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On the number of lattice points in a ball

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Abstract. Let $\mathcal{A} \subseteq \mathbb{R}^M$ be a discrete subgroup of rank N, so that $\mathcal{A}$ is the $\mathbb{Z}$-module generated by the columns of an $M \times N$ real matrix A of rank N. Let $\mathcal{B} \subseteq \mathbb{R}^M$ be the $\mathbb{R}$-linear subspace spanned by the columns of A and let $|A|$ denote the norm of the matrix A as a linear map from $\mathbb{R}^N$ into $\mathbb{R}^M$. We prove an explicit inequality that estimates the number of points in $\mathcal{A}$ contained in a ball of radius R centered at a generic point in $\mathcal{B}$. The inequality we prove is uniform over all matrices A with norm bounded by a positive constant. A particularly simple form of the inequality occurs when $N = 3$.

1 Introduction

Let A be an $M \times N$ matrix with real entries and $1 \le N = \operatorname{rank} A \le M$. The columns of A generate the free $\mathbb{Z}$-module

$$\mathcal{A} = \left\{A\boldsymbol{m} : \boldsymbol{m} \in \mathbb{Z}^N\right\} \subseteq \mathbb{R}^M$$

of rank N. The columns of A also generate the $\mathbb{R}$-linear subspace

$$\mathcal{B} = \left\{A\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^N\right\} \subseteq \mathbb{R}^M$$

of dimension N. We consider the problem of estimating the number of points in $\mathcal{A}$ that are contained in a ball of positive radius R centered at a generic point $A\boldsymbol{x}$ in the subspace $\mathcal{B}$.

The general problem of estimating the number of lattice points in a ball is treated in [18] and [35]. Results for lattice points in $\mathbb{R}^3$ are proved in [16] and [33]. The papers [15], [25], [29] and [30] treat questions somewhat closer to the present work.

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We write

$$|\boldsymbol{x}| = \left(x_1^2 + x_2^2 + \cdots + x_N^2\right)^{\frac{1}{2}}$$

for the Euclidean norm of a (column) vector $\boldsymbol{x}$ in $\mathbb{R}^N$, and similarly for a vector in $\mathbb{R}^M$. We write

$$V_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}, \quad \text{and} \quad \omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{1}$$

for the volume and surface area of the unit ball, respectively, in $\mathbb{R}^N$. We define the normalized characteristic function

$$\chi_R : \mathbb{R}^M \to \left\{0, \tfrac{1}{2}, 1\right\}$$

of a ball of radius R centered at $\mathbf{0}$ by

$$\chi_R(\boldsymbol{y}) = \begin{cases} 1 & \text{if } |\boldsymbol{y}| < R, \\ \frac{1}{2} & \text{if } |\boldsymbol{y}| = R, \\ 0 & \text{if } R < |\boldsymbol{y}|. \end{cases} \tag{2}$$

We suppress reference to the dimension M (or to the dimension N) in our notation for χ_R as this should always be clear. And we write

$$|A| = \sup\left\{|A\boldsymbol{x}| : \boldsymbol{x} \in \mathbb{R}^N,\ |\boldsymbol{x}| \le 1\right\}$$

for the norm of the linear transformation

$$\boldsymbol{x} \mapsto A\boldsymbol{x}$$

from $\mathbb{R}^N$ into $\mathbb{R}^M$. Then a more precise statement of the problem we consider is to estimate the sum

$$\left(\det A^T A\right)^{\frac{1}{2}} \sum_{\boldsymbol{m} \in \mathbb{Z}^N} \chi_R\big(A(\boldsymbol{m} + \boldsymbol{x})\big) \tag{3}$$

by an expression that is independent of the point $\boldsymbol{x}$ in $\mathbb{R}^N$. We seek an estimate for (3) that depends on N, R, and on an upper bound for the norm $|A|$. As

$$\left(\det A^T A\right)^{\frac{1}{2}} \int_{\mathbb{R}^N} \chi_R\big(A(\boldsymbol{y} + \boldsymbol{x})\big)\ \mathrm{d}\boldsymbol{y} = V_N R^N,$$

it is natural to estimate (3) by establishing an upper bound for

$$\left|\left(\det A^T A\right)^{\frac{1}{2}} \sum_{\boldsymbol{m} \in \mathbb{Z}^N} \chi_R\big(A(\boldsymbol{m} + \boldsymbol{x})\big) - V_N R^N\right| \tag{4}$$

that depends on N, R, and on an upper bound for $|A|$.

An estimate for (4) follows from inequalities for extremal functions that were obtained in [17]. The Bessel functions $J_\nu(x)$ and $J_{\nu+1}(x)$ with $2\nu + 2 = N$ naturally occur in our estimates. However, as a convenient abuse of notation, we use both N and ν depending on the situation. Our first result on the number of lattice points in a ball is the following inequality.

Theorem 1.1. *Let A be an $M \times N$ matrix with real entries, and*

$$1 \le N = \operatorname{rank} A \le M.$$

Let $-1 < \nu$ and $0 < \delta$ be real parameters such that $2\nu + 2 = N$. Then for $0 < R$ and $|A| \le \delta^{-1}$ we have

$$\left|\big(\det A^T A\big)^{1/2} \sum_{\boldsymbol{m} \in \mathbb{Z}^N} \chi_R\big(A(\boldsymbol{m} + \boldsymbol{x})\big) - V_N R^N\right| \le \omega_{N-1} u_\nu(R, \delta)$$

for all $\boldsymbol{x}$ in $\mathbb{R}^N$, where $u_\nu(R,\delta)$ is positive and defined by

$$u_\nu(R,\delta) = \delta^{-1} R^{N-1}\left(1 - \tfrac{\pi}{2}(N-1)\int_{\pi\delta R}^{\infty} x^{-1} J_\nu(x) J_{\nu+1}(x)\ \mathrm{d}x\right)^{-1}. \tag{5}$$

In dimension $N = 3$ we get the following simpler bound.

Corollary 1.2. *Let A be an $M \times 3$ matrix with real entries,* $\operatorname{rank} A = 3$*, and let $0 < \delta$. Then for $0 < R$ and $|A| \le \delta^{-1}$, we have*

$$\begin{aligned} &\left|\big(\det A^T A\big)^{1/2} \sum_{\boldsymbol{m} \in \mathbb{Z}^3} \chi_R\big(A(\boldsymbol{m} + \boldsymbol{x})\big) - \tfrac{4}{3}\pi R^3\right| \\ &\qquad\qquad \le 4\pi\delta^{-1} R^2 \left(1 - \left(\frac{\sin \pi\delta R}{\pi\delta R}\right)^2\right)^{-1} \end{aligned} \tag{6}$$

at each point $\boldsymbol{x}$ in $\mathbb{R}^3$.

Proof. Bessel functions have elementary representations when the index ν is half of an odd integer. If $N = 3$ then $\nu = \frac{1}{2}$, and we find that

$$\pi x^{-1} J_{\frac{1}{2}}(x) J_{\frac{3}{2}}(x) = -\frac{d}{dx}\left(\frac{\sin x}{x}\right)^2.$$

The integral on the right of (5) is then

$$\pi \int_{\pi\delta R}^{\infty} x^{-1} J_{\frac{1}{2}}(x) J_{\frac{3}{2}}(x)\ \mathrm{d}x = \left(\frac{\sin \pi\delta R}{\pi\delta R}\right)^2.$$

Therefore (6) follows from (5). □

In section 2 we prove results that we need from the theory of entire functions of exponential type in one and several variables. We say that an entire function $F : \mathbb{C}^N \to \mathbb{C}$ of N complex variables is a *real entire* function if the restriction of F to $\mathbb{R}^N$ takes real values. Such functions are used throughout the paper. Section 3 contains an account of the Beurling-Selberg extremal problem for a ball and the solution that was found in [17]. In particular, this includes the initial identification of $u_\nu(R,\delta)$ as the solution to an extremal problem. In section 4 we establish a special case of the Poisson summation formula (essentially [32, §VII, Theorem 2.4]) that applies to integrable functions $F : \mathbb{R}^N \to \mathbb{R}$ which are the restriction to $\mathbb{R}^N$ of a real entire function $F : \mathbb{C}^N \to \mathbb{C}$ of exponential type. The proof of Theorem 1.1 is given in section 5.

2 Entire functions of exponential type

We recall that an entire function $F : \mathbb{C} \to \mathbb{C}$ has *exponential type* if F is not identically zero, and if

$$\limsup_{|z|\to\infty} |z|^{-1} \log |F(z)| = \tau(F) < \infty.$$

If F has exponential type, then the nonnegative number $\tau(F)$ is the *exponential type* of F. We write $e(z) = e^{2\pi i z}$ for a complex number z.

Lemma 2.1. *Let $0 < \delta$ and let $F : \mathbb{C} \to \mathbb{C}$ be an entire function of exponential type at most $2\pi\delta$. Assume that*

$$\|F\|_1 = \int_{\mathbb{R}} |F(x)| \, \mathrm{d}x < \infty,$$

and write $\widehat{F} : \mathbb{R} \to \mathbb{C}$ for the Fourier transform

$$\widehat{F}(t) = \int_{\mathbb{R}} F(x) e(-tx) \, \mathrm{d}x. \tag{7}$$

Then

$$\|F\|_\infty = \sup\bigl\{|F(x)| : x \in \mathbb{R}\bigr\} \le 2\pi\delta \|F\|_1, \tag{8}$$

and

$$\|F\|_2 = \left(\int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \le \bigl(2\pi\delta\bigr)^{\frac{1}{2}} \|F\|_1. \tag{9}$$

Moreover, the Fourier transform (7) is a continuous function supported on $[-\delta, \delta]$. And the entire function F is determined at each point z in $\mathbb{C}$ by the Fourier inversion formula

$$F(z) = \int_{-\delta}^{\delta} \widehat{F}(t) e(tz) \, \mathrm{d}t. \tag{10}$$

Proof. Let $E : \mathbb{C} \to \mathbb{C}$ be the entire function defined by

$$E(z) = \int_0^z F(w) \, \mathrm{d}w.$$

Let $0 < \varepsilon$ and let C_ε be a positive constant that satisfies both of the inequalities

$$|F(z)| \le C_\varepsilon \exp\bigl((2\pi\delta + \varepsilon)|z|\bigr), \quad \text{and} \quad |z| \le C_\varepsilon \exp\bigl(\varepsilon|z|\bigr), \tag{11}$$

at each point z in $\mathbb{C}$. Using (11) we find that

$$\begin{aligned} |E(z)| = \left| z \int_0^1 F(tz) \, \mathrm{d}t \right| &\le C_\varepsilon |z| \int_0^1 \exp\bigl((2\pi\delta + \varepsilon)t|z|\bigr) \, \mathrm{d}t \\ &\le C_\varepsilon^2 \exp\bigl((2\pi\delta + 2\varepsilon)|z|\bigr) \end{aligned} \tag{12}$$

at each point z in $\mathbb{C}$. As $0 < \varepsilon$ is arbitrary it follows from (12) that E has exponential type at most $2\pi\delta$. At each point x in $\mathbb{R}$ we also get the inequality

$$|E(x)| = \left|\int_0^x F(w)\ \mathrm{d}w\right| \le \int_{\mathbb{R}} |F(w)|\ \mathrm{d}w = \|F\|_1, \tag{13}$$

and therefore E is bounded on $\mathbb{R}$. As $E'(z) = F(z)$, Bernstein's inequality (see [1, §74], [3], or [4]) and (13) imply that

$$\begin{aligned}\sup\left\{|F(x)| : x \in \mathbb{R}\right\} &= \sup\left\{\left|E'(x)\right| : x \in \mathbb{R}\right\}\\ &\le 2\pi\delta \sup\left\{|E(x)| : x \in \mathbb{R}\right\}\\ &\le 2\pi\delta\|F\|_1.\end{aligned}$$

This proves (8). Then

$$\int_{\mathbb{R}} \left|F(x)\right|^2\ \mathrm{d}x \le \sup\left\{|F(x)| : x \in \mathbb{R}\right\} \int_{\mathbb{R}} |F(x)|\ \mathrm{d}x \le 2\pi\delta\|F\|_1^2$$

verifies the inequality (9).

Because F belongs to $L^1(\mathbb{R})$, it follows that the Fourier transform (7) is a function in $C_0(\mathbb{R})$. Because F belongs to $L^2(\mathbb{R})$ and has exponential type at most $2\pi\delta$, it follows from the Paley-Wiener theorem (see [1, §72] or [32, §3.4]) that $t \mapsto \widehat{F}(t)$ is supported on the interval $[-\delta, \delta]$. Therefore the Fourier inversion formula asserts that

$$F(x) = \int_{-\delta}^{\delta} \widehat{F}(t)e(xt)\ \mathrm{d}t \tag{14}$$

at each point x in $\mathbb{R}$. Using the integral on the right of (14) and Morera's theorem it is easy to show that

$$z \mapsto \int_{-\delta}^{\delta} \widehat{F}(t)e(zt)\ \mathrm{d}t \tag{15}$$

defines an entire function of $z = x + iy$. Plainly the entire function defined by (15) is equal to the entire function $z \mapsto F(z)$ for z in $\mathbb{R}$. Therefore we get

$$F(z) = \int_{-\delta}^{\delta} \widehat{F}(t)e(zt)\ \mathrm{d}t$$

at each point z in $\mathbb{C}$ by analytic continuation. This verifies (10). □

We write $\boldsymbol{z} = (z_n)$ for a (column) vector in $\mathbb{C}^N$ and

$$|\boldsymbol{z}| = \left(|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2\right)^{\frac{1}{2}}$$

for the usual Hermitian norm of $\boldsymbol{z}$. We define a second norm

$$\|\ \| : \mathbb{C}^N \to [0, \infty)$$

on vectors $\boldsymbol{z}$ in $\mathbb{C}^N$ by setting

$$\|\boldsymbol{z}\| = \sup\Big\{\big|z_1t_1 + z_2t_2 + \cdots + z_Nt_N\big| : \boldsymbol{t} \in \mathbb{R}^N \text{ and } |\boldsymbol{t}| \le 1\Big\}.$$

If $N = 1$ then $|z| = \|z\|$ at each point z in $\mathbb{C}$. But if $2 \le N$ we find that

$$\|\boldsymbol{z}\|^2 \le |\boldsymbol{z}|^2 \le 2\|\boldsymbol{z}\|^2 \tag{16}$$

at each point $\boldsymbol{z}$ in $\mathbb{C}^N$, where both inequalities in (16) are sharp. If

$$F : \mathbb{C}^N \to \mathbb{C}$$

is an entire function of N complex variables and not identically zero, we say that F has *exponential type* if

$$\limsup_{\|\boldsymbol{z}\|\to\infty} \|\boldsymbol{z}\|^{-1} \log|F(\boldsymbol{z})| = \tau(F) < \infty.$$

The nonnegative number $\tau(F)$ is the *exponential type* of F. If $N = 1$ this is the usual definition of exponential type discussed above. If $2 \le N$ our definition is a special case of a more general notion of exponential type introduced by Stein in [31], (see also [32, pp. 111-112]).

Next we define a map that sends an even entire function $F : \mathbb{C} \to \mathbb{C}$ having exponential type into a radial entire function

$$\psi_N(F) : \mathbb{C}^N \to \mathbb{C}$$

having exponential type in the sense of Stein [31]. This map was defined and used in [17, section 6].

Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function. Then the power series for F at 0 can be written as

$$F(z) = \sum_{m=0}^{\infty} c_{2m}(F) z^{2m}, \tag{17}$$

where

$$c_{2m}(F) = \frac{F^{(2m)}(0)}{(2m)!} \quad \text{for } m = 0, 1, 2, \ldots.$$

As is well known, the partial sums for the series in (17) converge uniformly on compact subsets of $\mathbb{C}$. For each positive integer N we define, as in [17, Lemma 18], the entire function

$$\psi_N(F) : \mathbb{C}^N \to \mathbb{C}$$

of N complex variables $z_1, z_2, \ldots, z_N$, by

$$\psi_N(F)(\boldsymbol{z}) = \sum_{m=0}^{\infty} \frac{c_{2m}(F)\big(z_1^2 + z_2^2 + \cdots + z_N^2\big)^m}{(2m)!}. \tag{18}$$

If $\boldsymbol{x}$ belongs to $\mathbb{R}^N$ we find that

$$\begin{aligned}\psi_N(F)(\boldsymbol{x}) &= \sum_{m=0}^{\infty} \frac{c_{2m}(F)\big(x_1^2 + x_2^2 + \cdots + x_N^2\big)^m}{(2m)!} \\ &= \sum_{m=0}^{\infty} \frac{c_{2m}(F)|\boldsymbol{x}|^{2m}}{(2m)!} \\ &= F\big(|\boldsymbol{x}|\big).\end{aligned} \tag{19}$$

It follows that $\psi_N(F)$ restricted to $\mathbb{R}^N$ is a radial function.

Lemma 2.2. *Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function, and for each positive integer N let*

$$\psi_N(F) : \mathbb{C}^N \to \mathbb{C}$$

be the entire function of N complex variables defined by (18). Define

$$\|F\|_\infty = \sup\big\{|F(x)| : x \in \mathbb{R}\big\},$$

and

$$\big\|\psi_N(F)\big\|_\infty = \sup\big\{\big|\psi_N(F)(\boldsymbol{x})\big| : \boldsymbol{x} \in \mathbb{R}^N\big\}.$$

Then F has exponential type if and only if $\psi_N(F)$ has exponential type. Moreover, if either F or $\psi_N(F)$ has exponential type, then

$$\tau(F) = \tau\big(\psi_N(F)\big). \tag{20}$$

Also, F restricted to $\mathbb{R}$ is bounded if and only if $\psi_N(F)$ restricted to $\mathbb{R}^N$ is bounded. Additionally, if either F is bounded on $\mathbb{R}$ or $\psi_N(F)$ is bounded on $\mathbb{R}^N$, then

$$\|F\|_\infty = \big\|\psi_N(F)\big\|_\infty. \tag{21}$$

Proof. That F has exponential type if and only if $\psi_N(F)$ has exponential type, together with the identity (20), were both established in [17, Lemma 18].

Because F is an even function, it follows from the identity (19) that

$$\big\{F(x) : x \in \mathbb{R}\big\} = \big\{F\big(|\boldsymbol{x}|\big) : \boldsymbol{x} \in \mathbb{R}^N\big\} = \big\{\psi_N(F)(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^N\big\},$$

and therefore

$$\big\{\big|F(x)\big| : x \in \mathbb{R}\big\} = \big\{\big|\psi_N(F)(\boldsymbol{x})\big| : \boldsymbol{x} \in \mathbb{R}^N\big\}. \tag{22}$$

The conclusion that F is bounded on $\mathbb{R}$ if and only if $\psi_N(F)$ is bounded on $\mathbb{R}^N$, and the proposed identity (21), follow immediately from (22). □

If we assume that the even, entire function

$$F : \mathbb{C} \to \mathbb{C}$$

has exponential type at most $2\pi\delta$, then it follows from Lemma 2.2 that the entire function $\psi_N(F)(\boldsymbol{z})$, defined using (17) and (18), has exponential type at most $2\pi\delta$. Moreover, [17, equation (6.3)] asserts that

$$\tfrac{1}{2}\omega_{N-1}\int_{\mathbb{R}} |F(x)||x|^{2\nu+1}\ \mathrm{d}x = \int_{\mathbb{R}^N} |\psi_N(F)(\boldsymbol{x})||\boldsymbol{x}|^{2\nu+2-N}\ \mathrm{d}\boldsymbol{x}, \tag{23}$$

where $-1 < \nu$. And if either of the integrals in (23) is finite then both integrals are finite, and we get (this is [17, equation (6.4)])

$$\tfrac{1}{2}\omega_{N-1}\int_{\mathbb{R}} F(x)|x|^{2\nu+1}\ \mathrm{d}x = \int_{\mathbb{R}^N} \psi_N(F)(\boldsymbol{x})|\boldsymbol{x}|^{2\nu+2-N}\ \mathrm{d}\boldsymbol{x}.$$

The special case $2\nu + 2 = N$ leads to the following result.

Lemma 2.3. *Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function of exponential type at most $2\pi\delta$. Assume that N is a positive integer such that*

$$\int_{\mathbb{R}} |F(x)||x|^{N-1}\ \mathrm{d}x < \infty, \tag{24}$$

and let

$$\psi_N(F) : \mathbb{C}^N \to \mathbb{C}$$

be the entire function of N complex variables defined by (18). Then the restriction of $\psi_N(F)$ to $\mathbb{R}^N$ belongs to

$$L^1\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big).$$

Moreover, the Fourier transform

$$\widehat{\psi_N(F)} : \mathbb{R}^N \to \mathbb{C}$$

defined by

$$\widehat{\psi_N(F)}(\boldsymbol{t}) = \int_{\mathbb{R}^N} \psi_N(F)(\boldsymbol{x}) e\big(-\boldsymbol{t}^T\boldsymbol{x}\big)\ \mathrm{d}\boldsymbol{x}, \tag{25}$$

is a continuous, radial function supported in the closed ball

$$\big\{\boldsymbol{t} \in \mathbb{R}^N : |\boldsymbol{t}| \le \delta\big\}. \tag{26}$$

Proof. By taking $2\nu + 2 = N$ in (23), we find that

$$\tfrac{1}{2}\omega_{N-1}\int_{\mathbb{R}} |F(x)||x|^{N-1}\ \mathrm{d}x = \int_{\mathbb{R}^N} |\psi_N(F)(\boldsymbol{x})|\ \mathrm{d}\boldsymbol{x} < \infty. \tag{27}$$

Therefore $\psi_N(F)$ belongs to $L^1\big(\mathbb{R}^N\big)$, and the Fourier transform (25) is a continuous, radial function (see [32, §IV, Corollary 1.2]).

Clearly (24) implies that F belongs to $L^1(\mathbb{R})$. Then using (8) and (21) we get

$$\big\|\psi_N(F)\big\|_\infty = \|F\|_\infty \le 2\pi\delta\|F\|_1 < \infty. \tag{28}$$

It follows from (27) and (28) that

$$\int_{\mathbb{R}^N} \big|\psi_N(F)(\boldsymbol{x})\big|^2 \,\mathrm{d}\boldsymbol{x} \le \big\|\psi_N(F)\big\|_\infty \int_{\mathbb{R}^N} \big|\psi_N(F)(\boldsymbol{x})\big| \,\mathrm{d}\boldsymbol{x} < \infty.$$

This verifies that $\psi_N(F)$ belongs to both $L^1\big(\mathbb{R}^N\big)$ and $L^2\big(\mathbb{R}^N\big)$. Hence the Fourier transform (25) is a continuous, radial function. And by the generalization of the Paley-Wiener theorem proved by Stein (see [31, §III, Theorem 4] or [32, §III Theorem 4.9]), we conclude that (25) is supported on the closed ball (26). □

3 Beurling-Selberg extremal problems

In this section we consider extremal problems first investigated by A. Beurling and later by A. Selberg and we describe the solution to a more general extremal problem that was obtained in [17]. The method used in [17] was based on earlier work of L. de Branges [12], [13]. Further information about these problems can be found in [23], [26], [27], [28], and [34].

Let ξ, δ, and ν be real numbers that satisfy $0 < \delta$ and $-1 < \nu$. Then let $z \mapsto S(z)$ and $z \mapsto T(z)$ be real entire functions such that

$$S(x) \le \operatorname{sgn}(x-\xi) \le T(x) \text{ for all real } x, \tag{29}$$

and

$$z \mapsto S(z) \text{ and } z \mapsto T(z) \text{ have exponential type at most } 2\pi\delta. \tag{30}$$

We define the real number $u_\nu(\xi,\delta)$ to be the infimum of the collection of positive numbers

$$\tfrac{1}{2}\int_{\mathbb{R}} \big(T(x)-S(x)\big)|x|^{2\nu+1} \,\mathrm{d}x \tag{31}$$

taken over the set of all pairs of real entire functions $S(z)$ and $T(z)$ that satisfy (29) and (30), and for which the value of the integral in (31) is finite. The problem considered by Beurling was the special case $\xi = 0$ and $\nu = -\frac{1}{2}$. Related extremal problems are considered in [10], [11], [6], [9], [7, §1.1 and §1.2], [8, §1.2 and §1.4], [5], [21], [20], [19], and [22].

In [17, Theorem 1] the authors proved that for $0 < \delta$ and $-1 < \nu$, the infimum $u_\nu(\xi,\delta)$ is positive. They also proved that there exists a *unique* pair of real entire functions

$$z \mapsto s_\nu(z;\xi,\delta) \quad \text{and} \quad z \mapsto t_\nu(z;\xi,\delta) \tag{32}$$

that satisfy the inequality

$$s_\nu(x;\xi,\delta) \le \operatorname{sgn}(x-\xi) \le t_\nu(x;\xi,\delta) \quad \text{for all real } x, \tag{33}$$

the functions

$$z \mapsto s_\nu(z;\xi,\delta) \text{ and } z \mapsto t_\nu(z;\xi,\delta) \text{ have exponential type at most } 2\pi\delta, \tag{34}$$

and satisfy the identity

$$u_\nu(\xi,\delta) = \tfrac{1}{2}\int_{\mathbb{R}} \big(t_\nu(x;\xi,\delta) - s_\nu(x;\xi,\delta)\big)|x|^{2\nu+1}\,\mathrm{d}x. \tag{35}$$

The functions (32) are defined in [17, equation (5.7) and (5.8)].

In [17, Theorem 1] the authors proved that $u_\nu(\xi,\delta)$ satisfies the following identities:

(i) $u_\nu(\xi,\delta) = u_\nu(-\xi,\delta)$,

(ii) if $0<\kappa$ then $u_\nu(\xi,\delta) = \kappa^{2\nu+2}u_\nu\big(\kappa^{-1}\xi,\kappa\delta\big)$,

(iii) $u_\nu(0,\delta) = \Gamma(\nu+1)\Gamma(\nu+2)\big(2/\pi\delta\big)^{2\nu+2}$,

(iv) if $0<\xi$ then (this is [17, equation (1.4)])

$$u_\nu\big(\xi,\pi^{-1}\big) = \frac{2\xi^{2\nu+1}}{\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu+1)J_\nu(\xi)J_{\nu+1}(\xi)}, \tag{36}$$

where $z \mapsto J_\nu(z)$ and $z \mapsto J_{\nu+1}(z)$ are Bessel functions.

Using elementary properties of Bessel functions (see [24] and [36, §3.2]) we find that

$$\lim_{\xi\to\infty} \xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu+1)J_\nu(\xi)J_{\nu+1}(\xi) = 2\pi^{-1}, \tag{37}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(xJ_\nu(x)^2 + xJ_{\nu+1}(x)^2 - (2\nu+1)J_\nu(x)J_{\nu+1}(x)\Big)&\\ = (2\nu+1)x^{-1}J_\nu(x)J_{\nu+1}(x).&\end{aligned} \tag{38}$$

If $0<\xi$ then (37) and (38) lead to the identity

$$\begin{aligned}\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 &- (2\nu+1)J_\nu(\xi)J_{\nu+1}(\xi)\\ &= 2\pi^{-1} - (2\nu+1)\int_\xi^\infty x^{-1}J_\nu(x)J_{\nu+1}(x)\,\mathrm{d}x.\end{aligned} \tag{39}$$

Combining (36) and (39) we find that

$$u_\nu\big(\xi,\pi^{-1}\big) = \xi^{2\nu+1}\left(\pi^{-1} - \big(\nu+\tfrac{1}{2}\big)\int_\xi^\infty x^{-1}J_\nu(x)J_{\nu+1}(x)\,\mathrm{d}x\right)^{-1}.$$

Then by applying the identity (ii) with $\kappa = (\pi\delta)^{-1}$, we get the general formula

$$u_\nu\big(\xi, \delta\big) = \delta^{-1}\xi^{2\nu+1}\left(1 - \tfrac{\pi}{2}(2\nu+1)\int_{\pi\delta\xi}^{\infty} x^{-1}J_\nu(x)J_{\nu+1}(x)\ \mathrm{d}x\right)^{-1}, \tag{40}$$

which holds for $0 < \xi$, $0 < \delta$ and $-1 < \nu$. In the particular case where $2\nu + 2 = N$ the map (40) is the same as (5) in the statement of Theorem 1.1.

In the special case $\nu = -\frac{1}{2}$ and $\xi = 0$ considered by Beurling, the two extremal functions (32) can be represented using interpolation formulas that were given in [34, Theorem 9 and Theorem 10].

Next we describe an extremal problem for a ball in $\mathbb{R}^N$ centered at $\mathbf{0}$ that was considered in [17]. Related results can be found in [14].

Let R, δ, and ν, be real numbers such that $0 < R$, $0 < \delta$, and $-1 < \nu$. Then let

$$F : \mathbb{C}^N \to \mathbb{C}, \quad \text{and} \quad G : \mathbb{C}^N \to \mathbb{C}, \tag{41}$$

be real entire functions such that

$$F(\boldsymbol{x}) \le \chi_R(\boldsymbol{x}) \le G(\boldsymbol{x}) \quad \text{at each point } \boldsymbol{x} \text{ in } \mathbb{R}^N, \tag{42}$$

and

$$F(\boldsymbol{z}) \text{ and } G(\boldsymbol{z}) \text{ have exponential type at most } 2\pi\delta. \tag{43}$$

Define $H_\nu(N, R, \delta)$ to be the infimum of positive numbers

$$\int_{\mathbb{R}^N} \big(G(\boldsymbol{x}) - F(\boldsymbol{x})\big)|\boldsymbol{x}|^{2\nu+2-N}\ \mathrm{d}\boldsymbol{x} \tag{44}$$

taken over the set of all ordered pairs (F, G) of real entire functions (41) that satisfy (42), (43), and for which the integral (44) is finite.

In the special case $\nu = -\frac{1}{2}$ and $N = 1$ the problem of evaluating (or estimating) the infimum $H_{-\frac{1}{2}}(1, R, \delta)$ was considered by Selberg [27], [28]. For $N = 1$ the simple identity

$$\chi_R(x) = \tfrac{1}{2}\big(\mathrm{sgn}(x + R) - \mathrm{sgn}(x - R)\big), \tag{45}$$

which holds for all real x, indicates the connection between Beurling's extremal problem and the extremal problem considered by Selberg. Selberg observed (at least in the case $\nu = -\frac{1}{2}$) that for all real x the inequalities

$$\tfrac{1}{2}\big(s_\nu(x; -R, \delta) - t_\nu(x; R, \delta)\big) \le \chi_R(x) \le \tfrac{1}{2}\big(t_\nu(x; -R, \delta) - s_\nu(x; R, \delta)\big), \tag{46}$$

follow from the two inequalities in (33) and the identity (45). Therefore we define the two real entire functions

$$f_\nu(z; R, \delta) = \tfrac{1}{2}\big(s_\nu(z; -R, \delta) - t_\nu(z; R, \delta)\big) \tag{47}$$

and

$$g_\nu(z;R,\delta) = \tfrac{1}{2}\big(t_\nu(z;-R,\delta) - s_\nu(z;R,\delta)\big). \tag{48}$$

It follows from (34) that both $z \mapsto f_\nu(z;R,\delta)$ and $z \mapsto g_\nu(z;R,\delta)$ have exponential type at most $2\pi\delta$. Clearly the inequality (46) is also

$$f_\nu(x;R,\delta) \le \chi_R(x) \le g_\nu(x;R,\delta)$$

for all real x and all positive values of R.

From (33) we find that *both* of the inequalities

$$s_\nu(x;R,\delta) \le \operatorname{sgn}(x-R) \le t_\nu(x;R,\delta) \quad \text{for all real } x,$$

and

$$-t_\nu(-x;R,\delta) \le \operatorname{sgn}(x+R) \le -s_\nu(-x;R,\delta) \quad \text{for all real } x,$$

must hold. As the extremal functions $x \mapsto s_\nu(x;R,\delta)$ and $x \mapsto t_\nu(x;R,\delta)$ that satisfy (33), (34), and (35) are *unique*, we also get

$$-s_\nu(-x;R,\delta) = t_\nu(x;-R,\delta) \tag{49}$$

for all real x. Then it follows from (49) that both (47) and (48) are *even* entire functions.

Next we describe the solution to the problem of evaluating (or estimating) the infimum $H_\nu(N,R,\delta)$ for a ball in $\mathbb{R}^N$ which was obtained in [17]. We write

$$\boldsymbol{z} \mapsto \mathcal{F}_\nu(\boldsymbol{z};R,\delta), \quad \text{and} \quad \boldsymbol{z} \mapsto \mathcal{G}_\nu(\boldsymbol{z};R,\delta), \tag{50}$$

for the two real entire functions defined in [17, equation (1.22) and (1.23)]. From our discussion of the maps (18) and (19), it follows that the functions (50) are also given by

$$\mathcal{F}_\nu(\boldsymbol{z};R,\delta) = \psi_N\big(f_\nu\big)(\boldsymbol{z};R,\delta), \quad \text{and} \quad \mathcal{G}_\nu(\boldsymbol{z};R,\delta) = \psi_N\big(g_\nu\big)(\boldsymbol{z};R,\delta), \tag{51}$$

where f_ν and g_ν are the even functions defined in (47) and (48). It follows from [17, Theorem 3] that the functions (50) are both real entire functions of exponential type at most $2\pi\delta$. And it follows from the representations (51) that the restriction of these functions to $\mathbb{R}^N$ are both radial functions that satisfy the inequalities

$$\mathcal{F}_\nu(\boldsymbol{x};R,\delta) \le \chi_R(\boldsymbol{x}) \le \mathcal{G}_\nu(\boldsymbol{x};R,\delta) \tag{52}$$

for all $\boldsymbol{x}$ in $\mathbb{R}^N$. Moreover, these functions satisfy the integral identity (this is [17, equation (1.27)])

$$\omega_{N-1}u_\nu(R,\delta) = \int_{\mathbb{R}^N} \big(\mathcal{G}_\nu(\boldsymbol{x};R,\delta) - \mathcal{F}_\nu(\boldsymbol{x};R,\delta)\big)|\boldsymbol{x}|^{2\nu+2-N}\,\mathrm{d}\boldsymbol{x}, \tag{53}$$

where ω_{N-1} (the surface area of a unit ball in $\mathbb{R}^N$) was defined in (1). Now (53) (this is [17, equation (1.25)]) implies that

$$H_\nu(N,R,\delta) \le \omega_{N-1}u_\nu(R,\delta). \tag{54}$$

It was also shown in [17, Theorem 3] that there is equality in the inequality (54) if and only if

$$J_\nu(\pi\delta R)J_{\nu+1}(\pi\delta R) = 0.$$

We now restrict our attention to the case $2\nu + 2 = N$.

Lemma 3.1. *Let R, δ and ν be real numbers such that $0 < R$, $0 < \delta$, and $-1 < \nu$, and let N be a positive integer such that $2\nu + 2 = N$. Let*

$$\mathcal{F}_\nu(\boldsymbol{z}; R, \delta) = \psi_N\big(f_\nu\big)(\boldsymbol{z}; R, \delta), \quad \textit{and} \quad \mathcal{G}_\nu(\boldsymbol{z}; R, \delta) = \psi_N\big(g_\nu\big)(\boldsymbol{z}; R, \delta),$$

be the real entire functions defined by (51), or defined in [17, equation (1.22) and (1.23)]. Then both the restriction of $\mathcal{F}_\nu(\boldsymbol{z}; R, \delta)$ to $\mathbb{R}^N$, and the restriction of $\mathcal{G}_\nu(\boldsymbol{z}; R, \delta)$ to $\mathbb{R}^N$, belong to

$$L^1\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big). \tag{55}$$

Moreover, both of the Fourier transforms

$$\widehat{\mathcal{F}}_\nu : \mathbb{R}^N \to \mathbb{C}, \quad \textit{and} \quad \widehat{\mathcal{G}}_\nu : \mathbb{R}^N \to \mathbb{C},$$

defined by

$$\widehat{\mathcal{F}}_\nu(\boldsymbol{t}; R, \delta) = \int_{\mathbb{R}^N} \mathcal{F}_\nu(\boldsymbol{z}; R, \delta) e\big(-\boldsymbol{t}^T\boldsymbol{x}\big)\ \mathrm{d}\boldsymbol{x}, \tag{56}$$

and

$$\widehat{\mathcal{G}}_\nu(\boldsymbol{t}; R, \delta) = \int_{\mathbb{R}^N} \mathcal{G}_\nu(\boldsymbol{z}; R, \delta) e\big(-\boldsymbol{t}^T\boldsymbol{x}\big)\ \mathrm{d}\boldsymbol{x}, \tag{57}$$

are continuous functions supported on the closed ball

$$\big\{\boldsymbol{t} \in \mathbb{R}^N : |\boldsymbol{t}| \le \delta\big\}.$$

Proof. At each point $\boldsymbol{x}$ in $\mathbb{R}^N$ the restrictions of $\mathcal{F}_\nu$ and $\mathcal{G}_\nu$ to $\mathbb{R}^N$ satisfy the inequality (52). It follows from (53), and our assumption that $2\nu + 2 = N$, that

$$\int_{\mathbb{R}^N} \big(\mathcal{G}_\nu(\boldsymbol{x}; R, \delta) - \mathcal{F}_\nu(\boldsymbol{x}; R, \delta)\big)\ \mathrm{d}\boldsymbol{x} < \infty. \tag{58}$$

Let $\mathcal{F}_\nu^+ : \mathbb{R}^N \to [0, \infty)$ and $\mathcal{F}_\nu^- : \mathbb{R}^N \to [0, \infty)$ be defined by

$$\mathcal{F}_\nu^+(\boldsymbol{x}; R, \delta) = \max\big\{0, \mathcal{F}_\nu(\boldsymbol{x}; R, \delta)\big\},\ \text{and}\ \mathcal{F}_\nu^-(\boldsymbol{x}; R, \delta) = \max\big\{0, -\mathcal{F}_\nu(\boldsymbol{x}; R, \delta)\big\}.$$

It follows that

$$\mathcal{F}_\nu(\boldsymbol{x}; R, \delta) = \mathcal{F}_\nu^+(\boldsymbol{x}; R, \delta) - \mathcal{F}_\nu^-(\boldsymbol{x}; R, \delta)$$

and

$$\big|\mathcal{F}_\nu(\boldsymbol{x}; R, \delta)\big| = \mathcal{F}_\nu^+(\boldsymbol{x}; R, \delta) + \mathcal{F}_\nu^-(\boldsymbol{x}; R, \delta).$$

From (52) we also get the inequality

$$\mathcal{F}_\nu^+(\boldsymbol{x}; R, \delta) \le \chi_R(\boldsymbol{x}) \tag{59}$$

at each point $\boldsymbol{x}$ in $\mathbb{R}^N$. Then it follows using (59) that

$$\begin{aligned}\big|\mathcal{G}_\nu(\boldsymbol{x};R,\delta)\big| + \big|\mathcal{F}_\nu(\boldsymbol{x};R,\delta)\big| \\ &= \mathcal{G}_\nu(\boldsymbol{x};R,\delta) + \mathcal{F}_\nu^+(\boldsymbol{x};R,\delta) + \mathcal{F}_\nu^-(\boldsymbol{x};R,\delta) \\ &= \mathcal{G}_\nu(\boldsymbol{x};R,\delta) - \mathcal{F}_\nu(\boldsymbol{x};R,\delta) + 2\mathcal{F}_\nu^+(\boldsymbol{x};R,\delta) \\ &\le \mathcal{G}_\nu(\boldsymbol{x};R,\delta) - \mathcal{F}_\nu(\boldsymbol{x};R,\delta) + 2\chi_R(\boldsymbol{x})\end{aligned} \tag{60}$$

at each point $\boldsymbol{x}$ in $\mathbb{R}^N$. From (52) and (58) we find that the nonnegative valued function

$$\boldsymbol{x} \mapsto \mathcal{G}_\nu(\boldsymbol{x};R,\delta) - \mathcal{F}_\nu(\boldsymbol{x};R,\delta)$$

belongs to $L^1(\mathbb{R}^N)$, and it is obvious that

$$\boldsymbol{x} \mapsto \chi_R(\boldsymbol{x})$$

also belongs to $L^1(\mathbb{R}^N)$. Then (60) implies that both the restriction of $\mathcal{F}_\nu(\boldsymbol{z};R,\delta)$ to $\mathbb{R}^N$ and the restriction of $\mathcal{G}_\nu(\boldsymbol{z};R,\delta)$ to $\mathbb{R}^N$ belong to $L^1(\mathbb{R}^N)$.

Let

$$f_\nu : \mathbb{C} \to \mathbb{C}, \quad \text{and} \quad g_\nu : \mathbb{C} \to \mathbb{C} \tag{61}$$

be the real entire functions of exponential type at most $2\pi\delta$ defined by (47) and (48). The functions $\mathcal{F}_\nu$ and G_ν are the image of f_ν and g_ν, respectively, with respect to the map ψ_N. That is, we have

$$\mathcal{F}_\nu(\boldsymbol{z};R,\delta) = \psi_N\big(f_\nu\big)(\boldsymbol{z};R,\delta), \quad \text{and} \quad \mathcal{G}_\nu(\boldsymbol{z};R,\delta) = \psi_N\big(g_\nu\big)(\boldsymbol{z};R,\delta).$$

It follows from (23), our assumption that $N = 2\nu + 2$, and what we have already proved, that both

$$\tfrac{1}{2}\omega_{N-1}\int_{\mathbb{R}} |f_\nu(x;R,\delta)||x|^{N-1}\,\mathrm{d}x = \int_{\mathbb{R}^N} |\mathcal{F}_\nu(\boldsymbol{x};R,\delta)|\,\mathrm{d}\boldsymbol{x} < \infty,$$

and

$$\tfrac{1}{2}\omega_{N-1}\int_{\mathbb{R}} |g_\nu(x;R,\delta)||x|^{N-1}\,\mathrm{d}x = \int_{\mathbb{R}^N} |\mathcal{G}_\nu(\boldsymbol{x};R,\delta)|\,\mathrm{d}\boldsymbol{x} < \infty.$$

Therefore the two functions (61) satisfy the hypotheses of Lemma 2.3. Then it follows from the conclusion of Lemma 2.3 that the functions $\mathcal{F}_\nu = \psi_N\big(f_\nu\big)$ and $\mathcal{G}_\nu\big(g_\nu\big)$ belong to the set (55). The properties attributed to the Fourier transforms (56) and (57) also follow from Lemma 2.3. □

4 The Poisson formula

We require the following elementary lemma.

Lemma 4.1. *Let A be an $M \times N$ matrix with entries in $\mathbb{R}$ and*

$$1 \le N = \operatorname{rank} A \le M.$$

Then there exists a real $N \times N$, positive definite, symmetric matrix S such

$$|A\boldsymbol{x}| = |S\boldsymbol{x}| \tag{62}$$

for all vectors $\boldsymbol{x}$ in $\mathbb{R}^N$. Moreover, we have

$$|A|^{-1} \le \min\big\{\big|S^{-1}\boldsymbol{n}\big| : \boldsymbol{n} \in \mathbb{Z}^N,\ \boldsymbol{n} \ne \boldsymbol{0}\big\}. \tag{63}$$

Proof. The matrix A^TA is $N \times N$, positive definite and symmetric. Hence there exists (see [2, Theorem 8.6.10]) an $N \times N$ orthogonal matrix Φ and an $N \times N$ diagonal matrix $D = [d_n]$ with positive diagonal entries $d_1, d_2, \ldots, d_N$, such that

$$A^TA = \Phi^T D \Phi.$$

We set

$$S = \Phi^T D^{\frac{1}{2}} \Phi, \quad \text{where} \quad D^{\frac{1}{2}} = \big[d_n^{\frac{1}{2}}\big].$$

It follows that S is an $N \times N$, positive definite, symmetric matrix, and

$$\begin{aligned}|S\boldsymbol{x}|^2 &= \boldsymbol{x}^T S^T S \boldsymbol{x} = \boldsymbol{x}^T \Phi^T D^{\frac{1}{2}} \Phi \Phi^T D^{\frac{1}{2}} \Phi \boldsymbol{x} \\ &= \boldsymbol{x}^T A^T A \boldsymbol{x} = |A\boldsymbol{x}|^2.\end{aligned}$$

for all vectors $\boldsymbol{x}$ in $\mathbb{R}^N$. This establishes (62).

As is well known, we have

$$\begin{aligned}|A|^2 &= \sup\big\{\boldsymbol{x}^T\Phi^T D \Phi \boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^N,\ |\boldsymbol{x}| \le 1\big\} \\ &= \sup\big\{\boldsymbol{y}^T D \boldsymbol{y} : \boldsymbol{y} \in \mathbb{R}^N,\ |\boldsymbol{y}| \le 1\big\} \\ &= \sup\Big\{\sum_{n=1}^N d_n y_n^2 : \boldsymbol{y} \in \mathbb{R}^N,\ |\boldsymbol{y}| \le 1\Big\} \\ &= \max\big\{d_n : 1 \le n \le N\big\} \\ &= \Big(\min\big\{d_n^{-1} : 1 \le n \le N\big\}\Big)^{-1}.\end{aligned} \tag{64}$$

Let $\boldsymbol{n} \ne \boldsymbol{0}$ belong to $\mathbb{Z}^N$ and let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N$ denote the standard basis vectors in $\mathbb{Z}^N \subseteq \mathbb{R}^N$. Then

$$|\Phi\boldsymbol{n}|^2 = \sum_{n=1}^N \big(\boldsymbol{e}_n^T \Phi \boldsymbol{n}\big)^2 = |\boldsymbol{n}|^2 \ge 1, \tag{65}$$

and it follows from (64) and (65) that

$$\begin{aligned}\left|S^{-1}\boldsymbol{n}\right|^2 &= \boldsymbol{n}^T\Phi^T D^{-1}\Phi\boldsymbol{n}\\ &= \sum_{n=1}^{N} d_n^{-1}\big(\boldsymbol{e}_n^T\Phi\boldsymbol{n}\big)^2\\ &\ge \Big(\min\big\{d_n^{-1} : 1\le n\le N\big\}\Big)|\Phi\boldsymbol{n}|^2\\ &\ge |A|^{-2}.\end{aligned}$$

This shows that the inequality (63) holds. □

Let A and S be real matrices as in Lemma 4.1, which satisfy (62) and (63). Then we have

$$\big(\det A^TA\big)^{\frac{1}{2}}\sum_{\boldsymbol{m}\in\mathbb{Z}^N}\chi_R\big(A(\boldsymbol{m}+\boldsymbol{x})\big) = (\det S)\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\chi_R\big(S(\boldsymbol{n}+\boldsymbol{x})\big), \tag{66}$$

where $\boldsymbol{y}\mapsto\chi_R(\boldsymbol{y})$ is the normalized characteristic function defined in (2). It is obvious that the function defined by

$$\boldsymbol{x}\mapsto(\det S)\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\chi_R\big(S(\boldsymbol{n}+\boldsymbol{x})\big) - V_NR^N$$

is constant on cosets of the quotient group $\mathbb{R}^N/\mathbb{Z}^N$.

Suppose that $F:\mathbb{C}^N\to\mathbb{C}$ is a real entire function of exponential type at most $2\pi\delta$. We also assume that the restriction of F to $\mathbb{R}^N$ is a radial function that belongs to

$$L^1\big(\mathbb{R}^N\big)\cap L^2\big(\mathbb{R}^N\big).$$

It follows that the Fourier transform

$$\widehat{F}:\mathbb{R}^N\to\mathbb{C}$$

is a continuous function, and it follows from the generalization of the Paley-Wiener theorem established by Stein (see [31, §III, Theorem 4] or [32, §III Theorem 4.9]), that

$$\widehat{F}(\boldsymbol{t}) = \int_{\mathbb{R}^N} F(\boldsymbol{x})e\big(-\boldsymbol{t}^T\boldsymbol{x}\big)\,\mathrm{d}\boldsymbol{x} = 0,\quad \text{if } \boldsymbol{t}\in\mathbb{R}^N \text{ and } \delta\le|\boldsymbol{t}|. \tag{67}$$

The hypothesis (67) leads to a simple form of the Poisson summation formula. Because the $N\times N$ matrix S is symmetric the Poisson summation formula takes the form (see [32, Chapter VII, Corollary 2.6])

$$\begin{aligned}(\det S)\sum_{\boldsymbol{m}\in\mathbb{Z}^N}F\big(S(\boldsymbol{m}+\boldsymbol{x})\big) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{F}\big(S^{-1}\boldsymbol{n}\big)e\big((S\boldsymbol{x})^TS^{-1}\boldsymbol{n}\big)\\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{F}\big(S^{-1}\boldsymbol{n}\big)e\big(\boldsymbol{x}^TS^TS^{-1}\boldsymbol{n}\big)\\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{F}\big(S^{-1}\boldsymbol{n}\big)e\big(\boldsymbol{x}^T\boldsymbol{n}\big).\end{aligned} \tag{68}$$

And because $\widehat{F}$ is a continuous function with compact support, it follows that each sum on the right of (68) contains only finitely many nonzero terms. Applying (67) we find that

$$(\det S) \sum_{\boldsymbol{m} \in \mathbb{Z}^N} F\big(S(\boldsymbol{m} + \boldsymbol{x})\big) = \sum_{\substack{\boldsymbol{n} \in \mathbb{Z}^N \\ |S^{-1}\boldsymbol{n}| < \delta}} \widehat{F}\big(S^{-1}\boldsymbol{n}\big) e\big(\boldsymbol{x}^T \boldsymbol{n}\big). \tag{69}$$

The function on the right of (69) is a finite linear combination of continuous characters

$$\boldsymbol{x} \mapsto e\big(\boldsymbol{x}^T \boldsymbol{n}\big)$$

defined on the compact group $\mathbb{R}^N/\mathbb{Z}^N$. That is, the function of $\boldsymbol{x}$ on the right of (69) is a trigonometric polynomial. If we assume that $\delta \le |A|^{-1}$, then it follows from (63) in the statement of Lemma 4.1 that there is at most one nonzero term in the sum on the right of (69). That is, if we assume that $\delta \le |A|^{-1}$ then (69) simplifies to

$$(\det S) \sum_{\boldsymbol{m} \in \mathbb{Z}^N} F\big(S(\boldsymbol{m} + \boldsymbol{x})\big) = \widehat{F}(\boldsymbol{0}) \tag{70}$$

at each coset representative $\boldsymbol{x}$ in $\mathbb{R}^N/\mathbb{Z}^N$.

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying the Poisson formula (70) to the two real entire functions

$$\mathcal{F}_\nu(\boldsymbol{z}; R, \delta) = \psi_N\big(f_\nu\big)(\boldsymbol{z}; R, \delta), \quad \text{and} \quad \mathcal{G}_\nu(\boldsymbol{z}; R, \delta) = \psi_N\big(g_\nu\big)(\boldsymbol{z}; R, \delta)$$

which were initially defined in (51). We recall that f_ν and g_ν are even entire functions of exponential type at most $2\pi\delta$ defined in (47) and (48). The special functions $\mathcal{F}_\nu$ and $\mathcal{G}_\nu$ satisfy the basic inequality

$$\mathcal{F}_\nu(\boldsymbol{x}; R, \delta) \le \chi_R(\boldsymbol{x}) \le \mathcal{G}_\nu(\boldsymbol{x}; R, \delta)$$

at each point $\boldsymbol{x}$ in $\mathbb{R}^N$.

We recall that $2\nu + 2 = N$. It follows from Lemma 2.3 that $\mathcal{F}_\nu$ restricted to $\mathbb{R}^N$ and $\mathcal{G}_\nu$ restricted to $\mathbb{R}^N$ belong to

$$L^1\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big).$$

It also follows from Lemma 2.3 that their Fourier transforms

$$\widehat{\mathcal{F}}_\nu : \mathbb{R}^N \to \mathbb{C}, \quad \text{and} \quad \widehat{\mathcal{G}}_\nu : \mathbb{R}^N \to \mathbb{C},$$

are continuous functions supported on the closed ball

$$\big\{\boldsymbol{t} \in \mathbb{R}^N : |\boldsymbol{t}| \le \delta\big\}.$$

Using (52) and (70) we find that

$$\begin{aligned}\widehat{\mathcal{F}}_\nu(\mathbf{0}; R, \delta) &= (\det S) \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \mathcal{F}_\nu\big(S(\boldsymbol{m}+\boldsymbol{x}); R, \delta\big)\\ &\le (\det S) \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \chi_R\big(S(\boldsymbol{m}+\boldsymbol{x})\big)\\ &\le (\det S) \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \mathcal{G}_\nu\big(S(\boldsymbol{m}+\boldsymbol{x}); R, \delta\big)\\ &= \widehat{\mathcal{G}}_\nu(\mathbf{0}; R, \delta).\end{aligned} \tag{71}$$

We select the real parameter ν so that $2\nu + 2 = N$. Then it follows from (53) that

$$\widehat{\mathcal{G}}_\nu(\mathbf{0}; R, \delta) - \widehat{\mathcal{F}}_\nu(\mathbf{0}; R, \delta) = \omega_{N-1} u_\nu(R, \delta). \tag{72}$$

Also, using (52) again we get

$$\widehat{\mathcal{F}}_\nu(\mathbf{0}; R, \delta) \le \int_{\mathbb{R}^N} \chi_R(\boldsymbol{x})\, d\boldsymbol{x} = V_N R^N \le \widehat{\mathcal{G}}_\nu(\mathbf{0}; R, \delta), \tag{73}$$

where V_N is given by (1). Now (71), (72), and (73), imply that the inequality

$$\begin{aligned}V_N R^N - \omega_{N-1} u_\nu(R, \delta) &\le \widehat{\mathcal{F}}_\nu(\mathbf{0}; R, \delta)\\ &\le (\det S) \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \chi_R\big(S(\boldsymbol{m}+\boldsymbol{x})\big)\\ &\le \widehat{\mathcal{G}}_\nu(\mathbf{0}; R, \delta) \le V_N R^N + \omega_{N-1} u_\nu(R, \delta)\end{aligned} \tag{74}$$

holds at each point $\boldsymbol{x}$ in $\mathbb{R}^N$. Therefore (74) can be rewritten as the inequality

$$\left|(\det S) \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \chi_R\big(S(\boldsymbol{m}+\boldsymbol{x})\big) - V_N R^N\right| \le \omega_{N-1} u_\nu(R, \delta) \tag{75}$$

for all points $\boldsymbol{x}$ in $\mathbb{R}^N$. Finally, we can express (75) in terms of the original matrix A and the problem of estimating (3) by applying the identity (66). This completes the proof of Theorem 1.1.

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