

Transitions between root subsets associated with Carter diagrams

Rafael Stekolshchik

Abstract. For any two root subsets associated with two Carter diagrams that have the same ADE type and the same size, we construct the transition matrix that maps one subset to the other. The transition between these two subsets is carried out in some canonical way affecting exactly one root, so that this root is mapped to the minimal element in some root subsystem. The constructed transitions are involutions. It is shown that all root subsets associated with the given Carter diagram are conjugate under the action of the Weyl group. A numerical relationship is observed between enhanced Dynkin diagrams $\Delta(E_6)$, $\Delta(E_7)$ and $\Delta(E_8)$ (introduced by Dynkin-Minchenko) and Carter diagrams. This relationship echoes the $2 - 4 - 8$ assertions obtained by Ringel, Rosenfeld and Baez in completely different contexts regarding the Dynkin diagrams E_6 , E_7 , E_8 .

In memory of Semyon E. Konstein

1 Introduction

1.1 Diagrams with cycles

In 1972, R. Carter introduced the so-called admissible diagrams representing semi-Coxeter elements of conjugacy classes in the Weil group. The admissible diagrams also represent root subsets¹ of the root systems associated with the Weyl group. These root subsets sometimes form strange cycles, strange because the extended Dynkin diagram \tilde{A}_l cannot be part of any admissible diagram. The explanation for this fact was that in the case of extended Dynkin diagrams, the inner products of roots of cycle \tilde{A}_l are negative, while in the case of admissible diagrams, there are necessarily both positive and negative inner products, see [18, Lemma A.1]. Thus was born the concept of the Carter diagram, see [18]. They differ from admissible diagrams in that they take into account the sign of

MSC 2020: 17B22, 20F55, 16G70

Keywords: Root system, Carter diagram, enhanced Dynkin diagram

Contact information:

Email: r.stekol@gmail.com.

the inner product on the pair of roots; for this, the language of *solid and dotted* edges is used, see Section 2.2.

Carter diagrams distinguish between negative inner products (solid edges) and positive inner products (dotted edges). The number of solid edges must necessarily be odd, as well as the number of dotted edges. This agrees well with the fact that, by definition, the length of an admissible diagram is even. In the theorem on exclusion of long cycles [18, Theorem 3.1] it was shown that any Carter diagram with cycles of arbitrary even length can be reduced to another Carter diagram containing only 4-cycles. In proving this theorem, for each particular case, it is checked that a semi-Coxeter element associated with a Carter diagram with long cycles is conjugate to a semi-Coxeter element associated with another Carter diagram containing only cycles of length 4.

1.2 Homogeneous classes of Carter diagrams

The Dynkin diagram A_l , where $l \geq 1$ (resp. D_l , where $l \geq 4$; resp. E_l , where $l = 6, 7, 8$) is said to be the *Dynkin diagram of A-type* (resp. *D-type*, resp. *E-type*). The Carter diagram A_l , where $l \geq 1$ (resp. $D_l, D_l(a_k)$, where $l \geq 4, 1 \leq k \leq \lfloor \frac{l-2}{2} \rfloor$; resp. $E_l, E_l(a_k)$, where $l = 6, 7, 8, k = 1, 2, 3, 4$) is said to be the *Carter diagram of A-type* (resp. *D-type*, resp. *E-type*).

The Carter diagrams of the same type and the same index constitute a *homogeneous class* of Carter diagrams. Denote by $C(\Gamma)$ the homogeneous class containing the Carter diagram Γ , see (1) and Fig. 1.

$$\begin{aligned} C(E_6) &= \{E_6, E_6(a_1), E_6(a_2)\}, \\ C(E_7) &= \{E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4)\}, \\ C(E_8) &= \{E_8, E_8(a_1), E_8(a_2), E_8(a_3), E_7(a_4), E_8(a_5), E_8(a_6), E_8(a_7), E_7(a_8)\}, \\ C(D_l) &= \left\{ D_l, D_l(a_1), D_l(a_2), \dots, D_l \left(a_{\lfloor \frac{l-2}{2} \rfloor} \right) \right\}, \text{ where } l \geq 4. \end{aligned} \quad (1)$$

A root subset associated with some diagram $\tilde{\Gamma}$ is denoted by a $\tilde{\Gamma}$ -set. Let \tilde{S} (resp. S) be a $\tilde{\Gamma}$ -set (resp. Γ -set). In Tables 2-5 of Section 4.6, the *transition matrix* $M_I : \tilde{S} \mapsto S$ is constructed for the following homogeneous pairs $\{\tilde{\Gamma}, \Gamma\}$:

$$\begin{array}{ll} (1) & \{D_4(a_1), D_4\} \\ (2) & \{D_l(a_k), D_l\} \\ (3) & \{E_6(a_1), E_6\} \\ (4) & \{E_6(a_2), E_6(a_1)\} \\ (5) & \{E_7(a_1), E_7\} \\ (6) & \{E_7(a_2), E_7\} \\ (7) & \{E_7(a_3), E_7(a_1)\} \\ (8) & \{E_7(a_4), E_7(a_3)\} \\ (9) & \{E_8(a_1), E_8\} \\ (10) & \{E_8(a_2), E_8\} \\ (11) & \{E_8(a_3), E_8(a_2)\} \\ (12) & \{E_8(a_4), E_8(a_1)\} \\ (13) & \{E_8(a_5), E_8(a_4)\} \\ (14) & \{E_8(a_6), E_8(a_4)\} \\ (15) & \{E_8(a_7), E_8(a_5)\} \\ (16) & \{E_8(a_8), E_8(a_7)\} \end{array} \quad (2)$$

The diagrams obtained as images of the mapping M_I are considered up to equivalence of Carter diagrams, see [18, Section 1.3]. The list (2) is called the *adjacency list*. The

¹In the following, the phrase “root subset” always means the root subset of linearly independent roots.

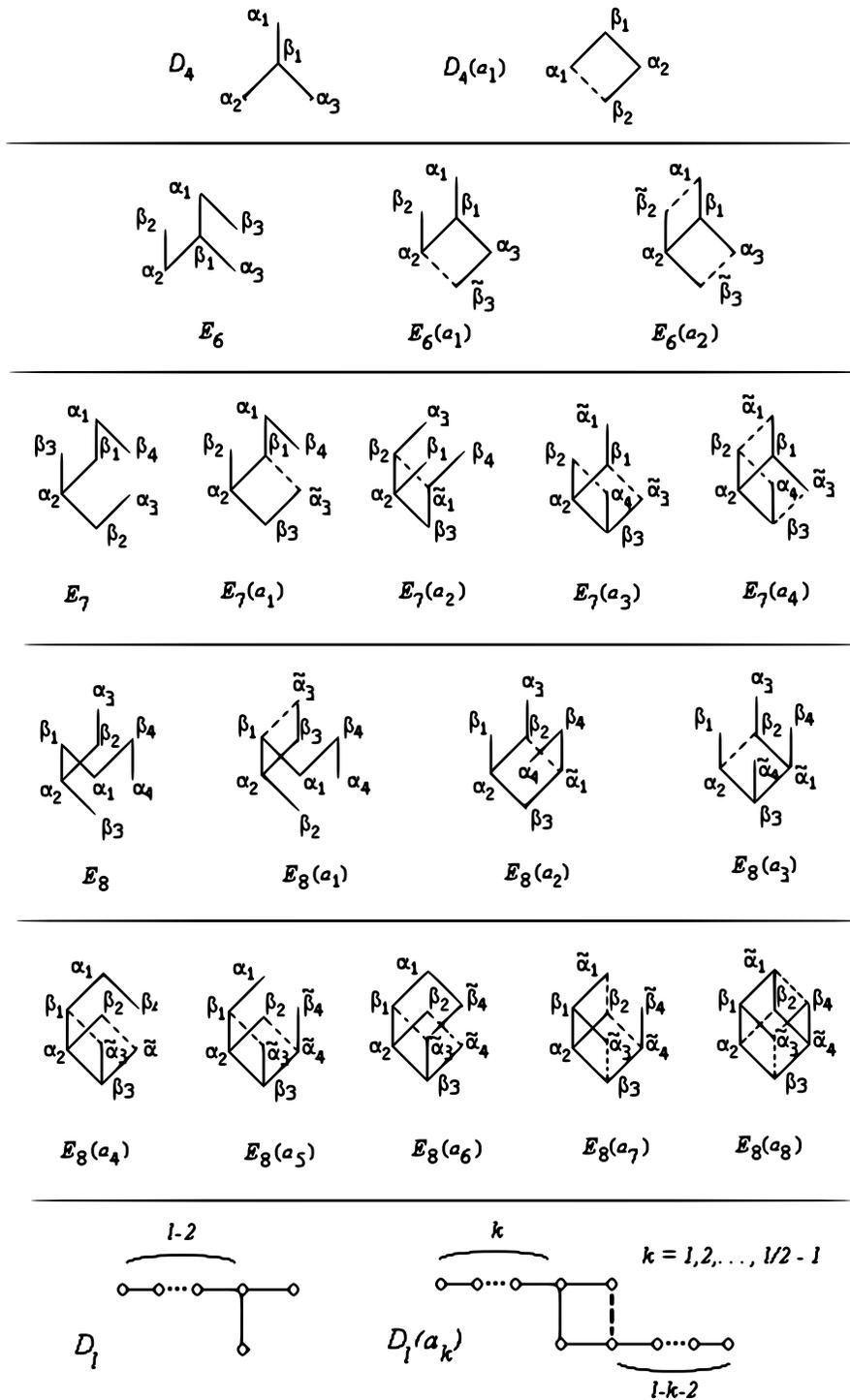


Figure 1: Carter diagrams of D and E types

adjacency list is not a complete list, but a minimal list that chains all Carter diagrams of to the same homogeneous class using the transition matrices constructed in Theorem 4.1.

1.3 A group of transitions \mathcal{M}

Let $C(\Gamma)$ be a homogeneous class of Carter diagrams out of (1), and \widehat{S} be the set of all root subsets associated with diagrams of $C(\Gamma)$. Let \mathcal{M} be some subgroup of the group generated by transitions of type M_I operating on \widehat{S} . The Weyl group W operates simply transitively on the set of bases of the corresponding root system, all bases are associated with a single Dynkin diagram (which is also the Carter diagram). This is not true for the group \mathcal{M} : any element $M_t \in \mathcal{M}$ (of type M_I) can map some root subset $S_1 \in \widehat{S}$ to another root subset $S_2 \in \widehat{S}$. The root subsets S_1 and S_2 are associated, generally speaking, with different Carter diagrams $\Gamma_1 \in C(\Gamma)$ and $\Gamma_2 \in C(\Gamma)$. We will say that S_1 (resp. S_2) is a Γ_1 -set (resp. Γ_2 -set).

1.4 Theorems on transition matrix

In this paper, we show that each homogeneous class of Carter diagrams essentially depicts the same subset of the root system given in different bases. There are several chains of diagrams containing homogeneous classes of Carter diagrams. The transition between neighboring bases in any chain can be performed in some *canonical way, affecting exactly one root*, which is mapped to the minimum element in some root subsystem. The transition matrix connecting two adjacent diagrams is an involution.

The main result of this paper can be formulated as follows: Let $\{\widetilde{\Gamma}, \Gamma\}$ be a pair out of the adjacency list (2), and let \widetilde{S} (resp. S) be a $\widetilde{\Gamma}$ -set (resp. Γ -set). We construct the matrix M_I having the following properties (Theorem 4.1, Theorem 4.3):

- (a) The matrix M_I is the transition matrix transforming the roots of \widetilde{S} to the roots of S .
- (b) The matrix M_I transforms only one element in $\widetilde{\alpha} \in \widetilde{S}$, the remaining elements of \widetilde{S} are left fixed. The element $\widetilde{\alpha}$ is transformed into the minimal element of some Dynkin subset $S(\widetilde{\alpha}) \subset \widetilde{S}$,
- (c) The matrix M_I acts as involution on \widetilde{S} :

$$M_I \widetilde{\alpha} = \alpha = -\widetilde{\alpha} + \sum_{\tau_i \in \widetilde{S}} t_i \tau_i, \quad M_I \tau_i = \tau_i \text{ for } \tau_i \neq \widetilde{\alpha}, \quad (3)$$

and M_I acts also as involution on S :

$$M_I \alpha = \widetilde{\alpha} = -\alpha + \sum_{\tau_i \in S} t_i \tau_i, \quad M_I \tau_i = \tau_i \text{ for } \tau_i \neq \alpha, \quad (4)$$

The values of t_i in (3) and (4) are given in Tables 2-5 of Section 4.6.

- (d) In most cases of Section 4.6, the mapping M_I given in (3) eliminates one circle (or one endpoint), the mapping M_I given in (4) builds one circle (or one endpoint). In case (16) of Section 4.6, M_I eliminates 3 cycles.

Only *ADE* root systems are considered.

Table 1: Cardinality of extra nodes

D_{2m}	D_{2m+1}	E_6	E_7	E_8
$m - 1$	$m - 1$	2	4	8

1.5 Relationship with other diagrams containing cycles

1.5.1 Dynkin-Minchenko diagrams: Procedure of completion

Let Γ be a Dynkin diagram of the complex semisimple Lie algebra \mathfrak{g} . Subdiagrams of Γ are Dynkin diagrams of regular subalgebras of \mathfrak{g} . However, not all regular subalgebras can be obtained in this way. Besides, non-conjugate regular subalgebras can have identical Dynkin diagrams. Both problems are efficiently solved by using enhanced Dynkin diagrams. In [6], Dynkin and Minchenko introduced a canonical enlargement of the basis called the *enhanced basis* and *enhanced Dynkin diagrams* representing an enhanced basis. They constructed an enhancement of Γ by a recursive procedure which they call the *completion*: At each step of the procedure, find a D_4 -subset in the already introduced nodes, add the maximal² root of this subset, and connect it by edges to the corresponding part of the already introduced nodes. For the enhanced Dynkin diagrams $\Delta(E_6)$ and $\Delta(E_7)$, see Section A.

In [19],[20] Vavilov and Migrin combined both types of considered diagrams: *Carter diagrams* and *enhanced Dynkin diagrams*, they applied the language of solid and dotted edges to enhanced Dynkin diagrams. The obtained diagrams are called *signed enhanced Dynkin diagrams*. They showed that any Carter diagram of the homogeneous class containing the Dynkin diagrams E_6, E_7, E_8 can be embedded into the signed enhanced Dynkin diagram $\Delta(\Gamma)$ associated with Γ such that the “solid and dotted” correspondence is preserved.

1.5.2 Values 2,4,8 and diagrams E_6, E_7, E_8

To the Vavilov-Margin observation mentioned above, I would like to add the following easily verifiable fact:

Remark 1.1. The number of extra nodes obtained by the Dynkin-Minchenko completion procedure for a simply-laced Dynkin diagram coincides with the number of Carter diagrams (with cycles) of the same type, see Fig. 1, Table 1.

For a further discussion of the relationship between enhanced Dynkin diagrams and Carter diagrams, see Section C.1.

In [13], in the context of Auslander-Reiten quivers, Ringel observed a completely different relation between values 2,4,8 and diagrams E_6, E_7, E_8 , see Section C.3.

In [1], Baez (in relation to Rosenfeld’s idea in [14]) points to another connection between values 2,4,8 and diagrams E_6, E_7, E_8 , see Section C.4.

²Adding a minimal root (instead of a maximal one) leads to a topologically isomorphic enhanced Dynkin diagram, but distinguished by solid and dotted edges.

1.5.3 McKee-Smyth diagrams: Eigenvalues in $(-2, 2)$

Much to my surprise, I found a complete list of 8-vertex Carter diagrams with circles in the paper of McKee and Smyth [11, Figs. 12-14]. The $\{0, 1\}$ -matrices with zeros on the diagonal can be regarded as adjacency matrices of graphs. Assume that the off-diagonal elements of such a matrix to be chosen from the set $\{-1, 0, 1\}$. Then, we get so-called a *signed* graph, a non-zero (α, β) th entry denotes a sign of -1 or 1 on the edge connecting vertices α and β . The signed graphs exactly correspond to our diagrams with solid and dotted edges. The matrix with zeros on the diagonal is called an *uncharged matrix*. By [11, Theorem 4], the signed graphs maximal with respect to having all their eigenvalues in $(-2, 2)$ are exactly Carter diagrams $E_8(a_i), 1 \leq i \leq 8$ and $D_l(a_i), i < l/2 - 1$, see Fig. 1. If the diagonal matrix $2I$ is added to such an uncharged matrix, then exactly *partial Cartan matrix* will be obtained, see Section 3.2. Then, the eigenvalues of the partial Cartan matrices should lie in the interval $(0, 4)$. Using eigenvalues one can get an invariant description of Carter diagrams, see [16, Section 4.4]. For some details on the relationship between Carter diagrams and eigenvalues of partial Cartan matrices, see Section C.2.

Similar results to [11] were also obtained by Mulas and Stanic in [12].

2 Diagrams containing cycles

2.1 Admissible diagrams: Conjugacy classes of W

Let Φ be the root system corresponding to the Weyl group W . Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \text{ where } \alpha_i \in \Phi \text{ for all } i. \quad (5)$$

Carter proved that k in the decomposition (5) is the smallest if and only if the subset of roots $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is linearly independent; such a decomposition is said to be *reduced*. The admissible diagram corresponding to the given element w is not unique, since the reduced decomposition of the element w is not unique.

Denote by $l_C(w)$ the smallest value k corresponding to any reduced decomposition (5). The corresponding set of roots $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ consists of linearly independent and not necessarily simple roots, see Lemma 2.1. If $l(w)$ is the smallest value k in any expression like (5) such that all roots α_i are *simple*, then $l_C(w) \leq l(w)$.

Lemma 2.1 ([4, Lemma 3]). *Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \Phi$. Then, $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ is reduced if and only if $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent.* \square

The Cartan matrix (resp. quadratic form) associated with Φ is denoted by \mathbf{B} (resp. \mathcal{B}). The inner product induced by \mathcal{B} is denoted by (\cdot, \cdot) .

Let us take the subset of linearly independent, but not necessarily simple roots $S \subset \Phi$. To the subset S we associate some diagram Γ that provides one-to-one correspondence between roots of S and nodes of Γ . The diagram Γ is said to be *admissible* if the following two conditions hold:

- (a) The nodes of Γ correspond to a set of linearly independent roots in Φ .
- (b) If a subdiagram of Γ is a cycle, then it contains an even number of nodes.

Note that the admissible diagram may contain cycles, since the roots of S are non necessarily simple, see [17, Section 1.2.1]. Let us fix some basis of roots corresponding to the given admissible diagram Γ :

$$S = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_h\}.$$

The admissible diagram is bicolored, i.e., the set of nodes can be partitioned into two disjoint subsets $S_\alpha = \{\alpha_1, \dots, \alpha_k\}$ and $S_\beta = \{\beta_1, \dots, \beta_h\}$, where roots of S_α (resp. S_β) are mutually orthogonal. The element

$$c = w_\alpha w_\beta, \quad \text{where } w_\alpha = \prod_{i=1}^k s_{\alpha_i}, \quad w_\beta = \prod_{j=1}^h s_{\beta_j}$$

is called the *semi-Coxeter element*; it represents the conjugacy class associated with the admissible diagram Γ and root subset S (not necessarily root system).

2.2 Carter diagrams: Language of “solid and dotted” edges

In [18], it was observed that the cycles in the admissible diagrams with necessity contains at least one pair of roots $\{\alpha_1, \beta_1\}$ with $(\alpha_1, \beta_1) > 0$ and at least one pair of roots $\{\alpha_2, \beta_2\}$ with $(\alpha_2, \beta_2) < 0$, where (\cdot, \cdot) is the Tits bilinear form associated with the root system Φ . This observation motivated me to distinguish such pairs of roots: Let us draw the *dotted* (resp. *solid*) edge $\{\alpha, \beta\}$ if $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$). The admissible diagram with dotted and solid edges is said to be the *Carter diagram*. Up to dotted edges, the classification of Carter diagrams coincides with the classification of admissible diagrams.

In the theorem on exclusion of long cycles [18], it was shown that any Carter diagram with cycles of arbitrary even length can be reduced to diagrams with cycles of length 4 only. This explains why the admissible diagrams $D_l(b_{\frac{1}{2}l-1})$, $E_7(b_2)$, $E_8(b_3)$, $E_8(b_5)$ listed in [4, Table 2] do not appear in the lists of conjugacy classes. The Carter diagrams with conjugate semi-Coxeter elements are said to be *equivalent*. The Carter diagrams (with cycles) representing non-Coxeter conjugacy classes are given in Fig. 1. For another view of these diagrams, see [18, Table 1].

2.3 Carter diagrams: Eliminating the cycle.

The semi-Coxeter elements generated by reflections $\{s_{\alpha_1}, s_{\alpha_2}, s_{\beta_1}, s_{\beta_2}\}$ constitute exactly two conjugacy classes with representatives w_t and w_o , see Fig. 2. Semi-Coxeter elements w_t and w_o are distinguished by orders of reflections in the decomposition of w_t , and w_o . Here, t is the bicolored order $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, and o is the cyclic order $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, see [18, Section 1.2].

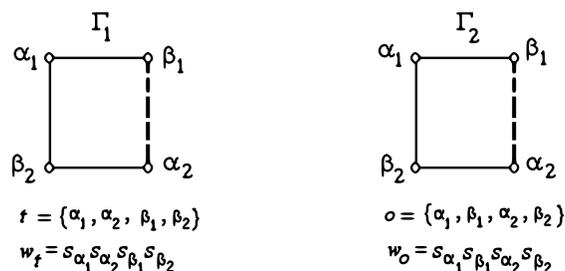


Figure 2: Diagram Γ_1 (resp. Γ_2) of type $D_4(a_1)$ (resp. equivalent to D_4)

The element w_o is conjugate to the $w_{\tilde{o}} = s_{\alpha_1} s_{\alpha_2} s_{-(\alpha_1 + \beta_1 + \beta_2)} s_{\beta_2}$ since

$$\begin{aligned} w_o &= s_{\alpha_1} s_{\beta_1} s_{\alpha_2} s_{\beta_2} = s_{\alpha_1 + \beta_1} s_{\alpha_1} s_{\alpha_2} s_{\beta_2} \stackrel{s_{\alpha_1 + \beta_1}}{\simeq} s_{\alpha_1} s_{\alpha_2} s_{\beta_2} s_{\alpha_1 + \beta_1} = \\ &= s_{\alpha_1} s_{\alpha_2} s_{\alpha_1 + \beta_1 + \beta_2} s_{\beta_2} = s_{\alpha_1} s_{\alpha_2} s_{-(\alpha_1 + \beta_1 + \beta_2)} s_{\beta_2} = w_{\tilde{o}}. \end{aligned}$$

The elements w_o and $w_{\tilde{o}}$ are conjugate, the corresponding sets of roots are as follows:

$$S_1 = \{\alpha_1, \beta_1, \alpha_2, \beta_2\} \text{ and } S_2 = \{\alpha_1, \alpha_2, -(\alpha_1 + \beta_1 + \beta_2), \beta_2\}.$$

There is a map $M : S_1 \rightarrow S_2$ acting as follows:

$$M\alpha_1 = \alpha_1, \quad M\alpha_2 = \alpha_2, \quad M\beta_2 = \beta_2, \quad \tilde{\beta}_1 = M\beta_1 = -(\alpha_1 + \beta_1 + \beta_2).$$

Note that M is an involution, $M : S_1 \rightarrow S_2$ and $M : S_2 \rightarrow S_1$, since

$$\begin{aligned} M^2\beta_1 &= -(\alpha_1 + M\beta_1 + \beta_2) = -(\alpha_1 + \beta_2) + (\alpha_1 + \beta_1 + \beta_2) = \beta_1, \\ M\beta_1 &= \tilde{\beta}_1 \quad \text{and} \quad M\tilde{\beta}_1 = \beta_1. \end{aligned}$$

Thus, M transforms the root β_1 into the minimal element of the root subsystem $\{\alpha_1, \beta_1, \beta_2\}$. In this paper, we will encounter a number of *involution mappings* M that map a certain element to the *minimal element* of some root subsystem of Φ . So, we observe that there are two different orders of reflections:

- The cyclic order of reflections o . Then, we get a 4-cycle leading to the Coxeter class D_4 of $W(D_4)$, see Fig. 3.
- The bicolored order of reflections t . Then, we get an *indestructible* 4-cycle leading to the semi-Coxeter class $D_4(a_1)$.

2.4 Connection diagrams

In [18], in addition to Carter diagrams, the so-called *connection diagrams* were introduced. Let S be a set of linearly independent and not necessarily simple roots, o be the order of reflections in the decomposition (5) of element w associated with the set of roots

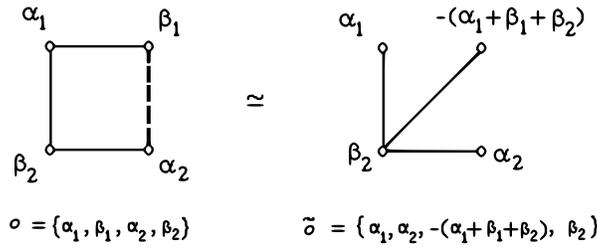


Figure 3: Conjugate elements $\{w_o, w_{\tilde{o}}\}$ corresponding to the Coxeter class D_4

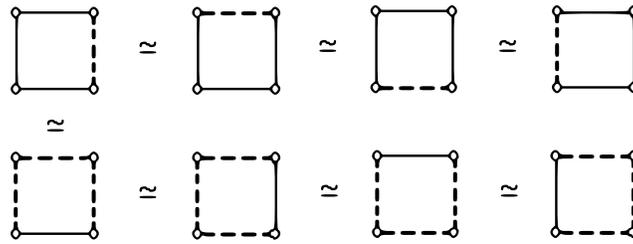


Figure 4: Eight similar 4-cycles equivalent to $D_4(a_1)$

S . The connection diagram is a pair (Γ, o) , where Γ corresponds to the set S . In the connection diagram Γ , edges are also solid and dotted as in Carter diagrams. The connection diagrams serve to transform one Carter diagram into another, since in the process of transformation we can get non-Carter diagrams – the evenness of the cycles may be violated, see [18, Section 1.2.2].

In [18], three equivalence transformations operating on the connection diagrams and Carter diagrams were introduced: similarities, conjugations and s-permutations. The Carter diagrams are studied there up to equivalence. In what follows, we only need *similarity*. Let α be a root in the Γ -set S . The similarity transformation L_α reflects the root α :

$$L_\alpha : \alpha \mapsto -\alpha. \tag{6}$$

Two connection diagrams obtained from each other by a sequence of reflections (6), are called *similar* connection diagrams, see Fig. 4.

2.5 Bicolored partition

Let Γ be a Carter diagram and

$$S = \{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_h\} \tag{7}$$

be any Γ -set (of not necessarily simple roots), where roots of the set

$$S_\alpha := \{\alpha_i \mid i = 1, \dots, k\} \quad (\text{resp. } S_\beta := \{\beta_j \mid j = 1, \dots, h\})$$

are mutually orthogonal. According to condition 2.1(a), there exists a certain set (7) of linearly independent roots, and thanks to condition 2.1(b), there exists a partition $S = S_\alpha \amalg S_\beta$ which is said to be the *bicolored partition*.

Let $w = w_1 w_2$ be the decomposition of w into the product of two involutions. By [4, Lemma 5] each of w_1 and w_2 can be expressed as a product of reflections as follows:

$$w = w_1 w_2, \quad \text{where} \quad w_1 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_h}, \quad (8)$$

where subset $S_\alpha = \{\alpha_1, \dots, \alpha_k\}$ (resp. $S_\beta = \{\beta_1, \dots, \beta_h\}$) consists of mutually orthogonal roots. Let

$$\Pi_w = \{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_h\} \quad (9)$$

be the linearly independent root subset. Then, the decomposition (8) is reduced, see Lemma 2.1, and $k + h = l_C(w)$. The decomposition (8) is said to be a *bicolored decomposition*.

3 The Cartan matrix

3.1 The generalized Cartan matrix

The $n \times n$ matrix $K = (k_{ij})$, where $1 \leq i, j \leq n$, such that

- (C1) $k_{ii} = 2$ for $i = 1, \dots, n$,
- (C2) $-k_{ij} \in \mathbb{Z} = \{0, 1, 2, \dots\}$ for $i \neq j$,
- (C3) $k_{ij} = 0$ implies $k_{ji} = 0$ for $i, j = 1, \dots, n$,

is called a *generalized Cartan matrix*, [10], [16, Section 2.1]. For the Carter diagram Γ , which is not a Dynkin diagram, the condition (C2) fails: The elements k_{ij} associated with dotted edges are positive.

If the Carter diagram does not contain any cycle, then the Carter diagram is the Dynkin diagram, the corresponding conjugacy class is the conjugacy class of the Coxeter element, and the partial Cartan matrix is the classical Cartan matrix, which is the particular case of a generalized Cartan matrix.

3.2 The partial Cartan matrix

Similarly to the Cartan matrix associated with Dynkin diagrams, we determine the Cartan matrix for each pair $\{\Gamma, S\}$ consisting of the connection or Carter diagram Γ and Γ -set S :

$$B_\Gamma := \begin{pmatrix} (\tau_1, \tau_1) & \dots & (\tau_1, \tau_n) \\ \dots & \dots & \dots \\ (\tau_n, \tau_1) & \dots & (\tau_n, \tau_n) \end{pmatrix}, \quad (10)$$

where $S = \{\tau_1, \dots, \tau_n\}$. We call the matrix B_Γ a *partial Cartan matrix* corresponding to the diagram Γ . The partial Cartan matrix B_Γ is well-defined since products (τ_i, τ_j) in (10)

do not depend on the choice of the Γ -set S . The elements of the partial Cartan matrix are uniquely determined by the diagram Γ as follows:

$$(\tau_i, \tau_j) = \begin{cases} 2, & \text{if } \tau_i = \tau_j, \\ 0, & \text{if } \tau_i \text{ and } \tau_j \text{ are not connected,} \\ -1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is solid,} \\ 1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is dotted.} \end{cases} \tag{11}$$

Let L be the subspace spanned by the vectors $\{\tau_1, \dots, \tau_n\}$. We write this fact as follows:

$$L = [\tau_1, \dots, \tau_n]. \tag{12}$$

The subspace L is said to be the S -associated subspace. Let \mathbf{B} be the Cartan matrix corresponding to the primary root system Φ .

Proposition 3.1. (i) *The restriction of the bilinear form associated with the Cartan matrix \mathbf{B} on the subspace L coincides with the bilinear form associated with the partial Cartan matrix B_Γ , i.e., for any pair of vectors $v, u \in L$, we have*

$$(v, u)_\Gamma = (v, u), \text{ and } \mathcal{B}_\Gamma(v) = \mathcal{B}(v). \tag{13}$$

(ii) *For every Carter diagram, the matrix B_Γ is positive definite.*

Proof. (i) From (10) we deduce:

$$(v, u)_\Gamma = \left(\sum_i t_i \tau_i, \sum_j q_j \tau_j \right)_\Gamma = \sum_{i,j} t_i q_j (\tau_i, \tau_j)_\Gamma = \sum_{i,j} t_i q_j (\tau_i, \tau_j) = (v, u).$$

(ii) This follows from (i). □

If Γ is a Dynkin diagram, the partial Cartan matrix B_Γ is the Cartan matrix associated with Γ . By (13) the matrix B_Γ is positive definite. The symmetric bilinear form associated with B_Γ is denoted by $(\cdot, \cdot)_\Gamma$ and the corresponding quadratic form is denoted by \mathcal{B}_Γ .

Remark 3.2. (i) D. Leites noticed that there are a number of other cases, where some off-diagonal elements of the Cartan matrices are positive. For example, this is so in the case of Lorentzian algebras, see [8], [5]. However, in these cases the Cartan matrices are of *hyperbolic type*, whereas the partial Cartan matrices are *positive definite*.

(ii) I would like to quote S. Brenner’s article: “...it is amusing to note that there is a surprisingly large intersection between the finite type quivers with commutativity conditions and the diagrams by Carter in describing conjugacy classes of the classical Weyl groups ...”, [3, p.43]. On various other cases arising in the representation theory of quivers, algebras and posets with Cartan matrices containing positive off-diagonal elements, see [2], [7, 10.7], [15].

4 Transitions

4.1 First transition theorem

Let $\{\tilde{\Gamma}, \Gamma\}$ be a homogeneous pair of Carter diagrams, \tilde{S} be a $\tilde{\Gamma}$ -set, and S be a Γ -set. In this section, we construct a mapping connecting \tilde{S} and S . This mapping represents the transition matrix connecting \tilde{S} and S as bases in the linear spaces. The transition matrix has some good properties that are presented in Theorems 4.1 and 4.3. Let Γ' be the subdiagram of Γ , and subset $S' \subset S$ be a Γ' -set. If Γ' is the Dynkin diagram, we call S' the Dynkin subset.

Theorem 4.1. *For each pair of diagrams $\{\tilde{\Gamma}, \Gamma\}$ out of the list (2), there exists the linear transformation matrix M_I mapping each $\tilde{\Gamma}$ -set \tilde{S} to some Γ -set S being the image of M_I , see Tables 2-5 of Section 4.6. The matrix M_I is the transition matrix binding \tilde{S} and S as bases in the linear spaces.*

- (i) *The matrix M_I acts only on one element $\tilde{\alpha} \in \tilde{S}$ and does not change remaining elements in \tilde{S} ; M_I transforms $\tilde{\alpha}$ into the minimal element α of some Dynkin subset $S(\tilde{\alpha})$ in \tilde{S} :*

$$\begin{cases} \tilde{\alpha} \in S(\tilde{\alpha}) \subset \tilde{S}, \\ M_I \tau_i = \tau_i \text{ for all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, \\ M_I \tilde{\alpha} = \alpha = -\tilde{\alpha} + \sum t_i \tau_i, \text{ where the sum is taken over } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, t_i \in \mathbb{Z}, \\ \alpha - \text{minimal element in } S(\tilde{\alpha}). \end{cases}$$

The image $S = M_I \tilde{S}$ is the set $\{\tilde{S} \setminus \tilde{\alpha}\} \sqcup \{\alpha\}$ that satisfies to the orthogonality relations of the Carter diagram Γ .

- (ii) *The transformation $M_I : \tilde{S} \mapsto S$ is an involution¹ on the set $\tilde{S} \sqcup \{\alpha\}$:*

$$M_I \tilde{\alpha} = \alpha \quad \text{and} \quad M_I \alpha = \tilde{\alpha}.$$

For each pair of diagrams $\{\tilde{\Gamma}, \Gamma\}$ from list (2), the matrix M_I is defined in Tables 2-5 of Section 4.6. The matrix M_I is the transition matrix transforming each basis \tilde{S} into some basis S .

The proof of Theorem 4.1 is given in Section 4.6. It is carried out separately for each pair $\{\tilde{\Gamma}, \Gamma\}$ in the adjacency list (2).

¹The index I in the symbol M_I originally appeared to indicate that M_I is an involution. In what follows, we will use also other notation for the matrix M and its index, which are more related to a specific situation.

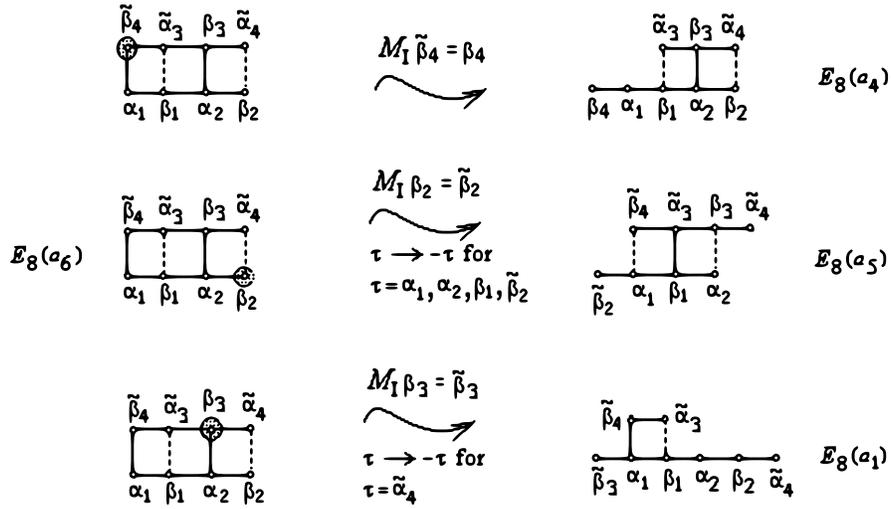


Figure 5: Alternative transitions $\{E_8(a_6), E_8(a_5)\}$ and $\{E_8(a_6), E_8(a_1)\}$

transition is followed by actions of similarities which result in the desired *edge type*¹. Note that the vertex names for $E_8(a_1)$ and $E_8(a_5)$ are different from those in Tables 4 – 5:

$$\{E_8(a_6), E_8(a_4)\} : M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3).$$

$$\{E_8(a_6), E_8(a_5)\} : M_I \beta_2 = \tilde{\beta}_2 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3),$$

followed by actions of L_τ , with $\tau = \alpha_1, \alpha_2, \beta_1, \tilde{\beta}_2$.

$$\{E_8(a_6), E_8(a_1)\} : M_I \beta_3 = \tilde{\beta}_3 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3).$$

followed by actions of L_τ , with $\tau = \tilde{\alpha}_4$.

4.2.3 The product of transition matrices

As in Section 4.2.1, for any chain (14), one can construct the matrices F and F^{-1} , where F is the product of corresponding transition matrices $M_I(n)$ and similarity matrices L_{τ_i} . The matrix F is invertible since all $M_I(n)$ and L_{τ_i} are invertible.

$$F : \tilde{\Gamma} \mapsto \Gamma, \quad F^{-1} : \Gamma \mapsto \tilde{\Gamma}. \tag{16}$$

This means that for any $\tilde{\Gamma}$ -set \tilde{S} there exists Γ -set S such that

$$F\tilde{S} = S, \quad F^{-1}S = \tilde{S}. \tag{17}$$

¹The property of the edge to be solid or dotted is called the edge type.

The matrix F does not depend on \tilde{S} and S . If $S = \{\tau_1, \dots, \tau_n\}$, F transforms τ_i so that

$$F\tau_i = \sum_{j=1}^n f_{ji}\tau_j, \quad (18)$$

where f_{ji} are some coefficients that depend only on diagrams $\tilde{\Gamma}$ and Γ .

4.3 Action of the Weyl group on a Carter diagram

We suppose that $\{\tilde{\Gamma}, \Gamma\}$ is the homogeneous pair of Carter diagrams, where Γ is the simply-laced Dynkin diagram, and W is the Weyl group associated with Γ .

Lemma 4.2. *Let F, F^{-1} be the matrices described in (16), (17). Then,*

(i) *The matrix F commutes with the Weyl group W on any Γ -set S as follows:*

$$wF = Fw \text{ for any } w \in W. \quad (19)$$

(ii) *Let Γ -sets S and S' be conjugate by some $w \in W$: $wS = S'$. Then, FS and FS' are conjugate by the same element $w \in W$, i.e.,*

$$wS = S' \quad \text{implies} \quad wFS = FS'.$$

Proof. (i) It suffices to prove eq. (19) for each element $\tau_i \in S$. Each element $w \in W$ transforms basis S to another basis $S' = wS$, where $w\tau_i = \tau'_i$, and $S' = \{\tau'_1, \dots, \tau'_n\}$. In our case,

$$Fw\tau_i = F\tau'_i = \sum_{j=1}^n f_{ji}\tau'_j \quad \text{and}$$

$$wF\tau_i = w \sum_{j=1}^n f_{ji}\tau_j = \sum_{j=1}^n f_{ji}w\tau_j = \sum_{j=1}^n f_{ji}\tau'_j.$$

Therefore, $Fw\tau_i = wF\tau_i$ for any $\tau_i \in S$.

(ii) If $wS = S'$ then by (i), we have $wFS = FwS = FS'$, i.e., FS and FS' conjugate by the same element $w \in W$. \square

4.4 Second transition theorem

Theorem 4.3. (i) *For any Carter diagram $\tilde{\Gamma}$, all $\tilde{\Gamma}$ -sets are conjugate under the Weyl group W .*

(ii) *Let $\{\tilde{\Gamma}, \Gamma\}$ be any homogeneous pair of Carter diagrams, where Γ is the Dynkin diagram, and let \tilde{S} be any $\tilde{\Gamma}$ -set, S be any Γ -set. Then, there exists F , the product of transition matrices $\tilde{M}_I(n)$ and some matrices of similarity maps like L_{τ_i} as in Theorem 4.1 and Section 4.2.1 and $w \in W$ such that $S = wF\tilde{S}$.*

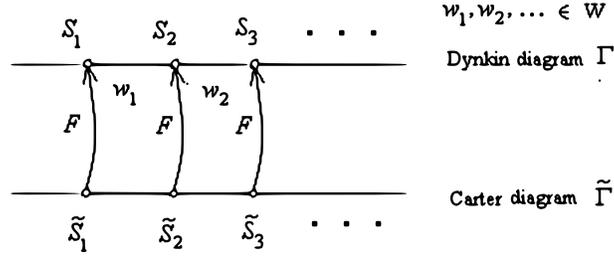


Figure 6: $F\tilde{S}_i = S_i$, where \tilde{S}_i (resp. S_i), are some $\tilde{\Gamma}$ -sets (resp. Γ -sets), $i = 1, 2, 3, \dots$

Proof. (i) The Carter diagram $\tilde{\Gamma}$ belongs to some homogeneous class $C(\tilde{\Gamma})$. Every homogeneous class contains a Dynkin diagram Γ . As in Section 4.2.1, there exists the mapping F from any $\tilde{\Gamma}$ -set \tilde{S} to some Γ -set S . By Theorem 4.1, the mapping F is the product of transition matrices $M_I(n)$ and some matrices of similarity maps like L_{τ_i} .

Let \tilde{S}' and \tilde{S}'' be any $\tilde{\Gamma}$ -sets. We will prove that \tilde{S}' and \tilde{S}'' are conjugate under the Weyl group W , i.e.,

$$w\tilde{S}' = \tilde{S}'', \quad \text{for some } w \in W. \tag{20}$$

There exist Γ -sets S' and S'' such that

$$F\tilde{S}' = S', \quad F\tilde{S}'' = S'', \tag{21}$$

see Fig. 6. Let Φ be the root system associated with Γ . All bases in Φ are conjugate, see [9, Theorem 1.4]. Then, there exists $w \in W$, such that $wS' = S''$. By (21),

$$wF\tilde{S}' = wS' = S'' = F\tilde{S}''. \tag{22}$$

By Lemma 4.2, transformations w and F in (22) commute, so

$$Fw\tilde{S}' = F\tilde{S}'', \quad \text{and} \quad w\tilde{S}' = \tilde{S}''. \tag{23}$$

(ii) First, by Theorem 4.1, we transform \tilde{S} to some Γ -set S' by the mapping F as in (i), see Fig. 6. Further, as in (i), there exists $w \in W$ such that $wS' = S$. Thus,

$$wF\tilde{S} = wS' = S. \quad \square$$

4.5 Conjugacy of all $\tilde{\Gamma}'$ -sets

As above, $\{\tilde{\Gamma}, \Gamma\}$ is a homogeneous pair of Carter diagrams, Γ is the simply-laced Dynkin diagram, W is the Weyl group associated with Γ .

Similarly to the fact that all Γ -bases (where Γ is the Dynkin diagram) are conjugate under the Weyl group [9, Theorem 1.4], the same fact holds for Carter diagrams.

Corollary 4.4. *All $\tilde{\Gamma}$ -sets are conjugate under the Weyl group W associated with the Dynkin diagram Γ .*

Proof. Let \tilde{S} and \tilde{S}' be two $\tilde{\Gamma}$ '-sets. By Theorem 4.3, there exist Γ -bases S and S' and $w \in W$ such that

$$F : \tilde{S} \mapsto S, \quad w : S \mapsto S', \quad F : \tilde{S}' \mapsto S'.$$

In other words,

$$\begin{aligned} Fw\tilde{S} &= wF\tilde{S} = S', & F\tilde{S}' &= S', & \text{i.e.,} \\ Fw\tilde{S} &= F\tilde{S}', & \text{and} & & w\tilde{S} &= \tilde{S}'. \end{aligned} \quad \square$$

Let $C(\Gamma)$, as above, be a homogeneous class of Carter diagrams out of (1), where Γ is the Dynkin diagram. Denote by $\nu(\Gamma')$ the number of all Γ' -sets.

Corollary 4.5. *The number $\nu(\Gamma')$ is the constant number for any homogeneous class. For any Carter diagram $\Gamma' \in C(\Gamma)$, the number $\nu(\Gamma')$ coincides with the number of bases in the root system associated with Γ , and coincides with the number of elements in the Weyl group:*

$$|\nu(\Gamma')| = |\nu(\Gamma)| = |W|. \quad (24)$$

4.6 Proof of Theorem 4.1

(i) Let us construct the matrix M_I for every pair $\{\tilde{\Gamma}, \Gamma\}$ in the adjacency list (2).

(1) Pair $\{D_4(a_1), D_4\}$: M_I maps $D_4(a_1)$ -set to D_4 -set.

- Mapping: $M_I\tilde{\alpha}_3 = \alpha_3 = -(\alpha_1 + \alpha_2 + \tilde{\alpha}_3)$.
- Root system: $S = \{\alpha_2, \alpha_1, \tilde{\alpha}_3\}$ and Φ is a root system of type A_3 .
- Minimal root: α_3 is the minimal root in Φ .
- Eliminated edges: $\{\tilde{\alpha}_3, \alpha_1\}$ and $\{\tilde{\alpha}_3, \alpha_4\}$.
- Emerging edge: $\{\alpha_3, \alpha_2\}$.
- Checking relations:

$$\begin{cases} \alpha_3 \perp \alpha_1 & (\alpha_3 \text{ is the minimal root}) \\ \alpha_3 \perp \alpha_4 & (\alpha_2 + \tilde{\alpha}_3 \perp \alpha_4) \\ (\alpha_3, \alpha_2) = -1 & (\alpha_3 \text{ is the minimal root}) \end{cases}$$

(2) Pair $\{D_l(a_k), D_l\}$: M_I maps $D_l(a_k)$ -set to D_l -set.

- Mapping:

$$M_I\bar{\tau}_{k+1} = \begin{cases} \beta_2 = -(\tau_1 + \tau_2 + \bar{\tau}_2), & \text{for } k = 1, \\ \beta_{k+1} = -(\tau_1 + 2\sum_{i=2}^k \tau_i + \tau_{k+1} + \bar{\tau}_{k+1}), & \text{for } k \geq 2. \end{cases}$$

- Root systems:

$$\begin{cases} S_1 = \{\tau_1, \dots, \tau_k, \tau_{k+1}, \bar{\tau}_{k+1}\}, & \text{for } k \geq 2, \\ S_2 = \{\tau_1, \tau_2, \bar{\tau}_2\}, & \text{for } k = 1, \end{cases}$$

$$\begin{cases} \Phi(S_1) \text{ is a root system of type } D_{k+2}, & \text{for } k \geq 2, \\ \Phi(S_2) \text{ is a root system of type } A_3, & \text{for } k = 1. \end{cases}$$

- Minimal roots: β_{k+1} is the minimal root in $\Phi(S_1)$ and β_2 is the minimal root in $\Phi(S_2)$.
- Eliminated edges: $\{\bar{\tau}_{k+1}, \tau_k\}$ and $\{\bar{\tau}_{k+1}, \tau_{k+2}\}$.
- Emerging edge: $\{\beta_{k+1}, \tau_2\}$.
- Checking relations: for $k \geq 2$,

$$\begin{cases} \beta_{k+1} \perp \tau_i \ (1 \leq i \leq k+1, i \neq 2) & (\beta_{k+1} \text{ is the minimal root}^1) \\ \beta_{k+1} \perp \tau_{k+2} & ((\beta_{k+1}, \tau_{k+2}) = (\tau_{k+1} + \bar{\tau}_{k+1}, \tau_{k+2}) = 0) \\ \beta_{k+1} \perp \tau_i \ (i > k+2) & (\text{disconnected}^2) \\ (\beta_{k+1}, \tau_2) = -1 & (\beta_{k+1} \text{ is the minimal root}), \end{cases}$$

and for $k = 1$,

$$\begin{cases} \beta_2 \perp \tau_1, \ (\beta_2, \tau_2) = -1 & (\beta_2 \text{ is the minimal root}) \\ \beta_2 \perp \tau_i \ (i > 3) & (\text{disconnected}). \end{cases}$$

(3) Pair $\{E_6(a_1), E_6\}$: M_I maps $E_6(a_1)$ -set to E_6 -set.

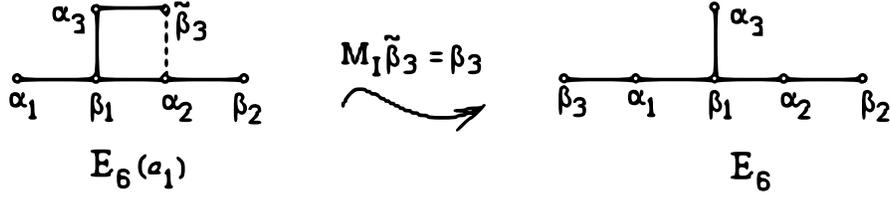
- Mapping: $M_I \tilde{\beta}_3 = \beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \tilde{\beta}_3)$.
- Root system: $S = \{\alpha_1, \beta_1, \alpha_3, \tilde{\beta}_3\}$ and Φ is a root system of type A_4 .
- Minimal root: β_3 is the minimal root in Φ .
- Eliminated edges: $\{\tilde{\beta}_3, \alpha_3\}, \{\tilde{\beta}_3, \alpha_2\}$.
- Emerging edge: $\{\beta_3, \alpha_1\}$.
- Checking relations:

$$\begin{cases} \beta_3 \perp \alpha_3, \beta_1 & (\beta_3 \text{ is the minimal root}) \\ \beta_3 \perp \alpha_2 & (\tilde{\beta}_3 + \beta_1 \perp \alpha_2) \\ \beta_3 \perp \beta_2 & (\text{disconnected}) \\ (\beta_3, \alpha_1) = -1 & (\beta_3 \text{ is the minimal root}) \end{cases}$$

(4) Pair $\{E_6(a_2), E_6(a_1)\}$: M_I maps $E_6(a_2)$ -set to $E_6(a_1)$ -set.

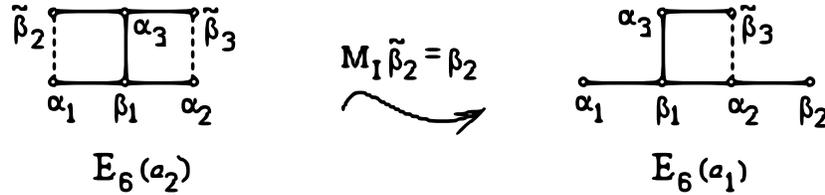
¹Hereinafter, the reason for the relation is indicated in parentheses.

²Let α be the sum of several roots α_i : $\alpha = \sum \alpha_i$. Hereinafter, the line “ $\alpha \perp \beta$ (disconnected)” means the case, where each summand α_i in α is orthogonal to β .


 Figure 7: Case (3). Mapping $M_I : E_6(a_1) \mapsto E_6$.

- Mapping: $M_I \tilde{\beta}_2 = \beta_2 = -(\alpha_3 + \beta_1 + \alpha_2 + \tilde{\beta}_2)$.
- Root systems: $S = \{\alpha_3, \beta_1, \alpha_2, \tilde{\beta}_2\}$ and $\Phi(S)$ is a root system of type A_4 .
- Minimal roots: β_2 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\tilde{\beta}_2, \alpha_3\}$ and $\{\tilde{\beta}_2, \alpha_1\}$.
- Emerging edges: $\{\beta_2, \alpha_2\}$.
- Checking relations:

$$\begin{cases} \beta_2 \perp \alpha_3, \beta_1 & (\beta_2 - \text{minimal root}) \\ \beta_2 \perp \alpha_1, \tilde{\beta}_3 & (\tilde{\beta}_2 + \beta_1 \perp \alpha_1 \text{ and } \alpha_2 + \alpha_3 \perp \tilde{\beta}_3) \\ (\beta_2, \alpha_2) = -1 & (\beta_2 - \text{minimal root}) \end{cases}$$


 Figure 8: Case (4). Mapping $M_I : E_6(a_2) \mapsto E_6(a_1)$.

(5) Pair $\{E_7(a_1), E_7\}$: M_I maps $E_7(a_1)$ -set to E_7 -set.

- Mapping: $M_I \tilde{\alpha}_3 = \alpha_3 = -(\beta_2 + \alpha_2 + \beta_3 + \tilde{\alpha}_3)$.
- Root system: $S = \{\beta_2, \alpha_2, \beta_3, \tilde{\alpha}_3\}$ and Φ is a root system of type A_4 .
- Minimal root: α_3 is the minimal root in Φ .
- Eliminated edges: $\{\tilde{\alpha}_3, \beta_1\}$ and $\{\tilde{\alpha}_3, \beta_3\}$.
- Emerging edge: $\{\beta_2, \alpha_3\}$.
- Checking relations:

$$\begin{cases} \alpha_3 \perp \alpha_2, \beta_3 & (\alpha_3 \text{ is the minimal root}) \\ \alpha_3 \perp \beta_1 & (\tilde{\alpha}_3 + \alpha_2 \perp \beta_1) \\ \alpha_3 \perp \alpha_1, \beta_4 & (\text{disconnected}) \\ (\alpha_3, \beta_2) = -1 & (\alpha_3 \text{ is the minimal root}) \end{cases}$$

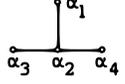
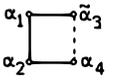
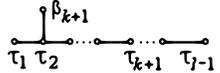
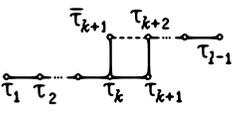
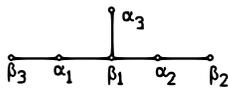
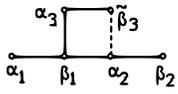
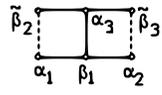
D_4		$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
(1) $D_4(\mathbf{a}_1)$	$\{\tilde{\Gamma}, \Gamma\} = \{D_4(a_1), D_4\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \alpha_4\},$ $M_I \tilde{\alpha}_3 = \alpha_3 = -(\alpha_1 + \alpha_2 + \tilde{\alpha}_3)$
D_l		$S = \{\tau_1, \tau_2, \dots, \tau_{l-1}, \beta_{k+1}\}$
(2) $D_l(\mathbf{a}_k)$	$\{\tilde{\Gamma}, \Gamma\} = \{D_l(a_k), D_l\}$ 	$\tilde{S} = \{\tau_1, \dots, \tau_k, \tau_{k+1}, \bar{\tau}_{k+1}, \tau_{k+2}, \dots, \tau_{l-1}\},$ $M_I \tau = \tau$ for $\tau \in \{\tau_1, \tau_2, \dots, \tau_{l-1}\},$ $M_I \bar{\tau}_{k+1} =$ $\begin{cases} \beta_{k+1} = -(\tau_1 + 2 \sum_{i=2}^k \tau_i + \tau_{k+1} + \bar{\tau}_{k+1}), \\ \beta_2 = -(\tau_1 + \tau_2 + \bar{\tau}_2). \end{cases} \quad k \geq 2,$
E_6		$S = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$
(3) $E_6(\mathbf{a}_1)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_6(a_1), E_6\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \tilde{\beta}_3\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\},$ $M_I \tilde{\beta}_3 = \beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \tilde{\beta}_3)$
(4) $E_6(\mathbf{a}_2)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_6(a_2), E_6(a_1)\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \tilde{\beta}_2, \tilde{\beta}_3\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \tilde{\beta}_3\}$ $M_I \tilde{\beta}_2 = \beta_2 = -(\alpha_3 + \beta_1 + \alpha_2 + \tilde{\beta}_2)$

Table 2: $\{D_4(a_1), D_4\}, \{D_l(a_k), D_l\}, \{E_6(a_1), E_6\}, \{E_6(a_2), E_6(a_1)\}$

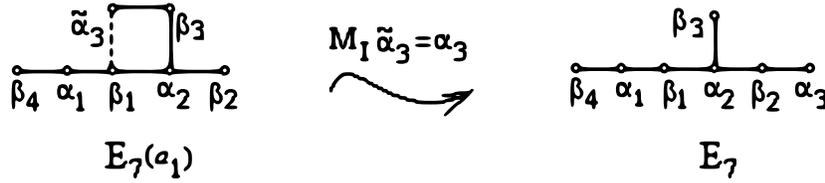


Figure 9: Case (5). Mapping $M_I : E_7(a_1) \mapsto E_7$.

(6) Pair $\{E_7(a_2), E_7\}$: M_I maps $E_7(a_2)$ -set to E_7 -set.

- Mapping: $M_I \tilde{\alpha}_1 = \alpha_1 = -(\beta_1 + \alpha_2 + \beta_3 + \tilde{\alpha}_1)$.
- Root system: $S = \beta_1, \alpha_2, \beta_3, \tilde{\alpha}_1$ and Φ is a root system of type A_4 .
- Minimal root: α_1 is the minimal root in Φ .
- Eliminated edges: $\tilde{\alpha}_1, \beta_3, \tilde{\alpha}_1, \beta_2$ and $\tilde{\alpha}_1, \tilde{\beta}_4$.
- Emerging edges: α_1, β_1 and $\alpha_1, \tilde{\beta}_4$.
- Checking relations:

$$\left\{ \begin{array}{ll} \alpha_1 \perp \alpha_2, \beta_3 & (\alpha_1 \text{ is the minimal root}) \\ \alpha_1 \perp \beta_2 & (\alpha_2 + \tilde{\alpha}_1 \perp \beta_2) \\ \alpha_1 \perp \alpha_3 & (\text{disconnected}) \\ (\alpha_1, \beta_1) = -1 & (\alpha_1 \text{ is the minimal root}) \\ (\beta_4, \alpha_1) = -1 & ((\beta_4, \alpha_1) = (\beta_4, \tilde{\alpha}_1)) \end{array} \right.$$

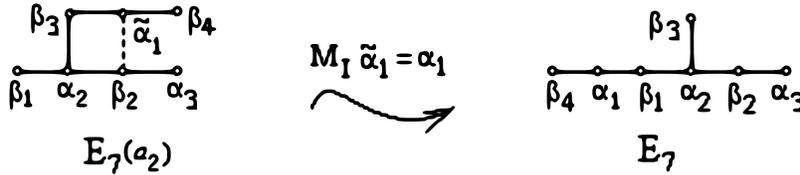


Figure 10: Case (6). Mapping $M_I : E_7(a_2) \mapsto E_7$.

(7) Pair $\{E_7(a_3), E_7(a_1)\}$: M_I maps $E_7(a_3)$ -set to $E_7(a_1)$ -set.

- Mapping: $M_I \alpha_4 = \beta_4 = -(\alpha_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1)$.
- Root system: $S = \{\alpha_4, \beta_3, \alpha_2, \beta_1, \alpha_1\}$ and $\Phi(S)$ is a root system of type A_5 .
- Minimal root: β_4 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\alpha_4, \beta_3\}$ and $\{\alpha_4, \beta_2\}$.
- Emerging edge: $\{\beta_4, \alpha_1\}$.
- Checking relations:

$$\left\{ \begin{array}{ll} \beta_4 \perp \beta_1, \alpha_2, \beta_3 & (\beta_4 \text{ is the minimal root}) \\ \beta_4 \perp \tilde{\alpha}_3, \beta_2 & (\beta_1 + \beta_3 \perp \tilde{\alpha}_3 \text{ and } \alpha_2 + \alpha_4 \perp \beta_2) \\ (\beta_4, \alpha_1) = -1 & (\beta_4 \text{ is the minimal root}) \end{array} \right.$$

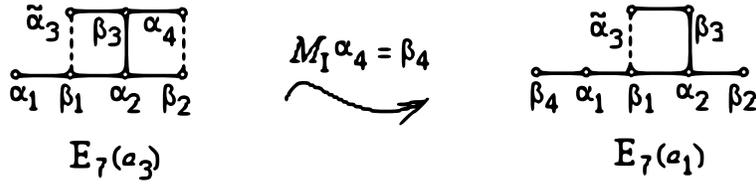


Figure 11: Case (7). Mapping $M_I : E_7(a_3) \mapsto E_7(a_1)$.

(8) Pair $\{E_7(a_4), E_7(a_3)\}$: M_I maps $E_7(a_4)$ -set to $E_7(a_3)$ -set.

- Mapping: $M_I \tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)$.
- Root system: $S = \{\beta_1, \tilde{\alpha}_1, \alpha_2, \alpha_3\}$ and $\Phi(S)$ is a root system of type D_4 .
- Minimal root: α_1 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\tilde{\alpha}_1, \beta_1\}$ and $\{\tilde{\alpha}_1, \beta_2\}$.
- Emerging edge: $\{\alpha_1, \beta_1\}$.
- Checking relations:

$$\begin{cases} \alpha_1 \perp \alpha_2, \tilde{\alpha}_3 & (\beta_1 \text{ is the minimal root}) \\ \alpha_1 \perp \beta_3, \beta_2 & (\alpha_2 + \tilde{\alpha}_3 \perp \beta_3 \text{ and } \tilde{\alpha}_1 + \alpha_2 \perp \beta_2) \\ \alpha_1 \perp \alpha_4 & (\alpha_4 \perp \tilde{\alpha}_1, \tilde{\alpha}_3, \alpha_2, \beta_1) \\ (\alpha_1, \beta_1) = -1 & (\beta_1 \text{ is the minimal root}) \end{cases}$$

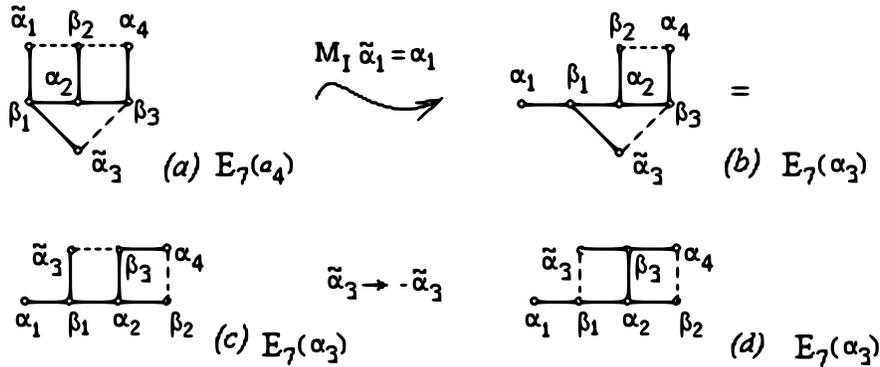


Figure 12: Case (8). Mapping $M_I : E_7(a_4) \mapsto E_7(a_3)$ (diagram (c)). Diagrams (c) and (d) are equivalent (by similarity $\tilde{\alpha}_3 \mapsto -\tilde{\alpha}_3$)

(9) Pair $\{E_8(a_1), E_8\}$: M_I maps $E_8(a_1)$ -set to E_8 -set.

- Mapping: $M_I \tilde{\alpha}_3 = \alpha_3 = -(\tilde{\alpha}_3 + \beta_3 + \alpha_2 + \beta_2)$.
- Root system: $S = \{\tilde{\alpha}_3, \beta_3, \alpha_2, \beta_2\}$ and Φ is a root system of type A_4 .
- Minimal root: α_3 is the minimal root in Φ .
- Eliminated edges: $\{\tilde{\alpha}_3, \beta_3\}$ and $\{\tilde{\alpha}_3, \tilde{\beta}_3\}$.

Table 3: $\{E_7(a_1), E_7\}$, $\{E_7(a_2), E_7\}$, $\{E_7(a_3), E_7(a_1)\}$, $\{E_7(a_4), E_7(a_3)\}$

	$\tilde{\Gamma}$ -basis \tilde{S} and Γ -basis S	Mapping M_I
E_7		$S = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\}$
(5) $E_7(a_1)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_7(a_1), E_7\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \beta_1, \beta_2, \beta_3, \beta_4\}$, $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4\}$, $M_I \tilde{\alpha}_3 = \alpha_3 = -(\beta_2 + \alpha_2 + \beta_3 + \tilde{\alpha}_3)$
(6) $E_7(a_2)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_7(a_2), E_7\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\}$, $M_I \tau = \tau$ for $\tau \in \{\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\}$, $M_I \tilde{\alpha}_1 = \alpha_1 = -(\beta_1 + \alpha_2 + \beta_3 + \tilde{\alpha}_1)$
(7) $E_7(a_3)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_7(a_3), E_7(a_1)\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$, $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \beta_1, \beta_2, \beta_3\}$, $M_I \alpha_4 = \beta_4 = -(\alpha_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1)$
(8) $E_7(a_4)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_7(a_4), E_7(a_3)\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$, $M_I \tau = \tau$ for $\tau \in \{\alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$, $M_I \tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)$

- Emerging edge: $\{\beta_2, \alpha_3\}$.
- Checking relations:

$$\begin{cases} \alpha_3 \perp \alpha_2, \beta_3 & (\alpha_3 \text{ is the minimal root}) \\ \alpha_3 \perp \beta_1 & (\tilde{\alpha}_3 + \alpha_2 \perp \beta_1) \\ \alpha_3 \perp \alpha_1, \alpha_4, \beta_1 & (\text{disconnected}) \\ (\alpha_3, \beta_2) = -1 & (\alpha_3 \text{ is the minimal root}) \end{cases}$$

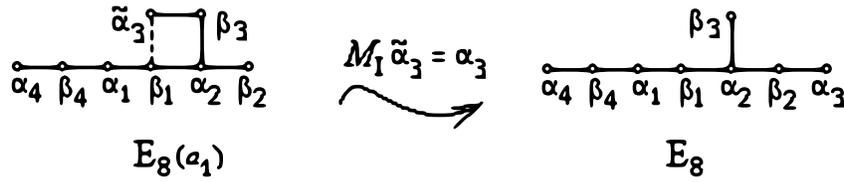


Figure 13: Case (9). Mapping $M_I : E_8(a_1) \mapsto E_8$.

(10) Pair $\{E_8(a_2), E_8\}$: M_I maps $E_8(a_2)$ -set to E_8 -set.

- Mapping: $M_I \tilde{\alpha}_1 = \alpha_1 = -(\tilde{\alpha}_1 + \beta_3 + \alpha_2 + \beta_1)$.
- Root system: $S = \{\tilde{\alpha}_1, \beta_3, \alpha_2, \beta_1\}$ and Φ is a root system of type A_4 .
- Minimal root: α_1 is the minimal root in Φ .
- Eliminated edges: $\{\beta_3, \tilde{\alpha}_1\}$, $\{\tilde{\alpha}_1, \beta_4\}$ and $\{\tilde{\alpha}_1, \beta_2\}$.
- Emerging edges: $\{\alpha_1, \beta_1\}$ and $\{\alpha_1, \beta_4\}$.
- Checking relations:

$$\begin{cases} \alpha_1 \perp \alpha_2, \beta_3 & (\alpha_1 \text{ is the minimal root}) \\ \alpha_1 \perp \beta_2 & (\tilde{\alpha}_1 + \alpha_2 \perp \beta_2) \\ \alpha_1 \perp \alpha_4 & (\text{disconnected}) \\ (\alpha_1, \beta_1) = -1 & (\alpha_1 \text{ is the minimal root}) \\ (\beta_4, \alpha_1) = +1 & ((\beta_4, \alpha_1) = -(\beta_4, \tilde{\alpha}_1)) \end{cases}$$

(11) Pair $\{E_8(a_3), E_8(a_2)\}$: M_I maps $E_8(a_3)$ -set to $E_8(a_2)$ -set.

- Mapping: $M_I \tilde{\alpha}_4 = \alpha_4 = -(3\tilde{\alpha}_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + \alpha_3 + \tilde{\alpha}_4)$.
- Root system: $S = \{\tilde{\alpha}_1, \beta_2, \beta_3, \beta_4, \alpha_3, \tilde{\alpha}_4\}$ and Φ is a root system of type E_6 .
- Minimal root: α_4 is the minimal root in Φ . The minimal root $M_I \tilde{\alpha}_4 = \alpha_4$ is connected only with β_4 .
- Eliminated edge: $\{\tilde{\alpha}_4, \beta_3\}$.
- Emerging edge: $\{\beta_4, \alpha_4\}$.

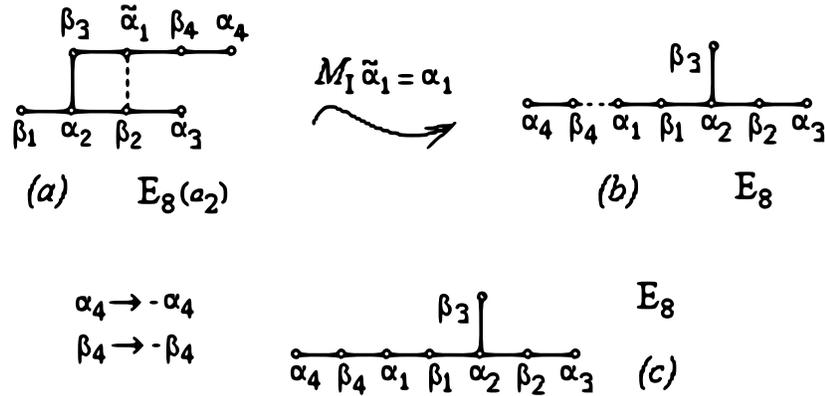


Figure 14: Case (10). Mapping $M_I : E_8(a_2) \mapsto E_8$ (diagram (b)). Diagrams (b) and (c) are equivalent (by similarity $\alpha_4 \mapsto -\alpha_4, \beta_4 \mapsto -\beta_4$)

- Checking relations:

$$\begin{cases} \alpha_4 \perp \tilde{\alpha}_1, \alpha_3, \beta_2, \beta_3 & (\alpha_4 - \text{minimal root}) \\ \alpha_4 \perp \alpha_2 & (\beta_2 + \beta_3 \perp \alpha_2) \\ \alpha_4 \perp \beta_1 & (\text{disconnected}) \\ (\alpha_4, \beta_4) = -1 & (\alpha_4 - \text{minimal root}) \end{cases}$$

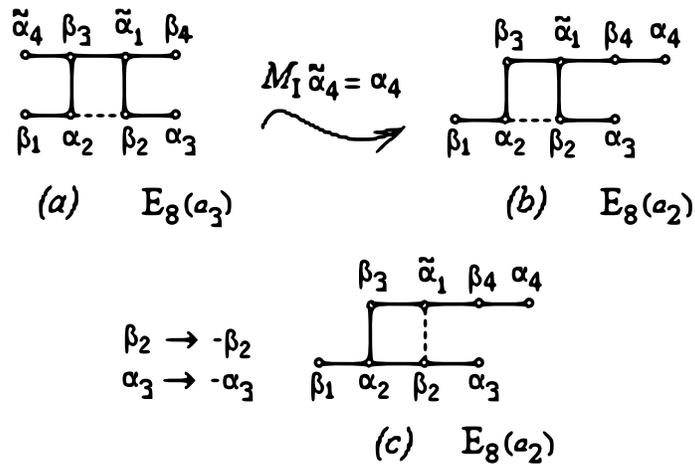


Figure 15: Case (11). Mapping $M_I : E_8(a_3) \mapsto E_8(a_2)$ (diagram (b)). Diagrams (b) and (c) are equivalent (by similarity $\beta_2 \mapsto -\beta_2, \alpha_3 \mapsto -\alpha_3$)

- (12) Pair $\{E_8(a_4), E_8(a_1)\}$: M_I maps $E_8(a_4)$ -set to $E_8(a_1)$ -set.

- Mapping: $M_I \tilde{\alpha}_4 = \alpha_4 = -(\tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1 + \beta_4)$.

- Root systems: $S = \{\tilde{\alpha}_4, \beta_3, \alpha_2, \beta_1, \alpha_1, \beta_4\}$ and $\Phi(S)$ is a root system of type A_4 .
- Minimal roots: α_4 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\tilde{\alpha}_4, \beta_2\}$ and $\{\tilde{\alpha}_4, \beta_3\}$.
- Emerging edge: $\{\alpha_4, \beta_4\}$.
- Checking relations:

$$\begin{cases} \alpha_4 \perp \beta_3, \alpha_2, \beta_1, \alpha_1 & (\alpha_4 - \text{minimal root}) \\ \alpha_4 \perp \tilde{\alpha}_3, \beta_2 & (\beta_3 + \beta_1 \perp \tilde{\alpha}_3, \text{ and } \tilde{\alpha}_4 + \alpha_2 \perp \beta_2) \\ (\alpha_4, \beta_4) = -1 & (\alpha_4 - \text{minimal root}) \end{cases}$$

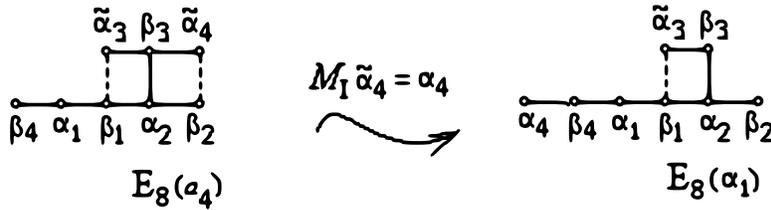


Figure 16: Case (12). Mapping $M_I : E_8(a_4) \mapsto E_8(a_1)$.

(13) Pair $\{E_8(a_5), E_8(a_4)\}$: M_I maps $E_8(a_5)$ -set to $E_8(a_4)$ -set.

- Mapping: $M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + \tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1)$
- Root systems: $S = \{\tilde{\beta}_4, \tilde{\alpha}_4, \beta_3, \alpha_2, \beta_1, \alpha_1\}$ and $\Phi(S)$ is a root system of type A_6 .
- Minimal roots: β_4 is the minimal root in $\Phi(S)$.
- Eliminated edge: $\{\tilde{\alpha}_4, \tilde{\beta}_4\}$.
- Emerging edge: $\{\beta_4, \alpha_1\}$.
- Checking relations:

$$\begin{cases} \beta_4 \perp \beta_1, \alpha_2, \beta_3, \tilde{\alpha}_4 & (\beta_4 - \text{minimal root}) \\ \beta_4 \perp \tilde{\alpha}_3, \beta_2 & (\beta_3 + \beta_1 \perp \tilde{\alpha}_3 \text{ and } \\ & \alpha_2 + \tilde{\alpha}_4 \perp \beta_2) \\ (\beta_4, \alpha_1) = -1 & (\beta_4 - \text{minimal root}) \end{cases}$$

(14) Pair $\{E_8(a_6), E_8(a_4)\}$: M_I maps $E_8(a_6)$ -set to $E_8(a_4)$ -set.

- Mapping: $M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3)$.
- Root systems: $S = \{\tilde{\beta}_4, \alpha_1, \beta_1, \alpha_2, \beta_2, \beta_3\}$ and Φ is a root system of type D_6 .
- Minimal root: β_4 is the minimal root in Φ .

Table 4: $\{E_8(a_1), E_8\}, \{E_8(a_2), E_8\}, \{E_8(a_3), E_8(a_2)\}, \{E_8(a_4), E_8(a_1)\}$

	$\tilde{\Gamma}$ -basis \tilde{S} and Γ -basis S	Mapping M_I
E_8		$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\}$
(9) $E_8(a_1)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_1), E_8\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tilde{\alpha}_3 = \alpha_3 = -(\tilde{\alpha}_3 + \beta_3 + \alpha_2 + \beta_2)$
(10) $E_8(a_2)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_2), E_8\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tilde{\alpha}_1 = \alpha_1 = -(\tilde{\alpha}_1 + \beta_3 + \alpha_2 + \beta_1)$
(11) $E_8(a_3)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_3), E_8(a_2)\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tau = \tau$ for $\tau \in \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tilde{\alpha}_4 = \alpha_4 =$ $- (3\tilde{\alpha}_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + \alpha_3 + \tilde{\alpha}_4)$
(12) $E_8(a_4)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_4), E_8(a_1)\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \beta_1, \beta_2, \beta_3, \beta_4\}$ $M_I \tilde{\alpha}_4 = \alpha_4 =$ $- (\tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1 + \beta_4)$

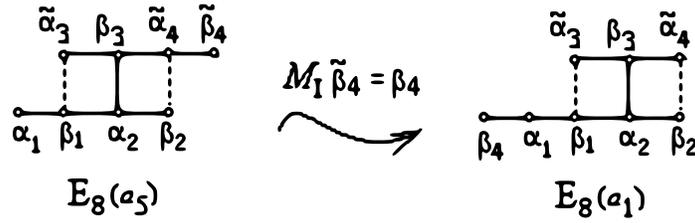


Figure 17: Case (13). Mapping $M_I : E_8(a_5) \mapsto E_8(a_4)$.

- Eliminated edges: $\{\tilde{\beta}_4, \tilde{\alpha}_3\}$ and $\{\tilde{\beta}_4, \alpha_1\}$.
- Emerging edge: $\{\beta_4, \alpha_1\}$.
- Checking relations:

$$\begin{cases} \beta_4 \perp \beta_1, \alpha_2, \beta_2, \beta_3, \tilde{\beta}_4 & (\beta_4 - \text{minimal root}) \\ \beta_4 \perp \tilde{\alpha}_3 & ((\beta_4, \tilde{\alpha}_3) = (\tilde{\beta}_4 + \beta_3 + 2\beta_1, \tilde{\alpha}_3) = -1 - 1 + 2 = 0) \\ \beta_4 \perp \tilde{\alpha}_4 & (\beta_3 + \beta_2 \perp \alpha_4) \\ (\beta_4, \alpha_1) = -1 & (\beta_4 - \text{minimal root}) \end{cases}$$

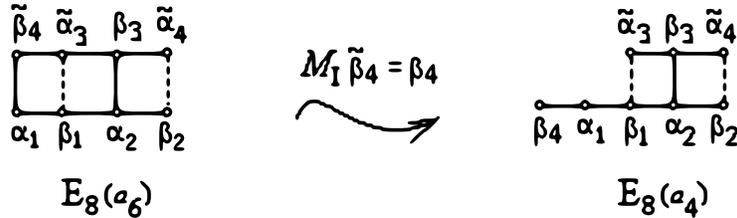


Figure 18: Case (14). Mapping $M_I : E_8(a_6) \mapsto E_8(a_4)$.

(15) Pair $\{E_8(a_7), E_8(a_5)\}$: M_I maps $E_8(a_7)$ -set to $E_8(a_5)$ -set.

- Mapping: $M_I \tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)$.
- Root systems: $S = \{\beta_1, \tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3\}$ and $\Phi(S)$ is a root system of type D_4 .
- Minimal root: α_1 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\tilde{\alpha}_1, \tilde{\beta}_1\}$ and $\{\tilde{\alpha}_1, \beta_2\}$.
- Emerging edge: $\{\beta_1, \alpha_1\}$.
- Checking relations:

$$\begin{cases} \alpha_1 \perp \alpha_2, \tilde{\alpha}_3 & (\alpha_1 - \text{minimal root}) \\ \alpha_1 \perp \tilde{\beta}_4 & (\text{disconnected}) \\ (\alpha_1, \beta_1) = -1 & (\alpha_1 - \text{minimal root}) \\ \alpha_1 \perp \beta_2, \beta_3 & (\alpha_2 + \tilde{\alpha}_1 \perp \beta_2, \text{ and } \alpha_2 + \tilde{\alpha}_3 \perp \beta_3) \end{cases}$$

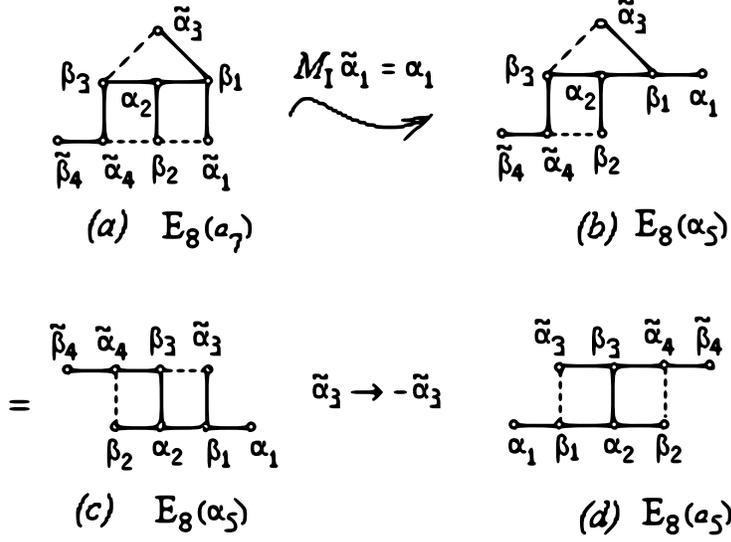


Figure 19: Case (15). Mapping $M_I : E_8(a_7) \mapsto E_8(a_5)$. Diagrams (c) and (d) are equivalent (by similarity $\tilde{\alpha}_3 \mapsto -\tilde{\alpha}_3$)

(16) Pair $\{E_8(a_8), E_8(a_7)\}$: M_I maps $E_8(a_8)$ -set to $E_8(a_7)$ -set.

- Mapping: $M_I\beta_4 = \tilde{\beta}_4 = -(2\tilde{\alpha}_4 + \beta_4 + \beta_2 + \beta_3)$.
- Root systems: $S = \{\tilde{\alpha}_4, \beta_4, \beta_2, \beta_3\}$ and $\Phi(S)$ is a root system of type D_4 .
- Minimal root: β_4 is the minimal root in $\Phi(S)$.
- Eliminated edges: $\{\beta_4, \tilde{\alpha}_4\}$, $\{\beta_4, \tilde{\alpha}_1\}$ and $\{\beta_4, \tilde{\alpha}_3\}$.
- Emerging edge: $\{\tilde{\beta}_4, \tilde{\alpha}_4\}$.
- Checking relations:

$$\begin{cases} \tilde{\beta}_4 \perp \beta_2, \beta_3 & (\beta_4 - \text{minimal root}) \\ \tilde{\beta}_4 \perp \tilde{\alpha}_1, \tilde{\alpha}_3 & (\tilde{\alpha}_1 \perp \beta_2 + \beta_4 \text{ and} \\ & \beta_3 + \beta_4 \perp \tilde{\alpha}_3) \\ (\tilde{\beta}_4, \tilde{\alpha}_1) = -1 & (\beta_4 - \text{minimal root}) \end{cases}$$

(ii) Let us prove that M_I is an involution. There exists a certain root $\tilde{\alpha} \in \tilde{S}$ such that

$$\begin{cases} M_I\tau_i = \tau_i \text{ for all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, \\ M_I\tilde{\alpha} = \alpha = -\tilde{\alpha} + \sum t_i\tau_i, \text{ where sum is taken over all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}. \end{cases}$$

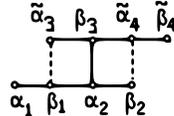
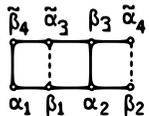
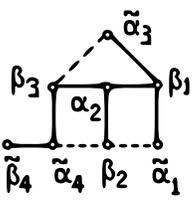
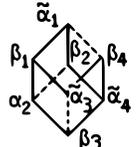
The image $M_I\tilde{\alpha} = \alpha$ is the root in S . Then,

$$M_I^2\tilde{\alpha} = -M\tilde{\alpha} + \sum t_i\tau_i = \tilde{\alpha} - \sum t_i\tau_i + \sum t_i\tau_i = \tilde{\alpha} \quad \text{and} \quad M_I^2\tilde{\alpha} = M_I\alpha = \tilde{\alpha}.$$

In other words,

$$M_I : \tilde{\alpha} \mapsto \alpha \quad \text{and} \quad M_I : \alpha \mapsto \tilde{\alpha}.$$

Table 5: $\{E_8(a_5), E_8(a_4)\}, \{E_8(a_6), E_8(a_4)\}, \{E_8(a_7), E_8(a_5)\}, \{E_8(a_8), E_8(a_7)\}$

	$\tilde{\Gamma}$ -basis \tilde{S} and Γ -basis S	Mapping M_I
(13) $E_8(a_5)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_5), E_8(a_4)\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \tilde{\beta}_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3\}$ $M_I \tilde{\beta}_4 = \beta_4 =$ $-(\tilde{\beta}_4 + \tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1)$
(14) $E_8(a_6)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_6), E_8(a_4)\}$ 	$\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \tilde{\beta}_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3\}$ $M_I \tilde{\beta}_4 = \beta_4 =$ $-(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3)$
(15) $E_8(a_7)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_7), E_8(a_5)\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \tilde{\beta}_4\},$ $M_I \tau = \tau$ for $\tau \in \{\alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \tilde{\beta}_4\}$ $M_I \tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)$
(16) $E_8(a_8)$	$\{\tilde{\Gamma}, \Gamma\} = \{E_8(a_8), E_8(a_7)\}$ 	$\tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3, \beta_4\},$ $M_I \tau = \tau$ for $\tau \in \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \beta_1, \beta_2, \beta_3\}$ $M_I \beta_4 = \tilde{\beta}_4 = -(2\tilde{\alpha}_4 + \beta_4 + \beta_2 + \beta_3)$

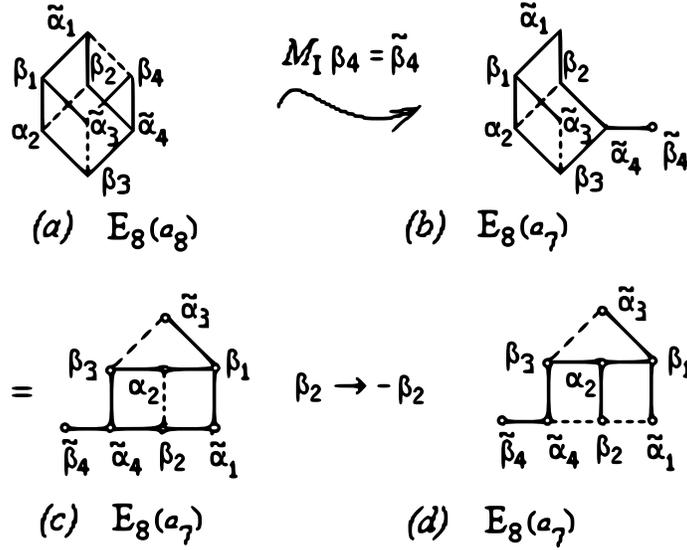


Figure 20: Case (16). Mapping $M_I : E_8(a_8) \mapsto E_8(a_7)$. Diagrams (c) and (d) are equivalent (by similarity $\tilde{\beta}_2 \mapsto -\tilde{\beta}_2$)

4.7 Relation of partial Cartan matrices

Consider pairs $\{\tilde{\Gamma}, \Gamma\}$ out of the adjacency list (2). The transition matrix M_I maps the $\tilde{\Gamma}$ -basis \tilde{S} to the Γ -basis S :

$$M\tilde{\tau}_i = \tau_i, \text{ where } \tilde{\tau}_i \in \tilde{S}, \tau_i \in S.$$

If the matrix M does not change a certain root $\tilde{\tau}_i$, the designation of this root and the corresponding node is the same for $\tilde{\Gamma}$ -basis and Γ -basis, namely: $M\tau_i = \tau_i$. The transition matrix M relates the partial Cartan matrices B_Γ and $B_{\tilde{\Gamma}}$ as follows:

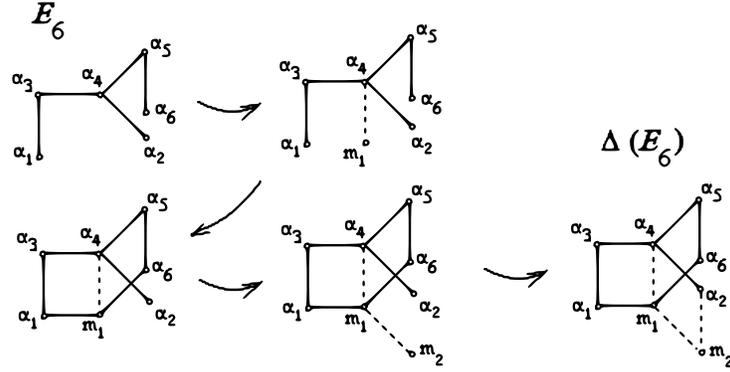
$${}^t M \cdot B_{\tilde{\Gamma}} \cdot M = B_\Gamma. \tag{25}$$

Eq. (25) is the relation of partial Cartan matrices B_Γ given in different bases.

4.7.1 Example: $\{E_6(a_1), E_6\}$

For the pair $\{E_6(a_1), E_6\}$ (case (3) in Appendix B),

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B_{E_6(a_1)} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 1 \\ 0 & 0 & 2 & -1 & 0 & -1 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

Figure 21: Enhanced Dynkin diagrams $\Delta(E_6)$

Then,

$${}^t M B_{E_6(a_1)} M = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The matrix ${}^t M B_{E_6(a_1)} M$ is the Cartan matrix B_{E_6} for Dynkin diagram E_6 .

A Dynkin-Minchenko completion procedure

A.1 Enhanced Dynkin diagram $\Delta(E_6)$

Extra nodes m_1 and m_2 for the enhanced Dynkin diagram $\Delta(E_6)$ are as follows (see Fig. 21):

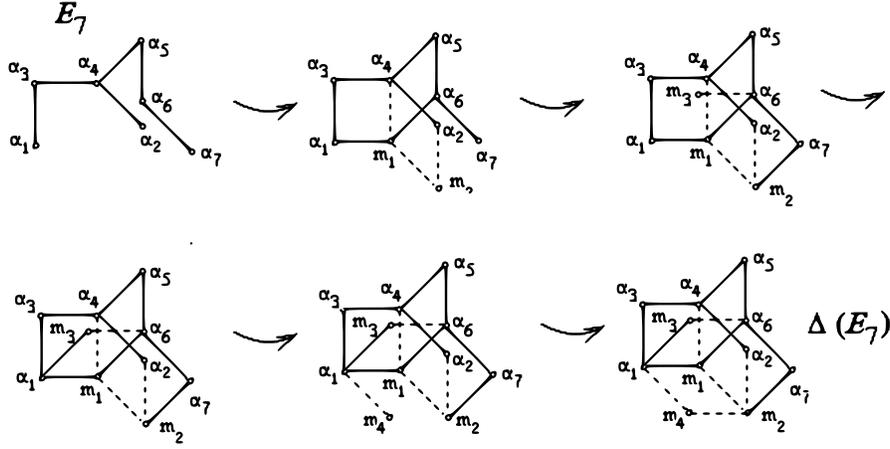
$$\begin{aligned} m_1 &= 2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5, \\ m_1 &\text{ is the maximal element in } \{\alpha_4, \alpha_2, \alpha_3, \alpha_5\}, \end{aligned} \tag{26}$$

$$\begin{aligned} m_2 &= 2m_1 - \alpha_4 + \alpha_1 + \alpha_6, \\ m_2 &\text{ is the maximal element in } \{m_1, -\alpha_4, \alpha_1, \alpha_6\}. \end{aligned} \tag{27}$$

From (26) and (27) we get

$$m_2 = 3\alpha_4 + 2\alpha_2 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6.$$

Then, m_2 is also the maximal element in the E_6 -set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$.


 Figure 22: Enhanced Dynkin diagrams $\Delta(E_7)$

A.2 Enhanced Dynkin diagram $\Delta(E_7)$

As in the case E_6 , the extra nodes m_1 and m_2 are as follows:

$$\begin{aligned} m_1 &= 2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5, \\ m_2 &= 3\alpha_4 + 2\alpha_2 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6. \end{aligned}$$

Further, the extra node m_3 is as follows:

$$\begin{aligned} m_3 &= 2\alpha_6 + \alpha_5 + \alpha_7 + m_1 = \\ &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7. \end{aligned}$$

Here, m_3 is the maximal element in the D_6 -set $\{\alpha_2, \alpha_3, \alpha_4, \alpha_4\alpha_6\}$. The extra node m_4 :

$$\begin{aligned} m_4 &= 2\alpha_1 + \alpha_3 + m_1 + m_3 = \\ &= 4\alpha_4 + 3\alpha_5 + 3\alpha_3 + 2\alpha_2 + 2\alpha_6 + 2\alpha_1 + \alpha_7. \end{aligned}$$

The node m_4 is the maximal element in the E_7 -set (see Fig. 22):

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}.$$

B Transition matrices M_I , cases (1) - (16)

(1) $\{D_4(a_1), D_4\}$:

$$\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & \\ 1 & 0 & -1 & 0 & \alpha_1 & \\ 0 & 1 & -1 & 0 & \alpha_2 & \\ 0 & 0 & -1 & 0 & \tilde{\alpha}_3 & \\ 0 & 0 & 0 & 1 & \alpha_4 & \end{array}$$

(2) $\{D_l(a_k), D_l\}$:

$$\begin{array}{cccccc}
\tau_1 & \tau_2 & \dots & \tau_{l-1} & \beta_{k+1} & \\
1 & 0 & \dots & 0 & -1 & \tau_1 \\
0 & 1 & \dots & -1 & -2 & \tau_2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & -1 & \tau_{k+1} \\
0 & 0 & \dots & 0 & -1 & \bar{\tau}_{k+1} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & \tau_{l-1}
\end{array}$$

(3) $\{E_6(a_1), E_6\}$:

$$\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 \\
1 & 0 & 0 & 0 & 0 & -1 & \alpha_1 \\
0 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \\
0 & 0 & 1 & 0 & 0 & -1 & \alpha_3 \\
0 & 0 & 0 & 1 & 0 & -1 & \beta_1 \\
0 & 0 & 0 & 0 & 1 & 0 & \beta_2 \\
0 & 0 & 0 & 0 & 0 & -1 & \tilde{\beta}_3
\end{array}$$

(4) $\{E_6(a_2), E_6(a_1)\}$:

$$\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \tilde{\beta}_3 \\
1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\
0 & 1 & 0 & 0 & -1 & 0 & \alpha_2 \\
0 & 0 & 1 & 0 & -1 & 0 & \alpha_3 \\
0 & 0 & 0 & 1 & -1 & 0 & \beta_1 \\
0 & 0 & 0 & 0 & -1 & 0 & \tilde{\beta}_2 \\
0 & 0 & 0 & 0 & 0 & 1 & \tilde{\beta}_3
\end{array}$$

(5) $\{E_7(a_1), E_7\}$:

$$\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & \alpha_2 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & \tilde{\alpha}_3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \beta_1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & \beta_2 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & \beta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \beta_4
\end{array}$$

(6) $\{E_7(a_2), E_7\}$:

α_1	α_2	α_3	β_1	β_2	β_3	β_4	
-1	0	0	0	0	0	0	$\tilde{\alpha}_1$
-1	1	0	0	0	0	0	α_2
0	0	1	0	0	0	0	α_3
-1	0	0	1	0	0	0	β_1
0	0	0	0	1	0	0	β_2
-1	0	0	0	0	1	0	β_3
0	0	0	0	0	0	1	β_4

(7) $\{E_7(a_3), E_7(a_1)\}$:

α_1	α_2	$\tilde{\alpha}_3$	β_1	β_2	β_3	β_4	
1	0	0	0	0	0	-1	α_1
0	1	0	0	0	0	-1	α_2
0	0	1	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	-1	α_4
0	0	0	0	1	0	-1	β_1
0	0	0	0	0	1	0	β_2
0	0	0	0	0	0	-1	β_3

(8) $\{E_7(a_4), E_7(a_3)\}$:

α_1	α_2	$\tilde{\alpha}_3$	α_4	β_1	β_2	β_3	
-1	0	0	0	0	0	0	$\tilde{\alpha}_1$
-1	1	0	0	0	0	0	α_2
-1	0	1	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	α_4
-2	0	0	0	1	0	0	β_1
0	0	0	0	0	1	0	β_2
0	0	0	0	0	0	1	β_3

(9) $\{E_8(a_1), E_8\}$:

α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	
1	0	0	0	0	0	0	0	α_1
0	1	-1	0	0	0	0	0	α_2
0	0	-1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	0	α_4
0	0	0	0	1	0	0	0	β_1
0	0	-1	0	0	1	0	0	β_2
0	0	-1	0	0	0	1	0	β_3
0	0	0	0	0	0	0	1	β_4

(10) $\{E_8(a_2), E_8\}$:

α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	
-1	0	0	0	0	0	0	0	$\tilde{\alpha}_1$
-1	1	0	0	0	0	0	0	α_2
0	0	1	0	0	0	0	0	α_3
0	0	0	1	0	0	0	0	α_4
-1	0	0	0	1	0	0	0	β_1
0	0	0	0	0	1	0	0	β_2
-1	0	0	0	0	0	1	0	β_3
0	0	0	0	0	0	0	1	β_4

(11) $\{E_8(a_3), E_8(a_2)\}$:

$\tilde{\alpha}_1$	α_2	α_3	α_4	β_1	β_2	β_3	β_4	
1	0	0	-3	0	0	0	0	$\tilde{\alpha}_1$
0	1	0	0	0	0	0	0	α_2
0	0	1	-1	0	0	0	0	α_3
0	0	0	-1	0	0	0	0	$\tilde{\alpha}_4$
0	0	0	0	1	0	0	0	β_1
0	0	0	-2	0	1	0	0	β_2
0	0	0	-2	0	0	1	0	β_3
0	0	0	-2	0	0	0	1	β_4

(12) $\{E_8(a_4), E_8(a_1)\}$:

α_1	α_2	$\tilde{\alpha}_3$	α_4	β_1	β_2	β_3	β_4	
1	0	0	-1	0	0	0	0	α_1
0	1	0	-1	0	0	0	0	α_2
0	0	1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	-1	0	0	0	0	$\tilde{\alpha}_4$
0	0	0	-1	1	0	0	0	β_1
0	0	0	0	0	1	0	0	β_2
0	0	0	-1	0	0	1	0	β_3
0	0	0	-1	0	0	0	1	β_4

(13) $\{E_8(a_5), E_8(a_4)\}$:

α_1	α_2	$\tilde{\alpha}_3$	$\tilde{\alpha}_4$	β_1	β_2	β_3	β_4	
1	0	0	0	0	0	0	-1	α_1
0	1	0	0	0	0	0	-1	α_2
0	0	1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	-1	$\tilde{\alpha}_4$
0	0	0	0	1	0	0	-1	β_1
0	0	0	0	0	1	0	0	β_2
0	0	0	0	0	0	1	-1	β_3
0	0	0	0	0	0	0	-1	$\tilde{\beta}_4$

(14) $\{E_8(a_6), E_8(a_4)\}$:

α_1	α_2	$\tilde{\alpha}_3$	$\tilde{\alpha}_4$	β_1	β_2	β_3	β_4	
1	0	0	0	0	0	0	-2	α_1
0	1	0	0	0	0	0	-2	α_2
0	0	1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	0	$\tilde{\alpha}_4$
0	0	0	0	1	0	0	-2	β_1
0	0	0	0	0	1	0	-1	β_2
0	0	0	0	0	0	1	-1	β_3
0	0	0	0	0	0	0	-1	$\tilde{\beta}_4$

(15) $\{E_8(a_7), E_8(a_5)\}$:

α_1	α_2	$\tilde{\alpha}_3$	$\tilde{\alpha}_4$	β_1	β_2	β_3	$\tilde{\beta}_4$	
-1	0	0	0	0	0	0	0	$\tilde{\alpha}_1$
-1	1	0	0	0	0	0	0	α_2
-1	0	1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	0	$\tilde{\alpha}_4$
-2	0	0	0	1	0	0	0	β_1
0	0	0	0	0	1	0	0	β_2
0	0	0	0	0	0	1	0	β_3
0	0	0	0	0	0	0	1	$\tilde{\beta}_4$

(16) $\{E_8(a_8), E_8(a_7)\}$:

$\tilde{\alpha}_1$	α_2	$\tilde{\alpha}_3$	$\tilde{\alpha}_4$	β_1	β_2	β_3	$\tilde{\beta}_4$	
1	0	0	0	0	0	0	0	$\tilde{\alpha}_1$
0	1	0	0	0	0	0	0	α_2
0	0	1	0	0	0	0	0	$\tilde{\alpha}_3$
0	0	0	1	0	0	0	-2	$\tilde{\alpha}_4$
0	0	0	0	1	0	0	0	β_1
0	0	0	0	0	1	0	-1	β_2
0	0	0	0	0	0	1	-1	β_3
0	0	0	0	0	0	0	-1	β_4

C For discussion

C.1 Conjectures regarding the Dynkin-Minchenko completion procedure

The numerical relation in Remark 1.1 motivates to the following assumption:

Conjecture C.1. *There is a correspondence between Carter diagrams (with cycles) and extra nodes in the Dynkin-Minchenko completion procedure.*

One more conjecture on relationship between enhanced Dynkin diagrams and Carter diagrams is as follows:

Conjecture C.2. *If $\{\tilde{\Gamma}, \Gamma\}$ is a homogeneous pair of Carter diagrams, then the signed enhanced Dynkin diagrams associated with $\tilde{\Gamma}$ and Γ coincide up to similarities.*

$$\Delta(\tilde{\Gamma}) = \Delta(\Gamma).$$

Remark C.3. Conjecture C.2 is easily verified for Carter diagrams $E_6(a_1)$ and $E_6(a_2)$:

- Case $E_6(a_1)$. Extra nodes m_1 and m_2 are as follows:

$$m_1 = 2\beta_1 + \alpha_1 + \alpha_2 + \alpha_3, \quad m_2 = 2\alpha_2 + \beta_1 + \beta_2 - \beta_3.$$

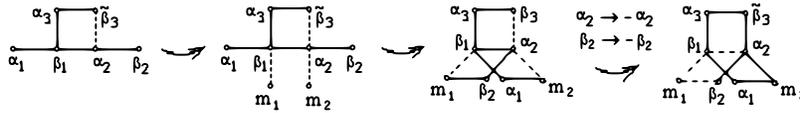


Figure 23: Enhanced Dynkin diagram $\Delta(E_6(a_1))$

Orthogonality relations are as follows:

$$\begin{aligned} m_1 \perp \alpha_1, \alpha_2, \alpha_3, \tilde{\beta}_3, \quad (m_1, \beta_2) = (\alpha_2, \beta_2) = -1, \\ (m_1, m_2) = (m_1, \beta_1 + \beta_2) = 1 - 1 = 0. \end{aligned}$$

- Case $E_6(a_2)$. Here, extra nodes m_1 and m_2 are as follows:

$$m_1 = 2\alpha_3 + \tilde{\beta}_3 + \tilde{\beta}_2 + \beta_1, \quad m_2 = 2\beta_1 + \alpha_1 + \alpha_2 + \alpha_3.$$

Orthogonality relations:

$$m_1 \perp \tilde{\beta}_3, \tilde{\beta}_2, \beta_1, \alpha_1, \alpha_2, \quad (m_1, m_2) = (m_1, \alpha_3) = 1.$$

Further, according to Vavilov-Mingin, see [19, Theorem 1],

$$E_7(a_i) \subset \Delta(E_7) \text{ and } E_8(a_i) \subset \Delta(E_8), \quad 1 \leq i \leq 8,$$

therefore

$$\Delta(E_7(a_i)) \subset \Delta(E_7) \text{ and } \Delta(E_8(a_i)) \subset \Delta(E_8).$$

Thus, to prove Conjecture C.2, it suffices to check only the reverse inclusions:

$$E_7 \subset \Delta(E_7(a_i)) \text{ and } E_8 \subset \Delta(E_8(a_i)).$$

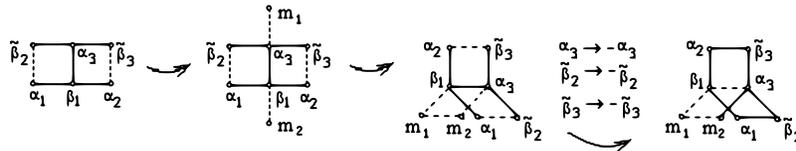


Figure 24: Enhanced Dynkin diagram $\Delta(E_6(a_2))$

C.2 Adjacency, complexity and eigenvalues

Let us define the *complexity* of the Carter diagram as $Nc+Ke$, where N is the number of cycles and K is the number of endpoints. Assume that one cycle contributes to complexity as two endpoints. One can select another proportion. In Table 6, Carter diagrams located side by side are the pairs from the adjacency list (2). The Carter diagrams from the adjacency list can be transformed to each other using the *transition matrix* M_I constructed in Theorem 4.1, see Section 4.2.1. Denote by *Max-E* the maximal eigenvalue of a partial Cartan matrix.

	$E_8(a_8)$	$E_8(a_7)$	$E_8(a_5)$	$E_8(a_4)$	$E_8(a_1)$	E_8	$E_8(a_2)$	$E_8(a_3)$
$Nc + Ke$	$6c$	$3c + 1e$	$2c + 2e$	$2c + 1e$	$1c + 2e$	$3e$	$1c + 3e$	$1c + 4e$
$2N + K$	12	7	6	5	4	3	5	6
<i>Max-E</i>	3.73	3.93	3.956	3.969	3.982	3.989	3.975	3.93

Table 6: Two chains arranged in ascending order of *Max-E*.

In Table 6, there are two chains which are arranged in ascending order of the *maximal eigenvalues* of the partial Cartan matrices and in descending order of the *complexity* parameter, see (28).

$$\begin{aligned}
 E_8(a_8) &\longrightarrow E_8(a_7) \longrightarrow E_8(a_5) \longrightarrow E_8(a_4) \longrightarrow E_8(a_1) \longrightarrow E_8, \\
 E_8(a_3) &\longrightarrow E_8(a_2) \longrightarrow E_8.
 \end{aligned}
 \tag{28}$$

The arrows in (28) point in the direction of increasing of maximal eigenvalue. It is not so clear the place of $E_8(a_6)$ in the homogeneous class $\{E, 8\}$:

- The complexity parameter for the diagram $E_8(a_6)$ is equal to $3c$.
- By alternative transition matrices from Section 4.2.2, the possible adjacent diagram for $E_8(a_6)$ can be $E_8(a_5)$, $E_8(a_4)$, $E_8(a_1)$.
- The maximum eigenvalue of the partial Cartan matrix for $E_8(a_6)$ is 3.902.

Based on these three factors, $E_8(a_6)$ can be placed, for example, between $E_8(a_5)$ and $E_8(a_4)$, or in the separated pair $\{E_8(a_5), E_8(a_6)\}$.

C.3 C. M. Ringel: Invariants with value 2, 4, 8

In the survey article [13], C. M. Ringel provided several notes regarding representations of Dynkin quivers. As Ringel writes: “*they shed some new light on properties of Dynkin and Euclidean quivers*”. The following is Ringel’s 2 – 4 – 8 assertion regarding Dynkin quivers E_n , $n = 6, 7, 8$.

Let $\bar{\Gamma}$ be the extended Dynkin diagram (=Euclidean quiver) for the Dynkin diagram Γ . If $\bar{\Gamma}$ is constructed from Γ by adding the new edge to the vertex y of Γ , then y is said

Γ	E_6	E_7	E_8
Γ'	A_5	D_6	E_7
n	2	4	8

Table 7: Ringel 2 – 4 – 8 assertion

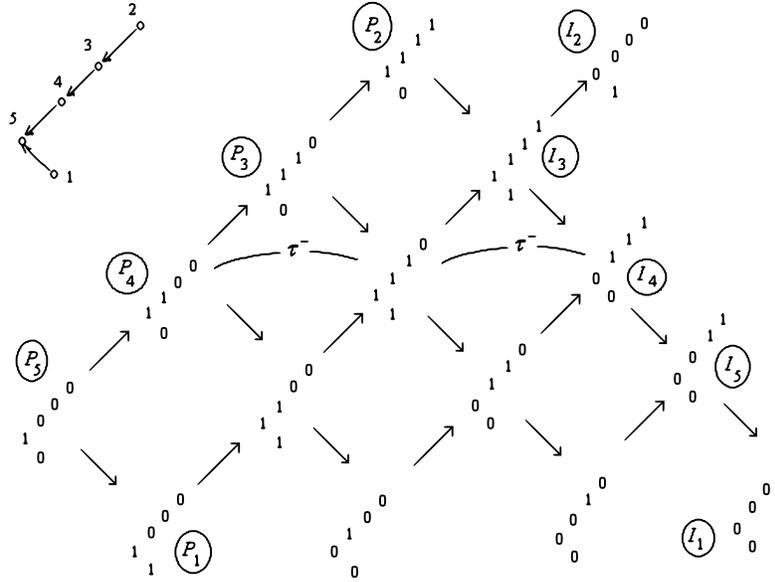


Figure 25: Auslander-Reiten quiver A_5 : $\tau^- = s_5s_4s_3s_2s_1$, $\tau^+ = s_1s_2s_3s_4s_5$, $\tau^{-2}P_4 = I_4$.

to be the exceptional vertex. For the Dynkin quivers E_n , $n = 6, 7, 8$, let x be the neighbor of y and $\Gamma' = \Gamma \setminus \{y\}$, see Table 7.

Let $P'(x)$ (resp. $I'(x)$) be the indecomposable projective (resp. injective) representation of Γ' corresponding to vertex x . Let τ' be the Auslander-Reiten translation in the category of finite-dimensional representations $\text{rep}(\Gamma)$. Then, $P'(x)$ and $I'(x)$ belong to the same τ' -orbit and there is the following 2 – 4 – 8 assertion:

$$P'(x) = (\tau')^n I'(x),$$

where n is the Ringel invariant given in the Table 7, see [13, Part 3]. For example, for the Auslander-Reiten A_5 , $n = 2$, see Fig. 25, and for the Auslander-Reiten D_6 , $n = 4$, see Fig. 26.

C.4 B. Rosenfeld: Isometry groups of the projective planes

J. Baez in [1] points out another connection between the invariants 2, 4, 8 and the diagrams E_6 , E_7 and E_8 . This connection was discovered by Rosenfeld, see [14]: “...

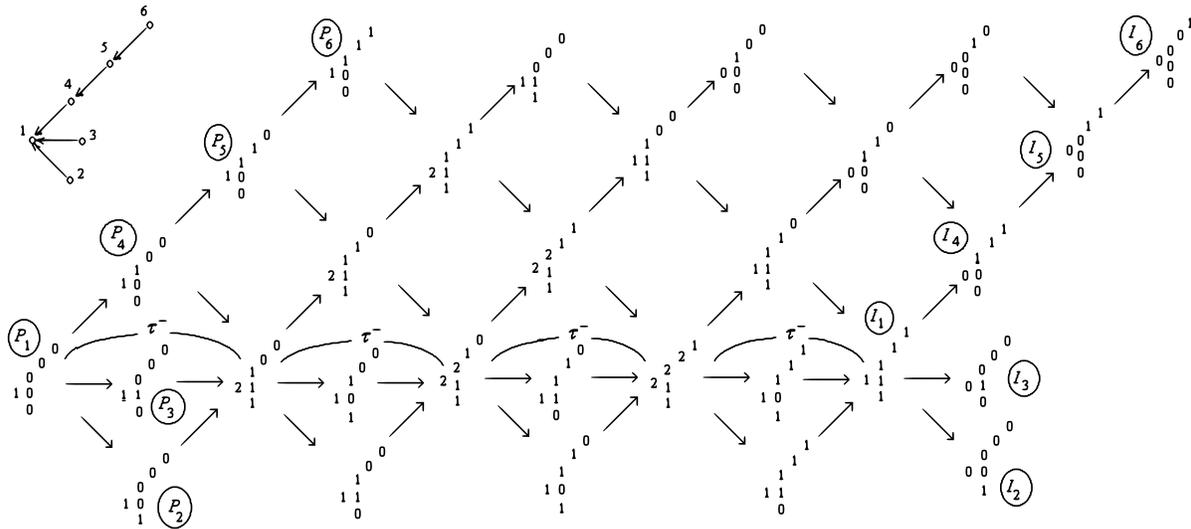


Figure 26: Auslander-Reiten quiver D_6 : $\tau^- = s_1s_2s_3s_4s_5s_6$, $\tau^+ = s_6s_5s_4s_3s_2s_1$. For each i , we have $\tau^{-4}P_i = I_i$.

Boris Rosenfeld had the remarkable idea ... that the exceptional Lie groups E_6 , E_7 and E_8 may be considered as the isometry groups of the projective planes over the following 3 algebras, respectively:

- the bioctonions $\mathbb{C} \otimes \mathbb{O}$,
- the quateroctonions $\mathbb{H} \otimes \mathbb{O}$,
- the octooctonions $\mathbb{O} \otimes \mathbb{O}$.

Any real finite-dimensional division algebra over the reals must be only one of these: \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O}^1 . The real numbers \mathbb{R} , complex numbers \mathbb{C} , the quaternions \mathbb{H} , and the octonions \mathbb{O} are division algebras of dimensions, respectively: 1, 2, 4, or 8.

References

[1] J. C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39(2):145–205, 2002.

[2] K. Bongartz. Algebras and quadratic forms. *J. London Math. Soc. (2)*, 28(3):461–469, 1983.

[3] S. Brenner. Quivers with commutativity conditions and some phenomenology of forms. In *Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974)*, Carleton Math. Lecture Notes, No. 9, pages Paper No. 5, 25. Carleton Univ., Ottawa, Ont., 1974.

¹This theorem is due to Kervaire and Bott-Milnor, 1958.

- [4] R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, 25:1–59, 1972.
- [5] D. Chapovalov, M. Chapovalov, A. Lebedev, and D. Leites. The classification of almost affine (hyperbolic) Lie superalgebras. *J. Nonlinear Math. Phys.*, 17(suppl. 1):103–161, 2010.
- [6] E. B. Dynkin and A. N. Minchenko. Enhanced Dynkin diagrams and Weyl orbits. *Transform. Groups*, 15(4):813–841, 2010.
- [7] P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras*. Springer-Verlag, Berlin, 1997. Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation.
- [8] V. A. Gritsenko and V. V. Nikulin. On the classification of Lorentzian Kac-Moody algebras. *Uspekhi Mat. Nauk*, 57(5(347)):79–138, 2002.
- [9] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [10] V. G. Kac. Infinite root systems, representations of graphs and invariant theory. *Invent. Math.*, 56(1):57–92, 1980.
- [11] J. McKee and C. Smyth. Integer symmetric matrices having all their eigenvalues in the interval $[-2, 2]$. *J. Algebra*, 317(1):260–290, 2007.
- [12] R. Mulas and Z. Stanić. Star complements for ± 2 in signed graphs. *Spec. Matrices*, 10:258–266, 2022.
- [13] C. M. Ringel. Representation theory of Dynkin quivers. Three contributions. *Front. Math. China*, 11(4):765–814, 2016.
- [14] B. Rosenfeld. *Geometry of Lie groups*, volume 393 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.
- [15] D. Simson. Integral bilinear forms, Coxeter transformations and Coxeter polynomials of finite posets. *Linear Algebra Appl.*, 433(4):699–717, 2010.
- [16] R. Stekolshchik. *Notes on Coxeter transformations and the McKay correspondence*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.
- [17] R. Stekolshchik. Classification of linkage systems, arxiv:1406.3049v2, 2014.
- [18] R. Stekolshchik. Equivalence of Carter diagrams. *Algebra Discrete Math.*, 23(1):138–179, 2017.
- [19] N. Vavilov and V. Migrin. Enhanced Dynkin diagrams done right. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 500(Voprosy Teorii Predstavlenii Algebr i Grupp. 37):11–29, 2021.
- [20] N. Vavilov and V. Migrin. Colourings of exceptional uniform polytopes of types e_6 and e_7 . *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 517(Teoriya Predstavlenii, Dinamicheski Sistemy, Kombinatornye Metody. XXXIV):36–54, 2022.

Received: July 5, 2022

Accepted for publication: June 6, 2023

Communicated by: Dmitry Leites and Sofiane Bouarroudj