Extended commutator algebra for the q-oscillator and a related Askey-Wilson algebra

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Abstract. Let q be a nonzero complex number that is not a root of unity. In the q-oscillator with commutation relation $aa^+ - qa^+a = 1$, it is known that the smallest commutator algebra of operators containing the creation and annihilation operators a^+ and a is the linear span of a^+ and a, together with all operators of the form $a^{+l}[a, a^+]^k$, and $[a, a^+]^k a^l$, where l is a nonnegative integer and k is a positive integer. That is, linear combinations of operators of the form a^h or $(a^+)^h$ with $h \geq 2$ or $h = 0$ are outside the commutator algebra generated by a and a^+. This is a solution to the Lie polynomial characterization problem for the associative algebra generated by a^+ and a. In this work, we extend the Lie polynomial characterization into the associative algebra $\mathcal{P} = \mathcal{P}(q)$ generated by a, a^+, and the operator $e^{\omega N}$ for some nonzero real parameter ω, where N is the number operator, and we relate this to a q-oscillator representation of the Askey-Wilson algebra $AW(3)$.

1 Introduction

Throughout, we consider operators on the Hilbert space ℓ_2 over the complex field $\mathbb{C}$, which consists of all sequences $(z_n)_{n=0}^\infty$ of elements of $\mathbb{C}$ with the property $\sum_{n=0}^\infty |z_n|^2 < \infty$. We use the complete orthonormal basis of ℓ_2 consisting of the ket vectors $|n\rangle$ for all $n \in \mathbb{N} := \{0, 1, \ldots\}$. Denote by $\mathcal{B}(\ell_2)$ the Banach algebra of all bounded operators on ℓ_2. In the traditional harmonic oscillator, the creation operator a^+ and the annihilation operator a can be represented as elements of $\mathcal{B}(\ell_2)$ acting on the basis ket vectors according

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to the equations

$$\begin{aligned}
a\left|0\right\rangle &= 0, \\
a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle, & (n \in \mathbb{N}\backslash\{0\}), \\
a^+\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle, & (n \in \mathbb{N}).
\end{aligned}$$

As a consequence, a^+ and a satisfy the canonical commutation relation

$$\left[a, a^+\right] = 1, \tag{1}$$

where the operation $[-,-] : \mathcal{B}(\ell_2) \times \mathcal{B}(\ell_2) \to \mathcal{B}(\ell_2)$ is the *commutator* given by the rule $[X, Y] := XY - YX$ for all $X, Y \in \mathcal{B}(\ell_2)$. The algebra $\mathcal{B}(\ell_2)$ is hence an abstract Lie algebra with respect to $[-,-]$ and the vector space operations.

Problem 1.1. What is the smallest space of operators containing a^+ and a that is closed under the commutator and vector space operations? In other words, viewing $\mathcal{B}(\ell_2)$ as a Lie algebra under said operations, what is the Lie subalgebra of $\mathcal{B}(\ell_2)$ generated by a^+ and a? Equivalently, what is the *commutator algebra* for the creation and annihilation operators of the classical harmonic oscillator?

The solution to Problem 1.1 is readily available from the theory of the classification of low-dimensional Lie algebras. As a consequence of the relation (1), the Lie subalgebra of $\mathcal{B}(\ell_2)$ generated by a^+ and a is isomorphic to the *Heisenberg algebra*, which is the three-dimensional Lie algebra whose derived (Lie) algebra is contained in the center. The operators a^+, a and $1 \in \mathcal{B}(\ell_2)$ form a basis [14, Section 3.2.1].

1.1   The commutator algebra for a^+, a in the q-oscillator

Several modifications or generalizations to the relation (1) have been proposed. One is the replacement of the undeformed commutator by the q-deformed commutator $[a, a^+]_q := aa^+ - qa^+a$ for some scalar parameter q. The result is the q-deformed harmonic oscillator or q-oscillator introduced in the classical works [4, 25]. Another is the ω-commutator $[a, a^+]_\omega := e^\omega aa^+ - e^{-\omega}a^+a$ for some nonzero real parameter ω, which shall arise in later sections of this work. The latter deformed commutator also leads to a q-oscillator representation [29]. See, for instance, [2, 26] for physical interpretations or realizations of the q-oscillator. The significance of the q-deformed oscillator is that, together with the deformed quantum group $SU_q(2)$, these two mathematical constructions were once considered as possible candidates for the role of oscillator and angular momentum, respectively, at very small scales, such as the Planck scale [29, p. 1155]. The natural continuation now is to extend Problem 1.1 for the q-oscillator.

Problem 1.2. Given a scalar $q \neq 1$, what is the commutator algebra for the creation and annihilation operators of the q-oscillator?

A solution to Problem 1.2 has recently been obtained in [6, 11]. Some generalizations and alternative approaches were studied in [7, 8, 9, 10]. Before we describe the solution, we

first discuss some consequences of, and some algebraic structures related to, the q-deformed commutation relation

$$aa^+ - qa^+a = 1. \tag{2}$$

The *q-deformed Heisenberg algebra* is the unital associative algebra $\mathcal{H}(q)$ (with unit 1) abstractly defined by a presentation with generators A, B and relation $AB - qBA = 1$. More succinctly, if the underlying field is $\mathbb{F}$, then $\mathcal{H}(q) := \mathbb{F}\langle A, B\rangle/(AB - qBA - 1)$.

The use of the symbol $\mathcal{H}(q)$ and of the term "q-deformed Heisenberg algebra" to both mean $\mathbb{F}\langle A, B\rangle/(AB - qBA - 1)$ is based on [16]. An excellent discussion may be found in [16, pp. 5–11] about which among the 410 references the authors cite, published not later than the year 2000, motivated the algebra $\mathbb{F}\langle A, B\rangle/(AB - qBA - 1)$, and how it has been consistently denoted by $\mathcal{H}(q)$ or referred to as the q-deformed Heisenberg algebra.

Some works published after [16] continue to refer to $\mathbb{F}\langle A, B\rangle/(AB - qBA - 1)$ as $\mathcal{H}(q)$ or as the q-deformed Heisenberg algebra. Said works come from varied fields of mathematics, such as Ring Theory [15, 17], Lie algebras [6, 8, 9, 10, 11], Mathematical Physics [7, 19], and algebraic curves [13]. All these, and most probably many more, refer to $\mathcal{H}(q) = \mathbb{F}\langle A, B\rangle/(AB - qBA - 1)$ as the q-deformed Heisenberg algebra. No confusion should arise with similar symbol and terminology from recent studies like [20, 21, 22, 23, 24], which are *not* the subject of this paper.

From this point onward, we assume that $\mathbb{F} = \mathbb{C}$. If $\mathcal{H}_0$ is the unital associative subalgebra of $\mathcal{B}(\ell_2)$ generated by a^+ and a, we immediately find that there exists a surjective homomorphism $\Psi : \mathcal{H}(q) \to \mathcal{H}_0$ of algebras such that $A \mapsto a$ and $B \mapsto a^+$. In the common realizations of a^+ and a as operators on ℓ_2, such as those in [1, 12, 27], it can be shown by routine arguments that the operator $[a, a^+]^n$ is not the zero operator for any $n \in \mathbb{N}$. This condition, together with an assumption that q is nonzero and is not a root of unity, is enough to conclude, using [17, Theorem 6.7], that the homomorphism Ψ is injective. i.e., The representation Ψ of $\mathcal{H}(q)$ is faithful. Thus, we have:

Proposition 1.3. *The subalgebra $\mathcal{H}_0$ of $\mathcal{B}(\ell_2)$ is isomorphic to $\mathcal{H}(q)$.*

By this isomorphism, the algebraic properties of $\mathcal{H}(q)$ such as bases, gradations, and others, carry over to $\mathcal{H}_0$ with $A \mapsto a$ and $B \mapsto a^+$. We discuss these properties in the succeeding remarks.

Remark 1.4. By [17, Corollary 4.5], the elements

$$[a, a^+]^k, \tag{3}$$
$$[a, a^+]^k a^l, \tag{4}$$
$$(a^+)^l [a, a^+]^k, \tag{5}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}\backslash\{0\}$, form a basis for $\mathcal{H}_0$.

Denote by $\mathfrak{L}(\mathcal{H}_0)$ the Lie subalgebra of $\mathcal{H}_0$ generated by a^+ and a, or equivalently, $\mathfrak{L}(\mathcal{H}_0)$ is the commutator algebra for the creation and annihilation operators of the q-oscillator.

The basis of $\mathcal{H}_0$ consisting of the elements (3)–(5) has a very important role in the solution of Problem 1.2 as obtained in [6]. To describe this solution, we divide (3)–(5) into two groups. For the first group, we take $1 = [a, a^+]^0$ from (3), and from (4),(5), we consider those in which the exponent k of $[a, a^+]$ is zero and at the same time, the exponent l of either a^+ or a is at least 2. That is,

$$1, \quad a^l, \quad (a^+)^l, \qquad (l \in \mathbb{N} \backslash \{0, 1\}). \tag{6}$$

Then the basis elements of $\mathcal{H}_0$ in (3)–(5) that are not in (6) are

$$[a, a^+]^k, \qquad\qquad (k \in \mathbb{N} \backslash \{0\}), \tag{7}$$

$$a, \quad [a, a^+]^k a^l, \qquad\qquad (k, l \in \mathbb{N} \backslash \{0\}), \tag{8}$$

$$a^+, \quad (a^+)^l [a, a^+]^k, \qquad\qquad (k, l \in \mathbb{N} \backslash \{0\}). \tag{9}$$

Remark 1.5. According to [6, Theorem 5.8], the elements (7)–(9) form a basis for $\mathfrak{L}(\mathcal{H}_0)$. That is, if Γ is the vector subspace of $\mathcal{H}_0$ spanned by (6), then we have the direct sum decomposition

$$\mathcal{H}_0 = \Gamma \oplus \mathfrak{L}(\mathcal{H}_0). \tag{10}$$

In other words, any finite linear combination of (6) cannot be expressed as a finite linear combination of nested commutators in a^+, a, or equivalently not a *Lie polynomial in a^+, a*. Thus, the answer to Problem 1.2 is that the commutator algebra $\mathfrak{L}(\mathcal{H}_0)$ is the set of all finite linear combinations of (7)–(9). The generators a^+, a of $\mathfrak{L}(\mathcal{H}_0)$ are clearly Lie polynomials in a^+, a, but then it is most natural to ask how the other basis elements of $\mathfrak{L}(\mathcal{H}_0)$ from (7)–(9) can be expressed as Lie polynomials in a^+, a. To show this, we shall exhibit some relations from [6, 11] that express such basis elements as Lie polynomials or linear combinations of nested commutators in a^+, a, but before exhibiting these relations, we first discuss a convenient notation using the *adjoint map*.

Given a Lie algebra L and $x \in L$, recall the adjoint map ad $x : L \to L$ defined by $y \mapsto [x, y]$. By linearity of the Lie bracket in the second argument, ad x is a linear map for any $x \in L$, and by the skew-symmetry of the Lie bracket, $(-\text{ad } x)(y) = -[x, y] = [y, x]$. For convenience in expressing nested commutators, we use the concept of the adjoint map in the following manner.

Remark 1.6. Let $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l \in L$. If $k = 2$, the nested commutator $[x_1, [x_2, y]]$ can be expressed as the composition of two adjoint maps given by $((\text{ad } x_1) \circ (\text{ad } x_2))(y)$. In general, we have

$$((\text{ad } x_1) \circ (\text{ad } x_2) \circ \cdots \circ (\text{ad } x_k))(y) = [x_1, [x_2, [\cdots, [x_k, y] \cdots]]]. \tag{11}$$

For the special case $x_1 = x_2 = \cdots = x_k$, we can write the left-hand side of (11) as $(\text{ad } x)^k(y)$. Similarly, if $l = 2$, the nested commutator of $[[x, y_1], y_2]$ can be expressed

as the composition of two adjoint maps given by $((-\text{ad } y_2) \circ (-\text{ad } y_1))(x)$. This can be generalized as

$$((-\text{ad } y_1) \circ (-\text{ad } y_2) \circ \cdots \circ (-\text{ad } y_l))(x) = [[[\cdots [x, y_l], \cdots], y_2], y_1]. \tag{12}$$

We also have the similar result that if $y_1 = y_2 = \cdots = y_l$, then we can write the left-hand side of (12) as $(-\text{ad } y)^l(x)$.

Remark 1.7. From [6, Section 5], if q is nonzero and is not a root of unity, then the basis elements of $\mathfrak{L}(\mathcal{H}_0)$ from (7)–(9) can be expressed as Lie polynomials in a^+ and a through the relations

$$[a, a^+]^{k+2} = \frac{-q^k(1-q)}{1-q^{k+1}} \sum_{i=0}^{k} \frac{\left((\text{ad } a^+) \circ (-\text{ad } [a, a^+])^k \circ (\text{ad } a)\right)([a, a^+])}{(q-1)^{1+i}}, \tag{13}$$

$$[a, a^+]^{k+1} a^l = -\frac{\left((-\text{ad } [a, a^+])^k \circ (-\text{ad } a)^{l+1}\right)(a^+)}{(1-q)^l(q^l-1)^k}, \tag{14}$$

$$(a^+)^l [a, a^+]^{k+1} = \frac{\left((\text{ad } a^+)^{l-1} \circ (\text{ad } [a, a^+])^{k+1}\right)(a^+)}{(q-1)^{k+1}(1-q^{k+1})^{l-1}}, \tag{15}$$

which hold for any $k \in \mathbb{N}$, and any $l \in \mathbb{N}\backslash\{0\}$. The operators a^+, a, and $[a, a^+]$ are clearly Lie polynomials in a^+, a, and so we need not consider them in the relations (13)–(15). The commutator table for these basis elements of $\mathfrak{L}(\mathcal{H}_0)$ can be gleaned from [6, Table 1], and a detailed discussion of the computational aspects of these relations can be found in [11, Section 2.1].

If q is a root of unity other than 1, there are special cases for the constructions in (13)–(15), and the basis (7)–(9) of $\mathfrak{L}(\mathcal{H}_0)$ is reduced according to some conditions in the exponents. These can be found in [11, Section 4.1]. If $q = 0$, an entirely different basis and nested commutator constructions can be found in [6, Section 4]. These two cases, however, are not covered by the representation theorem [17, Theorem 6.7], and so the possibly non-faithful representation $\mathcal{H}_0$ of $\mathcal{H}(q)$ for these two cases may lead to further restrictions in the algebraic structure of $\mathfrak{L}(\mathcal{H}_0)$. We reserve these two cases as for a possibly interesting continuation of the results presented in this paper, and so we have the following.

Assumption 1.8. *The scalar q is nonzero and is not a root of unity.*

1.2 The objective of this study: extending the commutator algebra for a^+, a

As a natural continuation of Problem 1.2, we consider extending the generating set of the operator Lie algebra by another important operator from the q-oscillator. One possibility is the *number operator N* defined by the action $N |n\rangle := n |n\rangle$ for any $n \in \mathbb{N}$. The canonical commutation relations that relate N to a^+ and a are

$$[N, a^+] = a^+, \quad [a, N] = a, \tag{16}$$

which hold even for the q-oscillator. We note the simple nature of the relations (16), in which we can see that the result of performing the commutator operation on any of a^+, a involving N simply results to linear combinations of a^+, a. By an easy argument, an immediate consequence is that the extended commutator algebra for a^+, a, N is simply the direct sum $\mathbb{C}N \oplus \mathfrak{L}(\mathcal{H}_0)$, where $\mathbb{C}N$ is the set of all complex scalar multiples of N. As will be demonstrated in the results in this paper, it is the operator $q^{\kappa N}$, defined by the action

$$q^{\kappa N} |n\rangle := q^{\kappa n} |n\rangle, \tag{17}$$

for some scalar κ, that gives a richer algebraic structure in extending the commutator algebra for a^+, a. By some choice of κ, our results for the extended commutator algebra for a^+, a, $q^{\kappa N}$ turn out to be intimately related to a q-oscillator realization of an *Askey-Wilson algebra*. The three-parameter Askey-Wilson algebra $AW(3)$ was introduced in [29], and is said to be important by itself as a dynamical symmetry algebra in problems in which the Askey-Wilson polynomials arise as eigenfunctions [29, p. 1147]. The algebra $AW(3)$ and its extension to generalized families of similar algebras have been extensively studied in algebraic combinatorics, among many other fields. See, for instance, the comprehensive discussion in [28, Section 1] about how the Askey-Wilson algebra and its variants are studied in algebraic combinatorics, integrable systems, and quantum mechanics, or the discussion in [18, Section 1] about how $AW(3)$ is related to several significant quantum groups. The algebraic perspective in this work, however, differs from that in [18] or [28]. We do not study ring-theoretic generalizations or finite-dimensional representation theory, but the Lie structure of associative algebras, such as the perspective in the studies [5, 6, 11]. We show that our results about the extended commutator algebra for the q-oscillator shed light on some related Lie structure in a q-oscillator representation of the algebra $AW(3)$.

2   Preliminaries: Algebraic structure of the q-deformed Heisenberg algebra $\mathcal{H}(q)$

We review some important results about the algebra $\mathcal{H}(q)$, which has a presentation with generators A, B and relation $AB - qBA = 1$. Let $C := [A, B]$. In terms of A, B, C, the basis (3)-(5) of $\mathcal{H}_0 = \mathcal{H}(q) \subseteq \mathcal{P}$ can be rewritten as

$$C^k, \quad C^k A^l, \quad C^k B^l, \qquad (k \in \mathbb{N}, \ l \in \mathbb{N}\backslash\{0\}). \tag{18}$$

For each $l \in \mathbb{N}$, define $\mathfrak{H}_{-l}$ as the span of all elements $C^k A^l$ ($k \in \mathbb{N}$), and $\mathfrak{H}_l$ as the span of $C^k B^l$ ($k \in \mathbb{N}$).

Remark 2.1. In the basis (18) of $\mathcal{H}(q)$, we use basis elements of the form $C^k B^l$ for the product of powers of C and B, instead of $B^l C^k$ which was used in [11, 17]. This may be justified by the reordering formula $B^l C^k = q^{-kl} C^k B^l$ (for any $k, l \in \mathbb{N}$) which is from [17, equation 18]. This change in normal form for this particular type of basis element is valid because of the assumption that q is nonzero (Assumption 1.8).

Denote the set of all integers by $\mathbb{Z}$. For any vector subspaces $\mathfrak{A}$ and $\mathfrak{B}$ of any algebra $\mathcal{A}$, we define $\mathfrak{A}\mathfrak{B}$ as the span of all elements ab where $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Following [17, Proposition 2.4, Corollary 4.5], the collection $\{\mathfrak{H}_k \ : \ k \in \mathbb{Z}\}$ is a $\mathbb{Z}$-gradation of $\mathcal{H}(q)$. That is, we have the direct sum decomposition

$$\mathcal{H}(q) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{H}_k, \tag{19}$$

and that for any $h, k \in \mathbb{Z}$,

$$\mathfrak{H}_h \mathfrak{H}_k \subseteq \mathfrak{H}_{h+k}. \tag{20}$$

Thus, for instance, if $m, k \in \mathbb{N}$ and $n, l \in \mathbb{N}\backslash\{0\}$ with $n > l$, then $C^m B^n \cdot C^k A^l \in \mathfrak{H}_{n-l}$. However, this does not give us any information about how to express $C^m B^n \cdot C^k A^l$ as a linear combination of the basis elements $C^h B^{n-l}$ ($h \in \mathbb{N}$) of $\mathfrak{H}_{n-l}$. In general, we are interested in the *structure constants* of $\mathcal{H}(q)$ with respect to the basis (18). We discuss these in the following subsections.

2.1 Structure constants of $\mathcal{H}(q)$ as an associative algebra

In this subsection, we recall the reordering formula and structure constants of $\mathcal{H}(q)$ as an associative algebra, which is studied in [6, 11, 16, 17]. We present here a simplified discussion of [11, Section 2.1]. Define $\{0\}_q := 0$, and for each $n \in \mathbb{N}\backslash\{0\}$, we recursively define $\{n\}_q := 1 + q\{n-1\}_q$. That is, $\{n\}_q = 1 + q + \cdots + q^{n-1}$. If $q \neq 1$, then $\{n\}_q = \frac{1-q^n}{1-q}$. The *Gaussian binomial coefficients* or *q-binomial coefficients* are recursively defined by

$$\binom{n}{0}_q = 1, \tag{21}$$

$$\binom{0}{k+1}_q = 0, \tag{22}$$

$$\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q, \tag{23}$$

for any $n, k \in \mathbb{N}$. These q-binomial coefficients satisy the symmetry property $\binom{n}{k}_q = \binom{n}{n-k}_q$ for any $k \in \{0, 1, \ldots, n\}$, and as a consequence also the properties

$$\binom{n}{0}_q = \binom{n}{n}_q = 1, \tag{24}$$

$$\binom{n}{1}_q = \binom{n}{n-1}_q = \{n\}_q. \tag{25}$$

For each $l \in \mathbb{N}\backslash\{0\}$ and each $i \in \{0, 1, \ldots, l\}$ we define

$$c_i(l) := (q-1)^{-l}(-1)^{l-i}q^{\binom{i+1}{2}}\binom{l}{i}_q. \tag{26}$$

Let $k, m \in \mathbb{N}$, and let $l, n \in \mathbb{N}\backslash\{0\}$. As derived in [11, Section 2.1], the structure constants of the basis (18), given $l, n \in \mathbb{N}\backslash\{0\}$, may be determined from the relations

$$
\begin{align}
C^m A^n \cdot C^k A^l &= q^{kn} C^{m+k} A^{n+l}, \tag{27}\\
C^m B^n \cdot C^k B^l &= q^{-kn} C^{m+k} B^{n+l}, \tag{28}\\
C^m A^n \cdot C^k B^l &= \sum_{i=0}^{\min\{l,n\}} q^{(i+k)n - il} c_i(l) \cdot C^{m+i+k} A^{n-l}, && (n \geq l), \tag{29}\\
&= \sum_{i=0}^{\min\{l,n\}} q^{kn} c_i(n) \cdot C^{m+i+k} B^{l-n}, && (l > n), \tag{30}\\
C^m B^n \cdot C^k A^l &= \sum_{i=0}^{\min\{l,n\}} q^{-(i+k)n} c_i(l) \cdot B^{n-l} C^{m+k+i}, && (n \geq l), \tag{31}\\
&= \sum_{i=0}^{\min\{l,n\}} q^{-(i+k)n} c_i(n) \cdot C^{i+m+k} A^{l-n}, && (l > n). \tag{32}
\end{align}
$$

There is some simplification in the scalar coefficients regarding $c_i(l)$, since in [11] some other scalar coefficients were defined but are actually dependent on $c_i(l)$. We summarize in Table 1 below how the relations (27) to (32) give the structure constants of $\mathcal{H}(q)$ with respect to the basis (18).

$\cdot$	$C^k A^l$	$C^k B^l$
$C^m A^n$	(27)	(29), (30)
$C^m B^n$	(31), (32)	(28)

Table 1: Relations that can be used to obtain the structure constants of the associative algebra $\mathcal{H}(q)$ with respect to the basis (18)

2.2   Structure constants of $\mathcal{H}(q)$ as a Lie algebra

We extend the information from Section 2.1 by determining the structure constants of $\mathcal{H}(q)$ as a Lie algebra, still with respect to the basis (18). Let $m, r \in \mathbb{N}$, and let $n, s \in \mathbb{N}\backslash\{0\}$. Using the computational techniques in the previous subsection, it is routine to show that the relations

$$[C^m A^n, C^r B^s] = \sum_{i=0}^{\min\{r,s\}} q^{rs} \left(1 - q^{E(i)}\right) c_i(n) C^{m+i+r}, \qquad (n = s), \quad (33)$$

$$= \sum_{i=0}^{\min\{r,s\}} q^{nr+(s-n)(n-i)} \left(1 - q^{F(i)}\right) c_i(n) C^{m+i+r} B^{s-n}, \quad (n < s), \quad (34)$$

$$= \sum_{i=0}^{\min\{r,s\}} q^{nr+i(n-s)} \left(1 - q^{G(i)}\right) c_i(s) C^{m+i+r} A^{n-s}, \qquad (n > s), \quad (35)$$

where

$$\begin{aligned}
E(i) &:= n(m + r - i), \\
F(i) &:= n(2m + r + n - s) - in - ms, \\
G(i) &:= s(m + 2r),
\end{aligned}$$

hold in $\mathcal{H}(q)$ where $i \in \{1, 2, \ldots, \min\{n, s\}\}$. Using (27), (28), we also have

$$\begin{aligned}
[C^m A^n, C^r A^s] &= q^{nr}(1 - q^{ms-nr}) C^{m+r} A^{n+s}, \qquad (36) \\
[C^m B^n, C^r B^s] &= q^{-nr}(1 - q^{nr-ms}) C^{m+r} B^{n+s}. \qquad (37)
\end{aligned}$$

Similar to that done in Table 1, we summarize the structure constants of $\mathcal{H}(q)$ as a Lie algebra with respect to the basis (18) in Table 2 below. Because of the skew-symmetry of the Lie bracket, the relations (33) to (37) give us all the structure constants of $\mathcal{H}(q)$ as a Lie algebra.

$[-,-]$	$C^r A^s$	$C^r B^s$
$C^m A^n$	(36)	(33), (34), (35)
$C^m B^n$	-	(37)

Table 2: Relations that can be used to obtain the structure constants of $\mathcal{H}(q)$ as a Lie algebra with respect to the basis (18)

3 An extended operator algebra for the q-oscillator

Let ω be a positive real number. Earlier, the parameter q was chosen to be any nonzero complex number that is not a root of unity. From this point onward, we further narrow down the choice of q by setting $q = e^{-2\omega}$. Following [29, equations (6.5)–(6.6)], we have the relations

$$\begin{aligned}
e^\omega a a^+ - e^{-\omega} a^+ a &= e^\omega, \qquad (38) \\
[a, a^+] &= e^{-2\omega N}, \qquad (39)
\end{aligned}$$

for the q-oscillator. Define K_0 as the operator $e^{\omega N}$, i.e., $K_0 |n\rangle := e^{\omega n} |n\rangle$ for any $n \in \mathbb{N}$. With reference to the discussion in Section 1.2 about (17), we are here setting $\kappa = \frac{\omega}{\ln q}$. Denote by $\mathcal{P}_q$ the algebra (of operators on ℓ_2) generated by K_0, a, a^+. As a consequence of (38),(39), it can be shown by routine calculations that the relations

$$aa^+ = \frac{1 - e^{-2\omega} \left[a, a^+\right]}{1 - e^{-2\omega}}, \tag{40}$$

$$a^+ a = \frac{1 - \left[a, a^+\right]}{1 - e^{-2\omega}}, \tag{41}$$

$$a \left[a, a^+\right] = e^{-2\omega} \left[a, a^+\right] a, \tag{42}$$

$$a^+ \left[a, a^+\right] = e^{2\omega} \left[a, a^+\right] a^+, \tag{43}$$

$$aK_0 = e^{\omega} K_0 a, \tag{44}$$

$$a^+ K_0 = e^{-\omega} K_0 a^+, \tag{45}$$

$$\left[a, a^+\right] K_0 = K_0 \left[a, a^+\right], \tag{46}$$

$$K_0^2 \left[a, a^+\right] = 1, \tag{47}$$

hold in $\mathcal{P}_q$. We are interested in the question of whether the relations (40)–(47) are enough to define a presentation for $\mathcal{P}_q$. In the interest of mathematical rigour, this is no trivial matter, and so in this section we prove that such a presentation can indeed be obtained for $\mathcal{P}_q$.

Definition 3.1. Given a nonzero real number ω, if $q = e^{-2\omega}$, then we define $\mathcal{P} = \mathcal{P}(q)$ as the unital associative algebra over $\mathbb{C}$ with generators K, A, B, C subject to the relations

$$AB = \frac{1 - qC}{1 - q}, \tag{48}$$

$$BA = \frac{1 - C}{1 - q}, \tag{49}$$

$$AC = qCA, \tag{50}$$

$$BC = q^{-1}CB, \tag{51}$$

$$AK = q^{-\frac{1}{2}}KA, \tag{52}$$

$$BK = q^{\frac{1}{2}}KB, \tag{53}$$

$$CK = KC, \tag{54}$$

$$K^2 C = 1. \tag{55}$$

By the defining relations (48) to (55) of the algebra $\mathcal{P}$, and using the Diamond Lemma for Ring Theory [3, Theorem 1.2], the elements

$$K^h C^k, \quad K^h C^k B^l, \quad K^h C^k A^l, \qquad (h, k \in \mathbb{N}, \ l \in \mathbb{N}\backslash\{0\}), \tag{56}$$

with the condition

$$k \in \mathbb{N}\backslash\{0\} \quad \Rightarrow \quad h \in \{0, 1\}, \tag{57}$$

form a basis for $\mathcal{P}$.

Comparing (40)-(47) with (48)-(55), if we consider the assignment

$$K \mapsto K_0, \qquad A \mapsto a, \qquad B \mapsto a^+, \qquad C \mapsto \left[a, a^+\right], \tag{58}$$

we find that the generators of $\mathcal{P}_q$ satisfy the defining relations of $\mathcal{P}$. Thus, we say that $\mathcal{P}_q$, a concrete algebra of operators on ℓ_2, is a representation of the abstractly defined algebra $\mathcal{P}$. More precisely, there exists a homorphism $\Psi_0 : \mathcal{P} \to \mathcal{P}_q$ of algebras that performs (58). Our goal, at this point, is to prove the faithfulness of the representation $\mathcal{P}_q$ of $\mathcal{P}$, or equivalently, the injectiveness of Ψ_0.

3.1   Deformed commutator maps and the faithfulness of the representation $\mathcal{P}_q$

Given an algebra $\mathcal{A}$ over $\mathbb{C}$, we recall, from [17], the *deformed commutator mappings*

$$\begin{aligned}
\theta_{\alpha,t} &: \gamma \mapsto \alpha\gamma - t\gamma\alpha, \\
\eta_{\beta,t} &: \gamma \mapsto \gamma\beta - t\beta\gamma,
\end{aligned}$$

given $\alpha, \beta \in \mathcal{A}$ and $t \in \mathbb{C}$. Consequently, any (two-sided) ideal of $\mathcal{A}$ is invariant under a deformed commutator mapping. We shall be concerned with such deformed commutator mappings on $\mathcal{P}$ of very specific types. Given $n \in \mathbb{Q}$, succeeding proofs and computations shall involve the deformed commutator mappings

$$\begin{aligned}
\theta_n &:= \theta_{A,q^n} : \gamma \mapsto A\gamma - q^n\gamma A, \\
\eta_n &:= \eta_{B,q^n} : \gamma \mapsto \gamma B - q^n B\gamma.
\end{aligned}$$

We collect all such maps in $\Phi := \{\theta_n, \ \eta_n \ : \ n \in \mathbb{Q}\}$. Some subsets of Φ shall be of importance in later proofs, such as $\Phi_{\mathbb{Z}} := \{\theta_n, \ \eta_n \in \Phi \ : \ n \in \mathbb{Z}\}$, and the set difference $\Phi \backslash \Phi_{\mathbb{Z}} = \{\theta_n, \ \eta_n \in \Phi \ : \ n \notin \mathbb{Z}\}$. Using the computational techniques in Section 2.1, the identities

$$\theta_n(C^k) = q^k(1 - q^{n-k})C^k A, \tag{59}$$
$$\theta_n(C^k A^l) = q^k(1 - q^{n-k})C^k A^{l+1}, \tag{60}$$
$$\theta_n(C^k B^l) = q^k\frac{1 - q^{n-k}}{1 - q}C^k B^{l-1} - q^{k+1}\frac{1 - q^{n-k-l}}{1 - q}C^{k+1}B^{l-1}, \tag{61}$$
$$\eta_n(C^k) = q^{-k}(1 - q^{n+k})C^k B, \tag{62}$$
$$\eta_n(C^k B^l) = q^{-k}(1 - q^{n+k})C^k B^{l+1}, \tag{63}$$
$$\eta_n(C^k A^l) = q^{-k}\frac{1 - q^{n+k}}{1 - q}C^k A^{l-1} - q^{-k}\frac{1 - q^{n+k+l}}{1 - q}C^{k+1}A^{l-1}, \tag{64}$$

hold in $\mathcal{H}(q)$ for any $k \in \mathbb{N}$, and $n \in \mathbb{Q}$ and any $l \in \mathbb{N}\backslash\{0\}$. Setting $n = k$ in (59) and $n = -k$ in (62), we have

$$\theta_k(C^k) = 0 = \eta_{-k}(C^k), \qquad (k \in \mathbb{N}). \tag{65}$$

Setting $n = k + 1$, $l = 1$ in (61) and $n = -k$, $l = 1$ in (64), we have

$$\begin{align}
\theta_{k+1}(C^k B) &= q^k C^k, \tag{66}\\
\eta_{-k}(C^k A) &= -q^{-k} C^{k+1}. \tag{67}
\end{align}$$

In order to exhibit some more interesting properties of the deformed commutator map θ_n that shall be useful in later computations and proofs, we first recall the notion of a q-derivative. It is the mapping

$$D_q : f(x) \mapsto \frac{f(x) - f(qx)}{x - qx},$$

where, in this work, we take f to be any element of the polynomial algebra $\mathbb{C}[x]$ on the indeterminate x. Throughout, we use the notation

$$\{i\}_q! := \prod_{j=1}^{i} \{j\}_q.$$

Using [16, Lemma C.3] and induction, it is routine to show that

$$\left(D_q^i\right)(x^n) = \{i\}_q! \binom{n}{i}_q x^{n-i}, \tag{68}$$

for any $i, n \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$, we have

$$\theta_k(C^k B^l) = q^{k-l+1} C^{k+1} \cdot D_q(B^l). \tag{69}$$

We note here that (69) is an adjusted form of a related formula from [17, Lemma 5.7]. As explained in Remark 2.1, the adjustment is due to our use of basis elements of the form $C^k B^l$ as normal form instead of using the elements of the form $B^l C^k$ which were used in [17]. It is possible to apply the mapping θ_k on both sides of (69) and use the identity (68) to simplify the resulting right-hand side. Continuing with this manner of computation in a way that the number $n \in \mathbb{N}$ of times the mapping θ_k is applied to $C^k B^l$ does not exceed l, and by some routine computations and induction, we have

$$\theta_k^n(C^k B^l) = q^{n(k-l+n)} \{n\}_q! \binom{l}{n}_q C^{k+n} B^{l-n}, \qquad (k, n \in \mathbb{N},\ l \in \mathbb{N}\backslash\{0\},\ l \leq n.) \tag{70}$$

The identity (70) suggests some computational significance of deformed commutator mappings in the sense that applying a deformed commutator mapping repeatedly reduces a basis element of $\mathcal{H}(q)$ into a power of C. In an interesting flow of computations and proofs, we show that this can be generalized—any nonzero element of $\mathcal{H}(q)$, after an application of a suitable number of particular types of deformed commutator mappings, reduces to a power of C.

We start with a few properties in the following proposition and develop our main algorithms in the three lemmas that follow. These are actually our reformulation of [17, Lemma 5.7-6.4]. In fact, we give a reformulation, an exposition, and a generalization all at the same time. Our techniques as presented here involve more explicit reference to computations and structure constants, as contrasted by the ring-theoretic approach in [17].

Lemma 3.2. *Let $b \in \mathbb{N}\backslash\{0\}$. For each nonzero $U \in \bigoplus_{i=1}^{b} \mathfrak{H}_i$, there exists a composition θ of elements of $\Phi_{\mathbb{Z}}$, and some nonzero $V \in \mathfrak{H}_0 = \mathbb{C}[C]$ such that $\theta(U) = V$.*

Proof. We use induction on b. Suppose $b = 1$. Let U be a nonzero element of $\mathfrak{H}_1$. Then there exists a nonzero $\beta \in \mathbb{C}[C]$ such that $U = \beta B$. Let k be the polynomial degree of β. Choose any positive integer $n > k + 1$. If c_k is the leading coefficient of the polynomial β, then by (61), $\theta_n(\beta B) = \beta'$ for some $\beta' \in \mathbb{C}[C]$ with polynomial degree $k + 1$, and with leading coefficient $-q^{k+1}\{n - k - 1\}_q c_k = -q^{k+1}\frac{1-q^{n-(k+1)}}{1-q}c_k$, which is nonzero by Assumption 1.8 and because $n > k + 1$.

Suppose that for all positive integers $c < b$, any nonzero element of $\bigoplus_{i=1}^{c} \mathfrak{H}_i$ satisfies the statement. Let U be a nonzero element of $\bigoplus_{i=1}^{b} \mathfrak{H}_i$. Then there exists $U' \in \bigoplus_{i=1}^{b-1} \mathfrak{H}_i$ and some $\beta \in \mathbb{C}[C]$ such that $U = U' + \beta B^b$. If $\beta = 0$, then we are done, and so we further assume $\beta \neq 0$. Let k be the polynomial degree of β. By computations similar to those described for the case $b = 1$, we have $\theta_n(U) = \theta_n(U') + \beta' B^{b-1}$ for some $\beta' \in \mathbb{C}[C]$ with polynomial degree $k + 1$, and with leading coefficient $-q^{k+1}\{n - k - b\}_q c_k \neq 0$. Whenever $b \in \mathbb{N}\backslash\{0, 1\}$, the element $\theta_n(U')$ is in $\bigoplus_{i=1}^{b-2} \mathfrak{H}_i$ by (61), and so no term in $\theta_n(U')$ can cancel the term $-q^{k+1}\{n - k - b\}_q c_k C^{k+1} B^{b-1}$ in $\beta' B^{b-1}$. Thus, $0 \neq \theta_n(U) \in \bigoplus_{i=1}^{b-1} \mathfrak{H}_i$, and the element $\theta_n(U)$ satisfies the inductive hypothesis. That is, there exists a composition ϕ of elements of $\Phi_{\mathbb{Z}}$, and some nonzero $V \in \mathfrak{H}_0 = \mathbb{C}[C]$ such that $\phi(\theta_n(U)) = V$. Take $\theta = \phi \circ \theta_n$. This completes the induction. $\qquad\square$

Lemma 3.3. *For any nonzero $U \in \mathbb{C}[C] = \mathfrak{H}_0$, there exists a nonzero scalar a, some positive integer N and a composition ϕ of elements of $\Phi_{\mathbb{Z}}$ such that $\phi(U) = aC^N$.*

Proof. For each nonzero $U \in \mathbb{C}[C]$, there exist numbers $k, n \in \mathbb{N}$ such that for some scalars $c_k, c_{k+1}, \ldots, c_{k+n}$, with c_k and c_{k+n} nonzero, we have

$$U = c_k C^k + c_{k+1} C^{k+1} + \cdots + c_{k+n} C^{k+n}. \tag{71}$$

That is, n is the difference between the highest exponent of C and the lowest exponent of C that appear with nonzero scalar coefficient in U. We use induction on n. First, we consider the case $n = 0$. If $k = 0$, then take $\phi = \eta_0 \circ \theta_1$, $a = q(q-1)c_0$ and $N = 1$, while if $k \neq 0$, then we take ϕ as the empty composition or the identity map, a as c_k and N as k, and we are done. Suppose that for some n, the statement holds for all nonzero elements of $\mathbb{C}[C]$ in the linear combination of which the difference m between the highest exponent of C and the lowest exponent is such that $m < n$. By routine computations that involve (61), (65), (66), there exist nonzero scalars e_i with $i \in \{k+1, \ldots, k+n\}$ such that

$$
\begin{aligned}
(\theta_{k+n+1} \circ \eta_{-k})(U) &= c_{k+1}e_{k+1}C^{k+1} + \cdots + c_{k+n-1}e_{k+n-1}C^{k+n-1} \\
&\quad + (c_{k+n-1}e_{k+n} + c_{k+n}(1 - q^n))C^{k+n}. \tag{72}
\end{aligned}
$$

The number of nonzero terms in the right-hand side of (72) is at most $k+n-(k+1) = n-1$. Thus, the inductive hypothesis applies to $V := (\theta_{k+n+1} \circ \eta_{-k})(U)$, and so there exists

a nonzero scalar a, some positive integer N and some composition ϕ_0 of elements of $\Phi_{\mathbb{Z}}$ such that $\phi_0(V) = aC^N$, and so

$$\left(\phi_0 \circ \theta_{k+n+1} \circ \eta_{-k}\right)(U) = \phi_0(V) = aC^N.$$

Take $\phi = \phi_0 \circ \theta_{k+n+1} \circ \eta_{-k}$, and this completes the induction. $\qquad\square$

Our Lemmas 3.2 and 3.3 now culminate into the following result, which is our generalization of the rule (70), and also our reformulation and generalization of [17, Theorem 6.4].

Corollary 3.4. *For any nonzero $U \in \mathcal{H}(q)$, there exists a composition ψ of elements of $\Phi_{\mathbb{Z}}$, some nonzero scalar a and some positive integer N such that $\psi(U) = aC^N$.*

Proof. Any nonzero $U \in \mathcal{H}(q)$ is a finite linear combination of the elements described in (18), and so there exist $a, b \in \mathbb{N}$ such that $U \in \bigoplus_{i=-a}^{b} \mathfrak{H}_i$. By (62)–(64), there exists a composition η of elements of $\Phi_{\mathbb{Z}}$ such that $\eta(U)$ satisfies the hypotheses of Lemma 3.2, by which there exists a composition θ of elements of $\Phi_{\mathbb{Z}}$ such that $(\theta \circ \eta)(U)$ satisfies the hypotheses of Lemma 3.3. Consequently, by Lemma 3.3, there exists a composition ϕ of elements of $\Phi_{\mathbb{Z}}$ such that if $\psi := \phi \circ \theta \circ \eta$, and thus there exists a nonzero scalar a and some positive integer N such that $\psi(U) = aC^N$. $\qquad\square$

As the reader may notice, our proof of Corollary 3.4 and our proof of the supporting lemmas constitute a logical flow that differs from the formulation in [17, Lemma 5.7-6.4]. First, we deal with the commutator mappings and basis elements, and eventually arbitrary nonzero elements, of $\mathcal{H}(q)$ without reference to any two-sided ideal. The proofs do not rely heavily on ring-theoretic notions and arguments such as divisibility in a ring. Our proofs are based on the structure constants and computational techniques as described in Section 2.1, which is based on the deeper discussion in [11]. Our only dilemma so far is that, since we are extending this reduction process into the bigger algebra $\mathcal{P}$ in which $\mathcal{H}(q)$ is embedded, we have to take into account the effect of K in the computations involving deformed commutator mappings. Thus, we have the following.

Proposition 3.5. *For any $n \in \mathbb{Q}$ and any $U \in \mathcal{H}(q)$, we have*

$$\theta_n(KU) = q^{-\frac{1}{2}} K \theta_{n+\frac{1}{2}}(U), \tag{73}$$

$$\eta_n(KU) = q^{\frac{1}{2}} K \eta_{n-\frac{1}{2}}(U). \tag{74}$$

Proof. By the linearity of deformed commutator mappings, letting U be a basis element of $\mathcal{H}(q)$ among those listed in (18) will suffice. If $U = C^k$ for some $k \in \mathbb{N}$, we have that $\theta_n(K^k) = AKC^k - q^n KC^k A$, which by (52), becomes

$$\theta_n(K^k) = q^{-\frac{1}{2}} K (AC^k - q^{n+\frac{1}{2}} C^k A) = q^{-\frac{1}{2}} K \theta_{n+\frac{1}{2}}(C^k).$$

The other cases for (73), which are $U = C^k A^l$ and the case $U = C^k B^l$ for some $k, l \in \mathbb{N}$ with $l \in \mathbb{N}\backslash\{0\}$, are proven similarly. The proof for the identity (74) involves similar routine computations. $\qquad\square$

Proposition 3.6. *If ψ is (finite) composition of elements of $\Phi\backslash\Phi_{\mathbb{Z}}$, then for any nonzero $U \in \mathcal{H}(q)$,*

$$0 \neq \psi(U) \in \mathcal{H}(q).$$

Proof. We use induction on the number t of deformed commutator mappings in the composition ψ. Suppose $t = 1$, that is, for some non-integer $m \in \mathbb{Q}$, either $\psi = \eta_m$ or $\psi = \theta_m$. Consider the case $\psi = \eta_m$. Let U be any nonzero element of $\mathcal{H}(q)$, and let $a, b \in \mathbb{Z}$ such that $U = V + W$ where $V \in \bigoplus_{i=a}^{b-1} \mathfrak{H}_i$ and $W \in \mathfrak{H}_b$. Without loss of generality, we assume $W \neq 0$. That is, b is the maximum index of a $\mathbb{Z}$-gradation subspace to which some terms in U belong. Consequently, there is a nonzero polynomial $\gamma \in \mathbb{C}[C]$ such that $W = \gamma B^b$ if $b \in \mathbb{N}$, or $W = \gamma A^{-b}$ if $-b \in \mathbb{N}\backslash\{0\}$. In either case, according to (63), (64), $\eta_m(W) \in \mathfrak{H}_{b+1}$.

Let k be the polynomial degree of γ and let c_k be the leading coefficient. Using the identities (63), (64), we find that there exists a polynomial $\gamma' \in \mathbb{C}[C]$ of degree either k or $k+1$, with leading coefficient $d = \frac{q^k}{(1-q)^i}(1 - q^{m+T})c_k$ (where k and T are integers, and i is either 0 or 1) such that $\eta_m(W) = \gamma'B^{b+1}$ if $b \in \mathbb{N}$, or $\eta_m(W) = \gamma'A^{-b+1}$ if $-b \in \mathbb{N}\backslash\{0\}$. Since $m + T$ is not an integer, $m + T$ is not zero, and by Assumption 1.8, $1 - q^{m+T}$ is nonzero, and so is the leading coefficient d of γ'. Then $\eta_m(W) \neq 0$. In the right-hand side of the equation $\eta_m(U) = \eta_m(V) + \eta_m(W)$, none of the terms in $\eta_m(V)$ can cancel the nonzero $\eta_m(W)$ because, by (62) and (63), $\eta_m(V) \in \bigoplus_{i=a+1}^{b} \mathfrak{H}_i$, while $\eta_m(W) \in \mathfrak{H}_{b+1}$. Therefore, $\eta_m(U) \neq 0$. The case $\psi = \theta_m$ is proven similarly. This completes the proof for $t = 1$.

Suppose the statement holds for any composition of $n < t$ elements of $\Phi\backslash\Phi_{\mathbb{Z}}$. We write $\psi = \zeta \circ \phi$ for some $\zeta \in \Phi\backslash\Phi_{\mathbb{Z}}$ and some composition ϕ of $t - 1$ elements from $\Phi\backslash\Phi_{\mathbb{Z}}$. By the inductive hypothesis, for any nonzero $U \in \mathcal{H}(q)$, the element $V := \phi(U)$ is nonzero. Then by the inductive hypothesis, $0 \neq \zeta(V) = \psi(U)$. Therefore, the statement holds for any value of t. □

Theorem 3.7. *The algebra $\mathcal{P}_q$ faithfully represents $\mathcal{P}$.*

Proof. Showing that $\mathcal{P}$ is simple will suffice. To do this, we prove that, given a (two-sided) ideal $\mathcal{T}$ of $\mathcal{P}$, the following are equivalent.

(i) $\mathcal{T} \neq 0$.

(ii) $C^M \in \mathcal{T}$ for some positive integer M.

(iii) $K \in \mathcal{T}$.

(iv) $\mathcal{T} = \mathcal{P}$.

The equivalence between *(iii)* and *(iv)* follows from the invertibility of K. To see why the implication *(ii)* $\Rightarrow$ *(iii)* is true, simply consider the fact that $K = K^{1+2h}C^h$ since $K^2C = 1$ and K commutes with C. Since K is a basis element of $\mathcal{P}$, it is nonzero, and so we immediately have *(iii)* $\Rightarrow$ *(i)*. We now show *(i)* $\Rightarrow$ *(ii)*. Suppose $\mathcal{T}$ is a nonzero

ideal of $\mathcal{P}$, and choose a nonzero $x \in \mathcal{T}$. If we write x as a linear combination of the basis elements (56) of $\mathcal{P}$, then each basis element is of the form $K^{2h+i}U$ for some $h \in \mathbb{N}$, some $i \in \{0, 1\}$ and some basis element U of $\mathcal{H}(q)$ from (18). Let N be the maximum possible h among the basis elements $K^{2h+i}U$ that appear in the linear combination. Since $K^2C = 1$ and $CK = KC$, by multiplying C^N to the left of x, every term on the right-hand side becomes $C^N K^{2h+i}U = C^k K^i U = K^i C^k U$, for some $k \in \mathbb{N}$. Observe that $C^k U$ is also another basis element of $\mathcal{H}(q)$ from (18). Since i is either 0 or 1, then

$$C^N x = f + Kg, \tag{75}$$

for some $f, g \in \mathcal{H}(q)$. If $f = 0$, then we multiply both sides of (75) by CK, and we obtain $KC^{N+1}x = g$. Here, g cannot be zero for this will contradict $x \neq 0$. Thus, we use Corollary 3.4 on g, and we are done. Suppose $f \neq 0$. If $g = 0$, then we can again make use of Corollary 3.4, and we are done for this case. Thus, we now assume that both f and g are nonzero. Using Corollary 3.4 on f, let ψ be the composition of elements of $\Phi_{\mathbb{Z}}$ such that, for some nonzero scalar c_1 and positive integer H, we have $\psi(f) = c_1 C^H$. Thus, $\psi(C^N x) = c_1 C^H + \psi(Kg)$, and by (65),

$$(\theta_H \circ \psi)(C^N x) = (\theta_H \circ \psi)(Kg). \tag{76}$$

Note here that $H \in \mathbb{Z}$ and so we can write $(\theta_H \circ \psi)$ as

$$(\theta_H \circ \psi) = \zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_r,$$

for some $r \in \mathbb{N}\backslash\{0\}$ and some $\zeta_1, \zeta_2, \ldots, \zeta_r \in \Phi_{\mathbb{Z}}$. Consequently,

$$(\theta_H \circ \psi)(Kg) = (\zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_r)(Kg). \tag{77}$$

Using the identities in Proposition 3.5 repeatedly on the right-hand side of (77), there exists $N \in \mathbb{Q}$ and some $\xi_1, \xi_2, \ldots, \xi_r \in \Phi\backslash\Phi_{\mathbb{Z}}$ such that

$$(\theta_H \circ \psi)(Kg) = q^N K (\xi_1 \circ \xi_2 \circ \cdots \circ \xi_r)(g). \tag{78}$$

Since $g \neq 0$, by Proposition 3.6, $g' := (\xi_1 \circ \xi_2 \circ \cdots \circ \xi_r)(g)$ is a nonzero element of $\mathcal{H}(q)$. Multiplying both sides of (78) by $q^{-N}KC$, we have $q^{-N}KC(\theta_H \circ \psi)(Kg) = g'$, and so (76) becomes

$$q^{-N}KC(\theta_H \circ \psi)(C^N x) = g'. \tag{79}$$

We apply Corollary 3.4 to the nonzero element g' of $\mathcal{H}(q)$, and so there exists a composition φ of elements of $\Phi_{\mathbb{Z}}$, some nonzero scalar c_2, and some positive integer M such that (79) becomes

$$c_2^{-1}\varphi(q^{-N}KC(\theta_H \circ \psi)(C^N x)) = C^M. \tag{80}$$

Since x is an element of the ideal $\mathcal{T}$, which is invariant under deformed commutator mappings, the left-hand side of (80) is an element of $\mathcal{T}$. Therefore, $C^M \in \mathcal{T}$. $\square$

By Proposition 1.3, we immediately have:

Corollary 3.8. *The subalgebra of $\mathcal{P}$ generated by A, B is isomorphic to $\mathcal{H}(q)$.*

Hence, we identify a as A and a^+ as B. All the properties of $\mathcal{H}(q)$ are then satisfied in the subalgebra of $\mathcal{P}$ generated by A, B.

Remark 3.9. We rewrite in here the relations (13) to (15) in terms of A, B, and $C = [A, B]$:

$$[A, B]^{k+2} = \frac{-q^k(1-q)}{1-q^{k+1}} \sum_{i=0}^{k} \frac{\left((\text{ad } B) \circ (-\text{ad } [A, B])^k \circ (\text{ad } A) \right)(C)}{(q-1)^{1+i}}, \tag{81}$$

$$[A, B]^{k+1} A^l = -\frac{\left((-\text{ad } [A, B])^k \circ (-\text{ad } A)^{l+1} \right)(B)}{(1-q)^l(q^l-1)^k}, \tag{82}$$

$$[A, B]^{k+1} B^l = q^{l(k+1)} \frac{\left((\text{ad } B)^{l-1} \circ (\text{ad } [A, B])^{k+1} \right)(B)}{(q-1)^{k+1}(1-q^{k+1})^{l-1}}, \tag{83}$$

where $k \in \mathbb{N}$ and $l \in \mathbb{N}\backslash\{0\}$. The relation (83) is (15) with the adjustment from $B^l C^k$ to $C^k B^l$ as discussed in Section 2. All such relations (81) to (83) hold in $\mathcal{P}$. Additionally, because of the embedding of $\mathcal{H}(q)$ in $\mathcal{P}$ as given in Corollary 3.8, by Remark 1.5, the elements

$$C^k, \qquad\qquad (k \in \mathbb{N}\backslash\{0\}), \tag{84}$$

$$A, \quad C^k A^l, \qquad (k, l \in \mathbb{N}\backslash\{0\}), \tag{85}$$

$$B, \quad C^k B^l, \qquad (k, l \in \mathbb{N}\backslash\{0\}), \tag{86}$$

form a basis for the Lie subalgebra of $\mathcal{P}$ generated by A, B (or by A, B, C), while any (finite) linear combination or

$$1, \quad A^l, \quad B^l, \qquad (l \in \mathbb{N}\backslash\{0, 1\}), \tag{87}$$

is not a Lie polynomial in in A, B (nor in A, B, C).

4   Commutator algebra for the generators K, A, B, C of $\mathcal{P}$

In this section we find a description for the commutator algebra or Lie subalgebra of $\mathcal{P}$ generated by K, A, B, C. In the next two lemmas, we exhibit some important properties of the basis elements (56).

Lemma 4.1. *Every basis element in (56), except the multiplicative identity 1, is a Lie polynomial in K, A, B, C.*

Proof. For convenience in later computations, we use the notation

$$p := q^{\frac{1}{2}}. \tag{88}$$

Thus, the defining relations of $\mathcal{P}$ become

$$AB = \frac{1 - p^2 C}{1 - p^2}, \tag{89}$$

$$BA = \frac{1 - C}{1 - p^2}, \tag{90}$$

$$AC = p^2 CA, \tag{91}$$

$$BC = p^{-2} CB, \tag{92}$$

$$AK = p^{-1} KA, \tag{93}$$

$$BK = pKB, \tag{94}$$

$$CK = KC, \tag{95}$$

$$K^2 C = 1. \tag{96}$$

By some routine computations and induction, the relations (91) to (94) can be generalized into:

$$A^l K^h = p^{-hl} K^h A^l, \tag{97}$$

$$B^l K^h = p^{hl} K^h B^l, \tag{98}$$

$$A^l C^k = p^{2kl} C^k A^l, \tag{99}$$

$$B^l C^k = p^{-2kl} C^k B^l, \tag{100}$$

which hold for any $h, k, l \in \mathbb{N}$. Our strategy is to segregate the basis elements $U \neq 1$ among (56) into the following:

$$C^k A^l, \quad C^k B^l, \quad C^k, \qquad (k, l \in \mathbb{N}\backslash\{0\}), \tag{101}$$

$$KC^k A^l, \quad KC^k B^l, \qquad (k, l \in \mathbb{N}\backslash\{0\}), \tag{102}$$

$$K^h A^l, \quad K^h B^l, \quad K^l, \quad KC^l, \qquad (h \in \mathbb{N},\ l \in \mathbb{N}\backslash\{0\}). \tag{103}$$

By (13) to (15), the basis elements (101) are all elements of $\mathcal{L}(\mathcal{P})$. Using the reordering formulas (97) and (98), and the commutativity of K with C in some routine computations, we have the relations

$$KC^k A^l = \frac{p^l}{1 - p^l} \left[C^k A^l, K \right], \tag{104}$$

$$KC^k B^l = \frac{1}{1 - p^l} \left[K, C^k B^l \right], \tag{105}$$

which hold for any $k, l \in \mathbb{N}\backslash\{0\}$. But since $C^k A^l,\ K,\ C^k B^l \in \mathcal{L}(\mathcal{P})$ for any k, l, the relations (104) and (105) imply that the basis elements (102) are all in $\mathcal{L}(\mathcal{P})$. We now

sketch the computations necessary to prove that basis elements (103) are in $\mathcal{L}(\mathcal{P})$. First, we consider those of the form $K^h A^l$, $K^h B^l$ for the case $h \in \mathbb{N}\backslash\{0\}$. Using induction and the reordering formulas (97) and (98), the relations

$$K^h A^l = \frac{p^{hl}}{(1-p)^l(1-p^l)^{h-1}} \left((-\text{ad } K)^{h-1} \circ (\text{ad } A)^l \right)(K), \quad (h, l \in \mathbb{N}\backslash\{0\}), \quad (106)$$

$$K^h B^l = \frac{1}{(1-p)^l(1-p^l)^{h-1}} \left((\text{ad } K)^{h-1} \circ (-\text{ad } A)^l \right)(K), \quad (h, l \in \mathbb{N}\backslash\{0\}), \quad (107)$$

hold in $\mathcal{P}$. In view of Remark 1.6, the relations (106), (107) imply that $K^h A^l$, $K^h B^l \in \mathcal{L}(\mathcal{P})$ whenever $h, l \in \mathbb{N}\backslash\{0\}$.

We are missing the case $h = 0$ in the relations (106) and (107), and so we turn our attention to some other types of Lie polynomials in K, A, B, C on which we can use the reordering formulas (97) to (100) and also the defining relations of $\mathcal{P}$ in order to show that $A^l, B^l \in \mathcal{L}(\mathcal{P})$. Using the reordering formulas (97) and (98), we have

$$p^{l+1} \left[KB, KA^{l+1} \right] = p^{l+2} K^2 (BA) A^l - K^2 A^l (AB). \quad (108)$$

Using (89), (90), we replace BA and AB in the right-hand side of (108). This gives us

$$p^{l+1}(1-p^2) \left[KB, KA^{l+1} \right] = K^2 (p^{l+2}(1-C)A^l - A^l(1-p^2 C)). \quad (109)$$

By further manipulations that also make use of the relation (96), we turn (109) into

$$A^l = \frac{1-p^2}{p(1-p^l)} \left[KA^{l+1}, KB \right] - \frac{1-p^{l+2}}{p^{l+2}(1-p^l)} K^2 A^l, \qquad (l \in \mathbb{N}\backslash\{0\}). \quad (110)$$

We have established previously that KA^{l+1}, KB, $K^2 A^l \in \mathcal{L}(\mathcal{P})$, and so (110) asserts that $A^l \in \mathcal{L}(\mathcal{P})$. By a computational pattern similar to that done in (108) to (110), the relations

$$B^l = \frac{1-p^2}{1-p^l} p^{l-1} \left[KA, KB^{l+1} \right] - \frac{1-p^{l+2}}{1-p^l} p^{l-2} K^2 B^l, \qquad (l \in \mathbb{N}\backslash\{0\}), \quad (111)$$

$$K^{h+2} = \frac{1-p^2}{1-p^{h+2}} \left[K^{h+2} A, B \right] + \frac{1-p^h}{1-p^{h+2}} p^2 K^h, \qquad (h \in \mathbb{N}), \quad (112)$$

$$KC^l = \frac{1-p^2}{1-p^{2l}} p^{2l-2} \left[KC^{l-1} A, B \right] + \frac{1-p^{2l-2}}{1-p^{2l}} KC^{l-1}, \qquad (l \in \mathbb{N}\backslash\{0\}), \quad (113)$$

can be shown to hold in $\mathcal{P}$ by routine calculations. As established earlier, KA, KB^{l+1}, $K^2 B^l \in \mathcal{L}(\mathcal{P})$. Then by (111), $B^l \in \mathcal{L}(\mathcal{P})$.

So far, the relations (106),(107),(110),(111) assert that all basis elements in (103) of the form $K^h A^l$, $K^h B^l$ (with $h \in \mathbb{N}$, $l \in \mathbb{N}\backslash\{0\}$) are in $\mathcal{L}(\mathcal{P})$. For those of the form K^l, we use (112). If $h = 0$ in (112), then we find a relation asserting that K^2 is a scalar multiple of $[K^2 A, B]$, where $K^2 A \in \mathcal{L}(\mathcal{P})$ as previously established. Thus, $K^2 \in \mathcal{L}(\mathcal{P})$. We can use the condition $K, K^2 \in \mathcal{L}(\mathcal{P})$ and (112) in an inductive argument which proves that $K^l \in \mathcal{L}(\mathcal{P})$ for all $l \in \mathbb{N}\backslash\{0\}$. By a similar argument, (113) can be used to show that $KC^l \in \mathcal{L}(\mathcal{P})$ for any $l \in \mathbb{N}\backslash\{0\}$. This completes all the cases needed in the proof. □

Lemma 4.2. *If U, V are any two basis elements from (56) neither of which is the multiplicative identity 1, and if $[U, V] = \sum_{i=1}^{t} c_t W_t$ for some nonzero scalars c_t and some basis elements W_t from (56), then $W_t \neq 1$ for any t.*

Proof. Since $[U, V] = 0$ if one of U, V is 1, we further assume that U and V are of the form $K^h C^k B^l$ or $K^h C^k A^l$ for some $h, k \in \mathbb{N}$ and some $l \in \mathbb{N} \backslash \{0\}$. If $U, V \in \mathcal{H}(q)$ (i.e., if $h = 0$), then by Remark 3.9, we find that it is not possible for W_t to be 1 for any t. Thus, we further assume the exponent of K in at least one of U, V is positive.

It is routine to show, using (97), (98), that under all such remaining cases, either $[U, V] = K^N [U', V']$ or $[U, V] = K^N W'$ for some positive integer N, where U', V', W' are among (84) to (86). If $[U, V] = K^N [U', V']$, then by Remark 3.9, $[U', V']$ is also one of (84) to (86), but K^N multiplied by any one of (84) to (86) is not 1. Neither is $K^N W'$ for the last remaining case. $\qquad \square$

Theorem 4.3. *The Lie subalgebra $\mathcal{L}(\mathcal{P})$ of $\mathcal{P}$ generated by K, A, B, C is precisely the span of all the basis elements (56) except the multiplicative identity 1. i.e., The algebra $\mathcal{P}$ can be decomposed as the direct sum*

$$\mathcal{P} = \mathbb{C}1 \oplus \mathcal{L}(\mathcal{P}). \tag{114}$$

Proof. Let $\mathcal{V}$ be the span of all basis elements (56) except 1. Thus, $\mathcal{V}$ contains the generators K, A, B, C of $\mathcal{L}(\mathcal{P})$. Lemma 4.2 asserts that $\mathcal{V}$ is a Lie subalgebra of $\mathcal{P}$, while Lemma 4.1 implies $\mathcal{V} \subseteq \mathcal{L}(\mathcal{P})$. By the minimality of $\mathcal{L}(\mathcal{P})$ among all Lie subalgebras of $\mathcal{P}$ that contain K, A, B, C, we have $\mathcal{V} = \mathcal{L}(\mathcal{P})$. $\qquad \square$

5   On a q-oscillator representation of the Askey-Wilson algebra

We now discuss how our algebraic results are related to a Lie algebra structure in some representation of an Askey-Wilson algebra. In particular, we shall consider the following.

Definition 5.1. The Askey-Wilson algebra, denoted by $AW(3)$, is the unital associative algebra over $\mathbb{C}$ with generators K_0, K_1, K_2 and relations

$$
\begin{align}
e^{\omega} K_0 K_1 - e^{-\omega} K_1 K_0 &= K_2, \tag{115} \\
e^{\omega} K_2 K_0 - e^{-\omega} K_0 K_2 &= b K_0 + c_1 K_1 + d_1, \tag{116} \\
e^{\omega} K_1 K_2 - e^{-\omega} K_2 K_1 &= b K_1 + c_0 K_0 + d_0, \tag{117}
\end{align}
$$

where $\omega \in \mathbb{R} \backslash \{0\}$. The scalars $b, c_0, c_1, d_0, d_1 \in \mathbb{C}$ are called the *structure constants* of $AW(3)$.

According to [29, pp. 1155–1156], if the conditions

$$b = c_1 = d_0 = d_1 = 0, \qquad c_0 = e^{2\omega} - 1, \tag{118}$$

are imposed on the structure constants of $AW(3)$, then the algebra $AW(3)$ has a q-oscillator representation given by

$$K_0 = e^{\omega N}, \tag{119}$$
$$K_1 = a + a^+. \tag{120}$$

In this representation, K_1 is said to be the ω-analogue of the position operator.

Definition 5.2. Define $AW_q(3)$ as the subalgebra of $\mathcal{P}_q$ generated by the representations of K_0 and K_1 given in (119),(120), with $q = e^{-2\omega}$.

Thus, the choice here of q is nonzero and not a root of unity, and the results of the previous sections apply. Recall our notation $p = q^{\frac{1}{2}}$. By $q = e^{-2\omega}$ and (118), the relations (115)-(117) can be rewritten as

$$p^{-1}K_0K_1 - pK_1K_0 = K_2, \tag{121}$$
$$p^{-1}K_2K_0 - pK_0K_2 = 0,$$
$$p^{-1}K_1K_2 - pK_2K_1 = \frac{1-p^2}{p^2}K_0,$$

By (121), the algebra $AW_q(3)$ contains the representation of $K_2 \in AW(3)$ as an operator on the q-oscillator. By Theorem 3.7, we can represent elements of $AW_q(3)$ in terms of the generators K, A, B, C of $\mathcal{P}$. Thus, (119), (120) become

$$K_0 = K,$$
$$K_1 = A + B.$$

As for the generator K_2, we use (121) and the reordering formulas (52), (53). By some routine calculations, we have

$$K_2 = p^{-1}(1-p)KA + p^{-1}(1-p^3)KB. \tag{122}$$

Our next goal is to determine the relationship between the algebras $\mathcal{P}$ and $AW_q(3)$, and as shall be seen in the computations and proofs that we will present, the Lie polynomials in the generators of these algebras play an important role.

Lemma 5.3. *The Lie subalgebra $\mathcal{L}(AW_q(3))$ of $AW_q(3)$ generated by K_0, K_1, K_2 is a Lie subalgebra of $\mathcal{L}(\mathcal{P})$.*

Proof. Using (93), (94), and by some routine calculations, we obtain

$$KA = \frac{p[A, K]}{1-p}, \tag{123}$$
$$KB = \frac{[K, B]}{1-p}. \tag{124}$$

From (122)-(124), we have enough information to express K_2 as a Lie polynomial in K, A, B by routine computations. To summarize all relevant equations, we have

$$K_0 = K, \tag{125}$$

$$K_1 = A + B, \tag{126}$$

$$K_2 = \left[A - p^{-1}(1 + p + p^2)B, K\right]. \tag{127}$$

An immediate consequence of (125)-(127) is that every Lie polynomial in K_0, K_1, K_2 is also a Lie polynomial in K, A, B, which are enough to generate $\mathcal{L}(\mathcal{P})$ since $C = [A, B]$.   □

The most natural continuation is to consider the converse: is it possible to write any Lie polynomial in the generators of $\mathcal{P}$ as a Lie polynomial in the generators of $AW_q(3)$? We now show that we have an answer in the affirmative.

Lemma 5.4. *The generators K, A, B of the Lie algebra $\mathcal{L}(\mathcal{P})$ are Lie polynomials in K_0, K_1, K_2.*

Proof. Since $K_0 = K$, we only need to show A, B are Lie polynomials in K_0, K_1, K_2. We start with evaluating $[K_0, K_1]$ using (125),(126) and the reordering formulas (52),(53). The result is a linear combination of KA and KB, and so we can use (122) to solve for KA and KB. From these routine computations, we have

$$KA = \frac{p^2 K_2 + p(1 + p + p^2)\left[K_1, K_0\right]}{(1 - p)(1 + p)^2}, \tag{128}$$

$$KB = \frac{pK_2 + p\left[K_0, K_1\right]}{(1 - p)(1 + p)^2}. \tag{129}$$

The right-hand sides of (128),(129) are Lie polynomials in K_0, K_1, K_2, and so we have $KA, KB \in \mathcal{L}(AW_q(3))$. Our next step is to compute the commutator of these new Lie polynomials KA, KB with $K_1 = A + B$. Using routine computations that make use of the reordering formulas (89),(90),(93),(94), we obtain

$$KA^2 - \frac{p^2}{1 - p^2}KC = \frac{p}{1 - p}[K_1, KA] + \frac{p}{1 - p^2}K_0, \tag{130}$$

$$KB^2 - \frac{1}{1 - p^2}KC = \frac{[KB, K_1]}{1 - p} + \frac{K_0}{p(1 - p^2)}. \tag{131}$$

We also compute the commutator of $K_1 = A + B$ with K_2 as given in (122). Moreover, using (89),(90),(93),(94), we obtain, by routine calculations, the relation

$$KA^2 - p(1 + p + p^2)KB^2 + p\frac{1 + p^2}{1 - p^2}KC = \frac{p^2}{(1 - p)^2}[K_1, K_2] - \frac{1 + p^2}{1 - p^2}K_0. \tag{132}$$

Because KA, KB are Lie polynomials in K_0, K_1, K_2, then so are the right-hand sides of (130)-(132). Denote these elements of $\mathcal{L}(AW_q(3))$ by X, Y, Z, respectively. We can

eliminate KA^2 and KB^2 from (132) using (130),(131). By routine computations, we have
the relation

$$KC = \frac{1+p}{p}X - (1+p)(1+p+p^2)Y - \frac{1+p}{p}Z \quad \in \quad \mathcal{L}(AW_q(3)).$$

We have thus shown that KA, KB, KC are Lie polynomials in K_0, K_1, K_2. The significance of this is that by some routine computations that involve the reordering formulas (89)-(96), we have

$$A = \frac{[KC, KA]}{1-p} \tag{133}$$

$$B = \frac{p\,[KB, KC]}{1-p}. \tag{134}$$

Therefore, $A, B \in AW_q(3)$ as desired. $\qquad\square$

Theorem 5.5. *The algebras $AW_q(3)$ and $\mathcal{P}$ are equal, and so are the corresponding Lie algebras $\mathcal{L}(AW_q(3))$ and $\mathcal{L}(\mathcal{P})$.*

Proof. By Lemma 5.4, the generators K, A, B and even $C = [A, B]$ of $\mathcal{L}(\mathcal{P})$ are in $\mathcal{L}(AW_q(3))$, and so we have the inclusion $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{L}(AW_q(3))$. Using Lemma 5.3, we further have $\mathcal{L}(\mathcal{P}) = \mathcal{L}(AW_q(3))$. By Lemma 4.1 each basis element $U \neq 1$ of $\mathcal{P}$ from (56) is a Lie polynomial in K, A, B, C, and so according to Lemma 5.4, $U \in \mathcal{L}(AW_q(3)) \subseteq AW_q(3)$. The remaining basis element $1 = K^2C$ is also in $AW_q(3)$ since $K, C \in \mathcal{L}(AW_q(3)) \subseteq AW_q(3)$. Therefore, $\mathcal{P}$ is contained in its subalgebra $AW_q(3)$, and is hence equal to $AW_q(3)$. $\qquad\square$

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