x]=e_2,\end{cases}$$

$$H_4:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [y_j,e_1]=y_{j+1}, & \qquad\qquad\qquad\quad 1\leq j\leq n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\leq j\leq n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\leq i\leq n,\\ [e_1,x]=\sum\limits_{k=3}^{n}a_{k-1}e_k, & [e_2,x]=e_2, \quad [e_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}e_k, \quad 3\leq i\leq n,\\ [y_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}y_k, & 1\leq i\leq n-1,\end{cases}$$

$$H_5(\gamma):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [y_j,e_1]=y_{j+1}, & \qquad\qquad\qquad\quad 1\leq j\leq n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\leq j\leq n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\leq i\leq n,\\ [e_1,x]=\sum\limits_{k=3}^{n}a_{k-1}e_k, & [e_2,x]=e_2, \quad [e_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}e_k, \quad 3\leq i\leq n,\\ [y_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}y_k, & 1\leq i\leq n,\\ [x,e_2]=-e_2, & [x,x]=\gamma e_2, \quad \gamma\in\{0;1\}.\end{cases}$$

Note that, the first non-vanishing parameter $\{a_2,a_3,\dots,a_n\}$ in the algebras H_4 and $H_5(\gamma)$ can be scaled to 1*.*

Proof. From Proposition 4.4, it is not difficult to see that the maximal number of nil-independent even derivations of the superalgebra $N=H(0,0,\dots,0,0)$ is equal to 2. Thus, for the dimension of the solvable Leibniz superalgebras with nilradical N, we have

$$\dim L-\dim N\leq 2.$$

Case $\dim L-\dim N=2$. Since the codimension of the nilradical N is equal to 2, we can choose a basis $\{e_1,e_2,\dots,e_n,x_1,x_2,y_1,y_2,\dots,y_m\}$ of L such that R_{x_1} and R_{x_2} are nil-independent even derivations of N. Then using Proposition 4.4, we have that

$$\begin{cases}[e_1,x_1]=2e_1+a_2e_3+a_3e_4+\dots+a_{n-1}e_n,\\ [e_i,x_1]=2(i-1)e_i+a_2e_{i+1}+a_3e_{i+2}+\dots+a_{n-i+1}e_n, & 3\leq i\leq n,\\ [y_i,x_1]=(2i-1)y_i+a_2y_{i+1}+a_3y_{i+2}+\dots+a_{n-i+1}y_n, & 1\leq i\leq n,\\ [e_1,x_2]=\alpha_2e_3+\alpha_3e_4+\dots+\alpha_{n-1}e_n,\\ [e_2,x_2]=e_2,\\ [e_i,x_2]=\alpha_2e_{i+1}+\alpha_3e_{i+2}+\dots+\alpha_{n-i+1}e_n, & 3\leq i\leq n-1,\\ [y_i,x_2]=\alpha_2y_{i+1}+\alpha_3y_{i+2}+\dots+\alpha_{n-i+1}y_n, & 1\leq i\leq n-1.\end{cases}$$

Taking the following change of basis

$$\begin{cases} y'_i = y_i + A_2 y_{i+1} + A_3 y_{i+2} + \cdots + A_{n-i+1} y_n, & 1 \le i \le n, \\ e'_1 = e_1 + A_2 e_2 + A_3 e_3 + \cdots + A_{n-1} e_n, \\ e'_2 = e_2, \\ e'_i = e_i + A_2 e_{i+1} + A_3 e_{i+2} + \cdots + A_{n-i+1} e_n, & 3 \le i \le n, \end{cases}$$

where

$$A_k = -\frac{a_k + A_2 a_{k-1} + A_3 a_{k-2} + \cdots + A_{k-1} a_2}{2(k-1)}, \ 2 \le k \le n,$$

we can assume $a_i = 0, \ 2 \le i \le n$.

From the multiplication of the nilradical and the properties of the right annihilator, we easily get $e_i, y_j \in \mathrm{Ann}_r(L)$ for $3 \le i \le n$ and $2 \le j \le n$, i.e.,

$$[x_1, e_i] = [x_2, e_i] = 0, \ 3 \le i \le n, \qquad [x_1, y_j] = [x_2, y_j] = 0, \ 2 \le j \le n.$$

Put

$$\begin{cases} [x_1, y_1] = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n, \\ [x_1, e_2] = p_1 e_1 + p_2 e_2 + \cdots + p_n e_n, \\ [x_2, y_1] = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n, \\ [x_2, e_2] = \delta_1 e_1 + \delta_2 e_2 + \cdots + \delta_n e_n, \\ [x_i, x_j] = c_{ij}^1 e_1 + c_{ij}^2 e_2 + \cdots + c_{ij}^n e_n, & 1 \le i, j \le 2. \end{cases}$$

Making the change

$$x'_1 = x_1 - 2m_2 e_1 - 2\sum_{k=3}^{n} m_k e_k, \quad x'_2 = x_2 - 2\gamma_2 e_1 - 2\sum_{k=3}^{n} \gamma_k e_k,$$

we can assume $m_i = \gamma_i = 0$ for $2 \le i \le n$.

Considering the Leibniz superidentity for $\{x_1, y_1, y_1\}$ and $\{x_2, y_1, y_1\}$, we derive

$$[x_1, e_1] = 2m_1 e_1, \quad [x_2, e_1] = 2\gamma_1 e_1.$$

Similarly, if we apply the Leibniz superidentity on the triples $\{x_1, e_2, e_1\}$, $\{x_2, e_2, e_1\}$, $\{e_1, x_1, e_1\}$, $\{e_1, x_2, e_1\}$, $\{x_1, x_1, e_1\}$, $\{x_2, x_2, e_1\}$, $\{x_1, y_1, x_1\}$, $\{x_2, y_1, x_2\}$, $\{e_1, x_1, x_2\}$, $\{e_1, x_2, x_1\}$, $\{x_1, x_2, e_1\}$, $\{x_2, x_1, e_1\}$, $\{x_1, x_2, y_1\}$ and $\{x_2, x_1, y_1\}$, we obtain

$$\begin{array}{llll} p_1 = 0, & p_i = 0, & 3 \le i \le n-1, & \\ \delta_1 = 0, & \delta_i = 0, & 3 \le i \le n-1, & \\ m_1 = -1, & \gamma_1 = 0, & \alpha_i = 0, & 2 \le i \le n, \\ c_{ii}^1 = 0, & c_{ii}^k = 0, & 1 \le i \le 2, & 3 \le k \le n, \\ c_{12}^1 = 0, & c_{12}^i = 0, & 3 \le i \le n, & \\ c_{21}^1 = 0, & c_{21}^i = 0, & 3 \le i \le n. & \end{array}$$

Moreover, the Leibniz superidentity for the triples $\{x_2, e_2, x_1\}$, $\{x_1, e_2, x_2\}$, $\{x_2, x_1, e_2\}$, $\{x_1, x_2, e_2\}$ and $\{x_2, x_2, e_2\}$ gives us

$$\delta_n = 0, \quad p_n = 0, \quad p_2 = 0, \quad \delta_2(\delta_2 + 1) = 0.$$

Changing the basis $x_1' = x_1 - c_{12}^2 e_2$ allows us to assume that $c_{12}^2 = 0$.

Now using the Leibniz superidentities for $\{x_1, x_1, x_2\}$, $\{x_2, x_1, x_2\}$, $\{x_2, x_2, x_2\}$, we have $c_{11}^2 = 0$, $c_{21}^2 = 0$, $\delta_2 c_{22}^2 = 0$.

- If $\delta_2 = 0$, then changing $x_2' = x_2 - c_{22}^2 e_2$, we can assume $c_{22}^2 = 0$ and obtain the superalgebra MH_1.
- If $\delta_2 = -1$, then $c_{22}^2 = 0$ and obtain the superalgebra MH_2.

Case $\dim L - \dim N = 1$. Since, the operator of right multiplication R_x is a derivation of $H(0, 0, \dots, 0)$, then using Proposition 4.4, we can assume that

$$\begin{cases} [e_1, x] = 2a_1 e_1 + a_2 e_3 + a_3 e_4 + \dots + a_{n-1} e_n, \\ [e_2, x] = b_2 e_2, \\ [e_i, x] = 2(i-1)a_1 e_i + a_2 e_{i+1} + a_3 e_{i+2} + \dots + a_{n-i+1} e_n, \quad 3 \le i \le n, \\ [y_i, x] = (2i-1)a_1 y_i + a_2 y_{i+1} + a_3 y_{i+2} + \dots + a_{n-i+1} y_n, \quad 1 \le i \le n. \end{cases}$$

Since $(a_1, b_2) \neq (0, 0)$, we divide this case into two subcases:

Subcase 1. Let $a_1 \neq 0$, then we may suppose $a_1 = 1$. Taking the change of basis

$$\begin{cases} y_i' = y_i + A_2 y_{i+1} + A_3 y_{i+2} + \dots + A_{n-i+1} y_n, \quad 1 \le i \le n, \\ e_1' = e_1 + A_2 e_2 + A_3 e_3 + \dots + A_{n-1} e_n, \\ e_2' = e_2, \\ e_i' = e_i + A_2 e_{i+1} + A_3 e_{i+2} + \dots + A_{n-i+1} e_n, \quad 3 \le i \le n, \end{cases}$$

where

$$A_k = -\frac{a_k + A_2 a_{k-1} + A_3 a_{k-2} + \dots + A_{k-1} a_2}{2(k-1)}, \ 2 \le k \le n,$$

we can assume $a_i = 0, \ 2 \le i \le n$.

Since $e_i, y_j \in \mathrm{Ann}_r(L)$ for $3 \le i \le n$ and $2 \le j \le n$, we conclude

$$[x, e_i] = 0, \ 3 \le i \le n, \quad [x, y_j] = 0, \ 2 \le j \le n.$$

Put

$$\begin{cases} [x, y_1] = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n, \\ [x, e_2] = \mu_1 e_1 + \mu_2 e_2 + \dots + \mu_n e_n, \\ [x, x] = \gamma_1 e_1 + \gamma_2 e_2 + \dots + \gamma_n e_n. \end{cases}$$

Using Leibniz superidentity we get:

$$[x, e_1] = [x, [y_1, y_1]] = 2[[x, y_1], y_1] = 2\alpha_1 e_1 + 2\alpha_2 e_3 + \cdots + 2\alpha_{n-1} e_n.$$

Taking the change $x' = x - 2(\alpha_3 e_3 + \cdots + \alpha_n e_n)$, one can assume $\alpha_i = 0$ for $3 \le i \le n$. From $0 = [y_1, [x, x]]$ and $[x, [y_1, x]] = [[x, y_1], x] - [[x, x], y_1]$, we obtain

$$\gamma_1 = 0, \quad \alpha_2 = 0, \quad \gamma_i = 0, \quad 3 \le i \le n,$$

Since $[e_1, x] + [x, e_1] = 2(\alpha_1 + 1)e_1 \in \mathrm{Ann}_r(L)$, we get $\alpha_1 = -1$.

Considering the Leibniz superidentity for $\{x, e_2, e_1\}$, $\{x, e_2, y_1\}$, $\{x, x, e_2\}$, $\{x, x, x\}$, we derive the following restrictions:

$$\begin{aligned} &\mu_1 = 0, && \mu_i = 0, \quad 3 \le i \le n, \\ &\mu_2(\mu_2 + b_2) = 0, && \gamma_2\mu_2 = 0. \end{aligned}$$

- Let $\mu_2 \neq 0$, then $\gamma_2 = 0$, $b_2 = -\mu_2$, and obtain the superalgebra $H_1(b)$.
- Let $\mu_2 = 0$,
  - If $b_2 \neq 0$, then taking $x' = x - \frac{\gamma_2}{b_2} e_2$, we can suppose $\gamma_2 = 0$, and obtain the superalgebra $H_2(b)$ for $b \neq 0$.
  - If $b_2 = 0$, then in case of $\gamma_2 = 0$, we have the superalgebra $H_2(b)$ for $b = 0$, in case of $\gamma_2 \neq 0$ making the change $e_2' = \gamma_2 e_2$, we obtain the superalgebra H_3.

Subcase 2. $a_1 = 0$, then $b_2 \neq 0$ and we may suppose $b_2 = 1$. Then

$$\begin{cases} [e_1, x] = a_2 e_3 + a_3 e_4 + \cdots + a_{n-1} e_n, \\ [e_2, x] = e_2, \\ [e_i, x] = a_2 e_{i+1} + a_3 e_{i+2} + \cdots + a_{n-i+1} e_n, & 3 \le i \le n, \\ [y_i, x] = a_2 y_{i+1} + a_3 y_{i+2} + \cdots + a_{n-i+1} y_n, & 1 \le i \le n. \end{cases}$$

Put

$$\begin{cases} [x, y_1] = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n, \\ [x, e_2] = \mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_n e_n, \\ [x, x] = \gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_n e_n. \end{cases}$$

Using Leibniz superidentity for triples $\{x, y_1, y_1\}$, we get

$$[x, e_1] = 2\alpha_1 e_1 + 2\alpha_2 e_3 + \cdots + 2\alpha_{n-1} e_n.$$

By changing the basis $x' = x - 2(\alpha_2 e_1 + \alpha_3 e_3 + \cdots + \alpha_n e_n)$, we can assume $\alpha_i = 0$ for $2 \le i \le n$.

Considering the Leibniz superidentity for elements $\{y_1, x, x\}$, $\{x, e_2, e_1\}$, $\{x, e_2, y_1\}$, $\{e_1, x, y_1\}$, $\{x, x, e_2\}$, $\{x, y_1, x\}$, we obtain the following restrictions:

$$\begin{cases} \gamma_1 = 0, \quad \mu_1 = 0, \quad \mu_i = 0, \quad 3 \leq i \leq n, \quad \alpha_1 = 0, \\ \mu_2(\mu_2 + 1) = 0, \quad \gamma_i = 0, \quad 3 \leq i \leq n. \end{cases}$$

Subcase 2.1. If $\mu_2 = 0$, then by $x' = x - \gamma_2 e_2$, we may suppose that $\gamma_2 = 0$ and obtain the superalgebra H_4.

Subcase 2.2. If $\mu_2 = -1$, then we have the superalgebra $H_5(\gamma)$. □

Now we consider the case when the nilradical is a non-split superalgebra from the class $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$, i.e., at least one of the parameters is not equal to zero.

Theorem 4.13. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to a non-split superalgebra N from the class $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$. Then L is isomorphic to one of the following pairwise non-isomorphic superalgebras:*

$$SH_1(t)(4 \leq t \leq n): \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n, \\ [e_1, e_2] = e_t, & [e_j, e_2] = e_{j+t-2}, & 3 \leq j \leq n-2, \\ [y_1, e_2] = y_{t-1}, & [y_j, e_2] = y_{j+t-2}, & 2 \leq j \leq n-2, \\ [y_i, x] = (2i-1) y_i, & & 1 \leq i \leq n, \\ [x, y_1] = -y_1, & & \\ [e_1, x] = 2e_1, & [x, e_1] = -2e_1, & \\ [e_2, x] = 2(t-2)e_2, & [x, e_2] = -2(t-2)e_2 - 2e_{t-1}, & \\ [e_i, x] = 2(i-1)e_i, & & 3 \leq i \leq n, \end{cases}$$

$$SH_2: \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n, \\ [y_1, e_2] = y_n, & & \\ [y_i, x] = (2i-1) y_i, & & 1 \leq i \leq n, \\ [x, y_1] = -y_1, & & \\ [e_1, x] = 2e_1, & [x, e_1] = -2e_1, & \\ [e_2, x] = 2(n-1)e_2, & [x, e_2] = -2(n-1)e_2 - 2e_n, & \\ [e_i, x] = 2(i-1)e_i, & & 3 \leq i \leq n. \end{cases}$$

$$SH_3(\gamma)(n \text{ is odd}) : \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \qquad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \qquad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \qquad 3\le i\le n,\\ [e_1,e_2]=e_{\frac{n+3}{2}}, & [e_j,e_2]=e_{j+\frac{n-1}{2}}, \qquad 3\le j\le n-2,\\ [y_1,e_2]=y_{\frac{n+1}{2}}, & [y_j,e_2]=y_{j+\frac{n-1}{2}}, \qquad 2\le j\le n-2,\\ [e_2,e_2]=\gamma e_n, & \gamma\neq 0,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n,\\ [x,y_1]=-y_1, & \\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=(n-1)e_2, & [x,e_2]=-(n-1)e_2-2e_{\frac{n+1}{2}},\\ [e_i,x]=2(i-1)e_i, & 3\le i\le n, \end{cases}$$

$$SH_4 : \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \qquad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-1,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \qquad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \qquad 3\le i\le n,\\ [e_2,e_2]=e_n, & \\ [y_i,x]=(2i-1)y_i, & 1\le i\le n,\\ [x,y_1]=-y_1, & \\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=(n-1)e_2, & [x,e_2]=-(n-1)e_2,\\ [e_i,x]=2(i-1)e_i, & 3\le i\le n, \end{cases}$$

Proof. From Proposition 4.4, we have that any non-split superalgebra from the class $H(\beta_4,\beta_5,\ldots,\beta_n,\delta,\gamma)$ has a maximum one nil-independent even derivation. Then we have that, the codimension of the solvable superalgebra L with such nilradicals may be equal to one. Let $\{e_1,e_2,\ldots,e_n,,x,y_1,y_2\ \ldots,y_n\}$ be a basis of the superalgebra $L=L_0\oplus L_1$, such that $L_0=\{e_1,e_2,\ldots,e_n,x\}$ and $L_1=\{y_1,y_2\ \ldots,y_n\}$. Since, the operator of right multiplication R_x is a derivation of $H(\beta_4,\beta_5,\ldots,\beta_n,\delta,\gamma)$, then using Proposition 4.4, we can assume that

$$\begin{aligned} &[y_i,x]=(2i-1)a_1y_i+a_2y_{i+1}+\cdots+a_{n-i+1}y_n, \quad 1\le i\le n,\\ &[e_1,x]=2a_1e_1+a_2e_3+\cdots+a_{n-1}e_n,\\ &[e_2,x]=b_2e_2,\\ &[e_i,x]=2(i-1)a_1e_i+a_2e_{i+1}+\cdots+a_{n-i+1}e_n, \quad 3\le i\le n. \end{aligned}$$

Moreover, if $\beta_i\neq 0$, for some $i(4\le i\le n)$, then $b_2=2(i-2)a_1$, if $\beta_i=0$, for any $i(4\le i\le n)$ and $\delta\neq 0$, then $b_2=2(n-1)a_1$, if $\beta_i=0$, for any $i(4\le i\le n)$, $\delta=0$ and $\gamma\neq 0$, then $b_2=(n-1)a_1$.

Thus, from non-nilpotency of even derivation of the non-split superalgebra from the class $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$, we conclude that $a_1 \neq 0$. Therefore, we can suppose $a_1 = 1$ and considering the following change of basis

$$\begin{cases} y_i' = y_i + A_2 y_{i+1} + A_3 y_{i+2} + \dots + A_{n-i+1} y_n, & 1 \leq i \leq n, \\ e_1' = e_1 + A_2 e_3 + A_3 e_4 + \dots + A_{n-1} e_n, \\ e_2 = e_2, \\ e_i' = e_i + A_2 e_{i+1} + A_3 e_{i+2} + \dots + A_{n-i+1} e_n, & 3 \leq i \leq n, \end{cases}$$

where

$$A_k = -\frac{a_k + A_2 a_{k-1} + A_3 a_{k-2} + \dots + A_{k-1} a_2}{2(k-1)}, \ 2 \leq k \leq n,$$

one can assume $a_i = 0, \ 2 \leq i \leq n$.

From (4.4), we can easily get that the basis elements $e_3, e_4, \dots, e_n, y_2, y_3, \dots, y_n$ belongs to the right annihilator of the superalgebra L. Thus, we have

$$[x, e_i] = 0, \ 3 \leq i \leq n, \qquad [x, y_j] = 0, \ 2 \leq j \leq n.$$

Put

$$\begin{cases} [x, y_1] = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n, \\ [x, e_2] = \mu_1 e_1 + \mu_2 e_2 + \dots + \mu_n e_n, \\ [x, x] = \gamma_1 e_1 + \gamma_2 e_2 + \dots + \gamma_n e_n. \end{cases}$$

Using Leibniz superidentity we get:

$$[x, e_1] = [x, [y_1, y_1]] = 2[[x, y_1], y_1] = 2\alpha_1 e_1 + 2\alpha_2 e_3 + \dots + 2\alpha_{n-1} e_n.$$

Making the change $x' = x - 2(\alpha_2 e_1 + \alpha_3 e_3 + \dots + \alpha_n e_n)$, we can assume that $\alpha_i = 0, \ 2 \leq i \leq n$.

Moreover, from $0 = [y_1, [x, x]]$, we obtain

$$\gamma_1 = 0, \quad \gamma_2 \beta_i = 0, \quad 4 \leq i \leq n, \quad \gamma_2 \delta = 0. \tag{4.6}$$

The Leibniz superidentity $[x, [y_1, x]] = [[x, y_1], x] - [[x, x], y_1]$ gives us the following

$$\gamma_i = 0, \quad 3 \leq i \leq n.$$

Since $[e_1, x] + [x, e_1] \in \mathrm{Ann}_r(L)$, we get that $\alpha_1 = -1$. Thus, we have the following multiplications

$$\begin{cases} [y_i, x] = (2i-1) y_i, & 1 \leq i \leq n, \\ [e_1, x] = 2e_1, & [e_2, x] = b_2 e_2, \qquad [e_i, x] = 2(i-1) e_i, \quad 3 \leq i \leq n, \\ [x, e_1] = -2e_1, & [x, e_2] = \sum\limits_{k=1}^{n} \mu_k e_k, \\ [x, y_1] = -y_1, & [x, x] = \gamma_2 e_2. \end{cases}$$

Now, using Corollary 4.10, consider the following cases:

Case 1. Let $N = H(0, 0, \ldots, \beta_t, 0, \ldots, 0, 0)$, where $4 \le t \le n$, i.e., nilradical N has the multiplication

$$\begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \le j \le n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \le j \le n, \\ [e_1, y_1] = \frac{1}{2}y_2, & [e_i, y_1] = \frac{1}{2}y_i, & 3 \le i \le n-1, \\ [e_1, e_2] = \beta_t e_t, & [e_j, e_2] = \beta_t e_{j+t-2}, & 3 \le j \le n-2, \\ [y_1, e_2] = \beta_t y_{t-1}, & [y_j, e_2] = \beta_t y_{j+t-2}, & 2 \le j \le n-2. \end{cases} \tag{4.7}$$

Since $\beta_t \neq 0$, then from (4.6), we get $\gamma_2 = 0$ and changing $e_2' = \frac{1}{\beta_t}e_2$, we have $\beta_t = 1$ and $b_2 = 2(t-2)$. Applying the Leibniz superidentity for the triples $\{e_1, x, e_2\}, \{x, e_2, y_1\}$, we have

$$\mu_1 = 0,\ \mu_2 = 2(2-t), \quad \mu_i = 0,\ 3 \le i \neq t-1 \le n, \qquad \mu_{t-1} = -2.$$

Therefore, we obtain the superalgebra $SH_1(t)$.

Case 2. Consider the case when nilradical is

$$H(0, 0, \ldots, 0, \delta, 0) : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \le j \le n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \le j \le n-1, \\ [e_1, y_1] = \frac{1}{2}y_2, & [e_i, y_1] = \frac{1}{2}y_i, & 3 \le i \le n, \\ [y_1, e_2] = \delta y_n. \end{cases}$$

Since $\delta \neq 0$, then from (4.6) we have $\gamma_2 = 0$ and by changing $e_2' = \frac{1}{\delta}e_2$, we can assume $\delta = 1$.

From the Leibniz superidentity for $\{x, e_2, y_1\}$, $\{y_1, x, e_2\}$, we obtain

$$\mu_1 = 0, \quad \mu_2 = 2(1-n), \quad \mu_i = 0, \quad 3 \le i \le n-1, \quad \mu_n = -2.$$

Therefore, we have the superalgebra SH_2.

Case 3. Let the nilradical of the superalgebra is

$$H(0, 0, \ldots, \beta_{\frac{n+3}{2}}, 0, \ldots, 0, \gamma) : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [y_j, e_1] = y_{j+1}, & 1 \le j \le n-1, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \le j \le n-1, \\ [e_1, y_1] = \frac{1}{2}y_2, & [e_i, y_1] = \frac{1}{2}y_i, & 3 \le i \le n, \\ [e_1, e_2] = \beta_{\frac{n+3}{2}} e_{\frac{n+3}{2}}, & [e_j, e_2] = \beta_{\frac{n+3}{2}} e_{j+\frac{n-1}{2}}, & 3 \le j \le n-2, \\ [y_1, e_2] = \beta_{\frac{n+3}{2}} y_{\frac{n+1}{2}}, & [y_j, e_2] = \beta_{\frac{n+3}{2}} y_{j+\frac{n-1}{2}}, & 2 \le j \le n-2, \\ [e_2, e_2] = \gamma e_n. \end{cases}$$

In this case, we have $b_2 = n-1$.

If $\beta_{\frac{n+3}{2}} \neq 0$, then from (4.6), we get $\gamma_2 = 0$. Taking the change $e_2' = \frac{1}{\beta_{\frac{n+3}{2}}} e_2$, we can suppose $\beta_{\frac{n+3}{2}} = 1$. Considering the Leibniz superidentity for $\{x, e_2, y_1\}$, $\{e_1, x, e_2\}$, we get

$$\mu_1 = 0, \quad \mu_2 = 1 - n, \quad \mu_{\frac{n+1}{2}} = -2, \quad \mu_i = 0, \quad 3 \leq i \leq n, (i \neq \frac{n+1}{2}).$$

Thus, we have the superalgebra $SH_3(\gamma)$.

If $\beta_{\frac{n+3}{2}} = 0$, then $\gamma \neq 0$ and by changing $e_2' = \frac{1}{\sqrt{\gamma}} e_2$, we can suppose $\gamma = 1$. Considering Leibniz superidentity for $\{x, e_2, y_1\}$, $\{e_2, x, x\}$, $\{e_2, x, e_2\}$, we get

$$\gamma_2 = 0, \quad \mu_1 = 0, \quad \mu_2 = 1 - n, \quad \mu_i = 0, \quad 3 \leq i \leq n.$$

Thus, we obtain the superalgebra SH_4. □

Now we give the description of solvable Leibniz superalgebras whose nilradical is isomorphic to the superalgebra from the class $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$. In the following theorem, we consider the case when nilradical is $G(0, 0, \ldots, 0)$.

Theorem 4.14. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to the superalgebra $G(0, 0, \ldots, 0)$. Then L is isomorphic to one of the following pairwise non-isomorphic superalgebras:*

$$MG_1 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-2, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n-1, \\ [e_1, x_1] = 2e_1, & [e_i, x_1] = 2(i-1)e_i, & 3 \leq i \leq n, \\ [y_i, x_1] = (2i-1)y_i, & & 1 \leq i \leq n-1, \\ [x_1, e_1] = -2e_1, & [x_1, y_1] = -y_1, & [e_2, x_2] = e_2, \end{cases}$$

$$MG_2 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, e_1] = y_{j+1}, & & 1 \leq j \leq n-2, \\ [y_1, y_1] = e_1, & [y_j, y_1] = e_{j+1}, & 2 \leq j \leq n-1, \\ [e_1, y_1] = \frac{1}{2} y_2, & [e_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n-1, \\ [e_1, x_1] = 2e_1, & [e_i, x_1] = 2(i-1)e_i, & 3 \leq i \leq n, \\ [y_i, x_1] = (2i-1)y_i, & 1 \leq i \leq n-1, & \\ [x_1, e_1] = -2e_1, & [x_1, y_1] = -y_1, & \\ [e_2, x_2] = e_2, & [x_2, e_2] = -e_2, & \end{cases}$$

$$G_1(b):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [e_2,x]=be_2, \quad [e_i,x]=2(i-1)e_i, \quad 3\le i\le n,\\ [x,e_1]=-2e_1, & [x,e_2]=-be_2, \quad [x,y_1]=-y_1, \quad b\neq 0,\end{cases}$$

$$G_2(b):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [e_2,x]=be_2, \quad [e_i,x]=2(i-1)e_i, \quad 3\le i\le n,\\ [x,e_1]=-2e_1, & [x,y_1]=-y_1,\end{cases}$$

$$G_3:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=2(n-1)e_2+e_n, & [e_i,x]=2(i-1)e_i, \quad 3\le i\le n,\\ [x,y_1]=-y_1,\end{cases}$$

$$G_4(\gamma,b):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=be_n, & [e_i,x]=2(i-1)e_i, \quad 3\le i\le n,\\ [x,y_1]=-y_1, & [x,x]=\gamma e_2, \quad (\gamma,b)=(0,1),(1,0),(1,1)\end{cases}$$

$$G_5: \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n-1,\\ [e_1,x]=\sum\limits_{k=3}^{n}a_{k-1}e_k, & [e_2,x]=e_2, & [e_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}e_k, \quad 3\le i\le n,\\ [x,x]=\gamma e_n, & [y_i,x]=\sum\limits_{k=i+1}^{n-1}a_{k+1-i}y, & 1\le i\le n-1, \end{cases}$$

$$G_6: \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n-1,\\ [e_1,x]=\sum\limits_{k=3}^{n}a_{k-1}e_k, & [e_2,x]=e_2, & [e_i,x]=\sum\limits_{k=i+1}^{n}a_{k+1-i}e_k, \quad 3\le i\le n,\\ [x,e_2]=-e_2, & [x,x]=\gamma e_n, & [y_i,x]=\sum\limits_{k=i+1}^{n-1}a_{k+1-i}y, \quad 1\le i\le n-1. \end{cases}$$

Note that, the first non-vanishing parameter $\{a_2,a_3,\dots,a_{n-1},\gamma\}$ *in the algebras* G_5 *and* G_6 *can be scaled to* 1.

Proof. The proof is similar to the proof of Theorem 4.12. □

Now we consider the case when the nilradical is a non-split superalgebra from the class $G(\beta_4,\beta_5,\dots,\beta_n,\gamma)$, i.e., at least one of the parameters is not equal to zero.

Theorem 4.15. *Let* $L=L_0\oplus L_1$ *be a solvable Leibniz superalgebra whose nilradical is isomorphic to a non-split superalgebra* N *from the class* $G(\beta_4,\beta_5,\dots,\beta_n,\gamma)$. *Then* L *is isomorphic to one of the following pairwise non-isomorphic superalgebras:*

$$SG_1(t)(4\le t\le n): \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, & 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, & 3\le i\le n-1,\\ [e_1,e_2]=e_t, & [e_j,e_2]=e_{j+t-2}, & 3\le j\le n-2,\\ [y_j,e_2]=y_{j+t-2}, & & 2\le j\le n-3,\\ [y_i,x]=(2i-1)y_i, & & 1\le i\le n-1,\\ [x,y_1]=-y_1, & [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=2(t-2)e_2, & [x,e_2]=-2(t-2)e_2-2e_{t-1}, & \\ [e_i,x]=2(i-1)e_i, & & 3\le i\le n, \end{cases}$$

$$SG_2(\gamma)(n\ is\ odd): \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [e_1,e_2]=e_{\frac{n+3}{2}}, & [e_j,e_2]=e_{j+\frac{n-1}{2}}, \quad 3\le j\le n-2,\\ [y_j,e_2]=y_{j+\frac{n-1}{2}}, & 1\le j\le n-3,\\ [e_2,e_2]=\gamma e_n,\gamma\neq 0, & [x,y_1]=-y_1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=(n-1)e_2, & [x,e_2]=-(n-1)e_2-2e_{\frac{n+1}{2}},\\ [e_i,x]=2(i-1)e_i, & 3\le i\le n. \end{cases}$$

$$SG_3: \begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\le i\le n-1,\\ [y_j,e_1]=y_{j+1}, & 1\le j\le n-2,\\ [y_1,y_1]=e_1, & [y_j,y_1]=e_{j+1}, \quad 2\le j\le n-1,\\ [e_1,y_1]=\frac{1}{2}y_2, & [e_i,y_1]=\frac{1}{2}y_i, \quad 3\le i\le n-1,\\ [e_2,e_2]=e_n, & [x,y_1]=-y_1,\\ [y_i,x]=(2i-1)y_i, & 1\le i\le n-1,\\ [e_1,x]=2e_1, & [x,e_1]=-2e_1,\\ [e_2,x]=(n-1)e_2, & [x,e_2]=-(n-1)e_2,\\ [e_i,x]=2(i-1)e_i, & 3\le i\le n, \end{cases}$$

The proof of this theorem is carried out similarly to the proof of Theorem 4.13.

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Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author (Khudoyberdiyev A.Kh) on reasonable request.

Conflict of Interest

The authors have no conflicts and interests to declare that are relevant to the content of this article.

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