

# Some generalizations of the variety of transposed Poisson algebras

*Bauyrzhan Sartayev*

**Abstract.** It is shown that the variety of transposed Poisson algebras coincides with the variety of Gelfand–Dorfman algebras in which the Novikov multiplication is commutative. The Gröbner–Shirshov basis for the transposed Poisson operad is calculated up to degree 4. Furthermore, we demonstrate that every transposed Poisson algebra is F-manifold. We verify that the special identities of GD-algebras hold in transposed Poisson algebras. Finally, we propose a conjecture stating that every transposed Poisson algebra is special, i.e., can be embedded into a differential Poisson algebra.

## 1 Introduction

A vector space  $A$  with a bilinear product  $\circ$  satisfying the identities

$$(x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3) = (x_2 \circ x_1) \circ x_3 - x_2 \circ (x_1 \circ x_3), \quad (1)$$

$$(x_1 \circ x_2) \circ x_3 = (x_1 \circ x_3) \circ x_2, \quad (2)$$

is called a Novikov algebra. A linear space  $V$  with two bilinear operations  $\circ$  and  $[\cdot, \cdot]$  is called a *Gelfand–Dorfman algebra* (or simply GD-algebra) [24], [25] if  $(V, \circ)$  is a Novikov algebra and  $(V, [\cdot, \cdot])$  is a Lie algebra, with the following additional identity:

$$[x_1, x_2 \circ x_3] - [x_3, x_2 \circ x_1] + [x_2, x_1] \circ x_3 - [x_2, x_3] \circ x_1 - x_2 \circ [x_1, x_3] = 0. \quad (3)$$

Let us provide the motivation for considering these algebras. The variety of Gelfand–Dorfman algebras is a natural generalization of the variety of Novikov algebras, while the

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*MSC 2020:* 17A30, 17A50, 17B63

*Keywords:* Poisson algebra, Transposed Poisson algebra, Gelfand–Dorfman algebra, Polynomial identities

*Contact information:*

B. Sartayev:

*Affiliation:* SDU University, Kaskelen, Kazakhstan and Narxoz University, Almaty, Kazakhstan.

*Email:* baurjai@gmail.com

variety of Poisson algebras is a generalization of the variety of commutative associative algebras. It is well-known that a free Novikov algebra can be embedded into a free commutative associative algebra with a derivation [8]. It was later discovered [17] that it is also possible to extend the mapping from free Novikov algebras to differential commutative associative algebras to the mapping from free Gelfand-Dorfman algebras to free Poisson algebras with a derivation, as follows:

$$\begin{aligned}\tau : \text{GD} &\rightarrow \text{PoisDer}, \\ x_1 \circ x_2 &\mapsto x_1 d(x_2), \\ [x_1, x_2] &\mapsto \{x_1, x_2\},\end{aligned}$$

where  $\text{PoisDer}$  is a Poisson algebra with derivation. However, it turns out [16] that the mapping  $\tau$  is not always injective, that is, there exist non-embeddable GD-algebras into  $\text{PoisDer}$  algebras with respect to the defined mapping  $\tau$ .

We say that a GD-algebra  $A$  is *special* if it can be embedded into some  $\text{PoisDer}$  algebra with respect to  $\tau$ . Denote by  $\text{SGD}$  the variety of algebras generated by special GD-algebras, namely, the class of all homomorphic images of all special GD-algebras. A polynomial identity is said to be *special* for the variety  $\text{SGD}$  if it holds on all special GD-algebras but it does not hold on all GD-algebras. In [17], a list of special identities for the variety  $\text{SGD}$  up to degree four was obtained:

$$[x_1, (x_2 \circ x_3) \circ x_4] - [x_1, x_2 \circ x_3] \circ x_4 - [x_1, x_2 \circ x_4] \circ x_3 + ([x_1, x_2] \circ x_3) \circ x_4 = 0, \quad (4)$$

$$\begin{aligned}[x_4 \circ x_1, x_3 \circ x_2] - [x_3 \circ x_1, x_4 \circ x_2] + [x_3, x_4 \circ x_1] \circ x_2 - [x_4, x_3 \circ x_2] \circ x_1 \\ - [x_4, x_3 \circ x_1] \circ x_2 + [x_3, x_4 \circ x_2] \circ x_1 + 2([x_4, x_3] \circ x_1) \circ x_2.\end{aligned} \quad (5)$$

A basis of a free special GD-algebra was constructed in [17]. In [9] was constructed the monomial basis of the free special GD-algebra in terms of  $\circ$  and  $[\cdot, \cdot]$ . Recall that a basis of free Novikov algebra was found in [8] using rooted trees, and the basis of free Lie algebra is the Lyndon-Shirshov words [23]. Therefore, one can easily construct a basis of a free algebra that involves the Novikov multiplication  $\circ$  and Lie product  $[\cdot, \cdot]$ . The problem of constructing a basis becomes more complicated when we consider those multiplications with additional identity (3).

In this paper, we consider the variety of Gelfand-Dorfman algebras whose Novikov multiplication is commutative, i.e.,

$$x_1 \circ x_2 = x_2 \circ x_1. \quad (6)$$

We observe that commutative Novikov algebra satisfies the associative identity:

$$(x_1 \circ x_2) \circ x_3 = (x_2 \circ x_1) \circ x_3 = (x_2 \circ x_3) \circ x_1 = x_1 \circ (x_2 \circ x_3).$$

We call this algebra commutative GD-algebra. In this case the identity (3) can be written as

$$[x_1, x_2 x_3] - [x_3, x_2 x_1] + [x_2, x_1] x_3 - [x_2, x_3] x_1 - [x_1, x_3] x_2 = 0. \quad (7)$$

Let us formulate the main results of this paper: for commutative Gelfand-Dorfman operad, we determine a set of basis elements up to degree 4. Using these obtained basis elements we show that every commutative Gelfand-Dorfman algebra is  $F$ -manifold, and we show that the variety of commutative GD-algebras coincides with the variety of transposed Poisson algebras, which has garnered considerable attention in recent years [1, 2, 11, 12, 13]. All mentioned results give

$$\text{GD}/(a \circ b - b \circ a) = \text{TP} \subset \text{F-manifold},$$

where TP is a variety of transposed Poisson algebras. In other words, in commutative GD-algebra the identity (3) can be rewritten to

$$2[x_1, x_2]x_3 = [x_1x_3, x_2] + [x_1, x_2x_3],$$

which is defining the identity of the variety of TP-algebras.

In addition, we prove that a free transposed Poisson algebra satisfies special identities (4) and (5). It raises a question about the speciality of transposed Poisson algebras but it is still an open question.

In this paper, all algebras are defined over a field of characteristic 0.

## 2 Gröbner base of commutative GD-operad

The operad theory is a powerful tool for computing a multilinear basis of fixed algebra. The main method for this calculation is the shuffling of monomials with so-called the forgetful functor  $f$ . For more details about this functor, see [3, 7].

**Definition 2.1.** ([3]) A shuffle operad is a monoid in the category of nonsymmetric collections of vector spaces with respect to the shuffle composition product defined as follows:

$$\mathcal{V} \circ_{\text{III}} \mathcal{W}(n) = \bigoplus_{r \geq 1} \mathcal{V}(r) \otimes \bigoplus_{\pi} \mathcal{W}(|I^{(1)}|) \otimes \mathcal{W}(|I^{(2)}|) \otimes \dots \otimes \mathcal{W}(|I^{(r)}|),$$

where  $\mathcal{W}$  and  $\mathcal{V}$  are nonsymmetric collections,  $\pi$  ranges in all set partitions  $\{1, \dots, n\} = \bigsqcup_{j=1}^r I^{(j)}$  for which all parts  $I^{(j)}$  are nonempty and  $\min(I^{(1)}) < \dots < \min(I^{(r)})$ .

To shuffle given monomials, we extend the alphabet of operations until it satisfies the definition of shuffle operad. For the case of operad  $\mathcal{A}s$  governed by the variety of associative algebras, we need to add additional operation  $y$  for given  $x$  as follows:

$$x(2\ 1) = y(1\ 2),$$

and we obtain the following relations of operad  $\text{As}_{\text{III}}(x, y)$

$$\begin{aligned} &x(x(1\ 2)\ 3) - x(1\ x(2\ 3)), \quad x(y(1\ 2)\ 3) - y(x(1\ 3)\ 2), \quad x(x(1\ 3)\ 2) - x(1\ y(2\ 3)), \\ &x(y(1\ 3)\ 2) - y(x(1\ 2)\ 3), \quad y(1\ x(2\ 3)) - y(y(1\ 3)\ 2), \quad y(1\ y(2\ 3)) - y(y(1\ 2)\ 3), \end{aligned}$$

where  $\text{As}_{\text{III}}(x, y)$  is a shuffled operad  $\text{As}$ . To obtain the defining relations of operad  $\text{Com}_{\text{III}}(x)$  (associative and commutative) we have to add to the given relations commutative identity:

$$x(1\ 2) - y(1\ 2).$$

In the same way, we write the defining relations of operad  $\text{Lie}_{\text{III}}(z)$  governed by the variety of Lie algebras:

$$z(z(1\ 2)\ 3) - z(1\ z(2\ 3)) - z(z(1\ 3)\ 2). \quad (8)$$

For our purposes, we write the defining identities of the operad  $\text{Com-GD}_{\text{III}}(x, z)$  governed by the variety of commutative GD-algebras. It remains only to rewrite the identity (7):

$$\begin{aligned} & z(1\ x(2\ 3)) + z(x(1\ 2)\ 3) - x(z(1\ 2)\ 3) - x(1\ z(2\ 3)) - x(z(1\ 3)\ 2), \\ & - z(x(1\ 3)\ 2) + z(x(1\ 2)\ 3) + x(z(1\ 2)\ 3) - x(z(1\ 3)\ 2) - x(1\ z(2\ 3)), \\ & - x(z(1\ 2)\ 3) + z(1\ x(2\ 3)) + z(x(1\ 3)\ 2) - x(z(1\ 3)\ 2) + x(1\ z(2\ 3)). \end{aligned}$$

Calculating the Gröbner base of  $\text{Com-GD}_{\text{III}}(x, z)$  by means of the package [6], we get the following result:

$n$	1	2	3	4	5	6
$\dim(\text{Com-GD}(n))$	1	2	6	20	74	301

Moreover, we obtain the following result:

**Theorem 2.2.** *The Gröbner basis of the operad  $\text{Com-GD}_{\text{III}}(x, z)$  up to degree 4 is defined by the following relations:*

$$x(x(1\ 3)\ 2) \rightarrow x(1\ x(2\ 3)), \quad x(x(1\ 2)\ 3) \rightarrow x(1\ x(2\ 3)), \quad (9)$$

$$z(x(1\ 2)\ 3) \rightarrow 2\ x(z(1\ 3)\ 2) - z(1\ x(2\ 3)), \quad (10)$$

$$x(z(1\ 2)\ 3) \rightarrow x(z(1\ 3)\ 2) - x(1\ z(2\ 3)), \quad (11)$$

$$z(z(1\ 2)\ 3) \rightarrow z(z(1\ 3)\ 2) + z(1\ z(2\ 3)), \quad (12)$$

$$z(x(1\ 3)\ 2) \rightarrow 2\ x(z(1\ 3)\ 2) - z(1\ x(2\ 3)) - 2\ x(1\ z(2\ 3)), \quad (13)$$

$$x(z(1\ 4)\ z(2\ 3)) \rightarrow x(z(1\ 3)\ z(2\ 4)) - x(z(1\ z(3\ 4))\ 2) + x(1\ z(2\ z(3\ 4))), \quad (14)$$

$$\begin{aligned} x(z(1\ 3)\ x(2\ 4)) & \rightarrow z(1\ x(2\ x(3\ 4))) + 3\ x(1\ x(z(2\ 4)\ 3)) - 2\ x(1\ z(2\ x(3\ 4))) \\ & - 2\ x(1\ x(2\ z(3\ 4))), \quad (15) \end{aligned}$$

$$\begin{aligned} x(z(1\ 4)\ x(2\ 3)) & \rightarrow z(1\ x(2\ x(3\ 4))) + 3\ x(1\ x(z(2\ 4)\ 3)) - 2\ x(1\ z(2\ x(3\ 4))) \\ & - x(1\ x(2\ z(3\ 4))), \quad (16) \end{aligned}$$

$$x(z(1 x(3 4)) 2) \rightarrow z(1 x(2 x(3 4))) + 2 x(1 x(z(2 4) 3)) - x(1 z(2 x(3 4))) - x(1 x(2 z(3 4))), \quad (17)$$

$$z(z(1 4) x(2 3)) \rightarrow z(z(1 x(3 4)) 2) - 2 x(z(1 3) z(2 4)) + z(1 x(z(2 4) 3)), \\ + z(1 z(2 x(3 4))) + 2 x(1 z(z(2 4) 3)), \quad (18)$$

$$z(z(1 3) x(2 4)) \rightarrow z(z(1 x(3 4)) 2) - 2 x(z(1 3) z(2 4)) + 2 x(z(1 z(3 4)) 2) \\ + z(1 x(z(2 4) 3)) + z(1 z(2 x(3 4))) - z(1 x(2 z(3 4))) + 2 x(1 z(z(2 4) 3)) \quad (19)$$

The rewriting rule (10) coincides with the identity of transposed Poisson algebra which gives  $\text{Com-GD} \subseteq \text{TP}$ . Analogically, calculating the Gröbner base of  $\text{TP}_{\text{III}}(x, z)$ , we obtain the same result as for  $\text{Com-GD}_{\text{III}}(x, z)$  which means  $\text{Com-GD} = \text{TP}$ .

### 3 Examples of transposed Poisson algebras

Let us turn our attention to an important class of algebras which is called  $F$ -manifold.  $F$ -manifold algebras appear in many fields of mathematics such as singularity theory [10], quantum  $\mathbb{K}$ -theory [19], integrable systems [4, 5, 21], operad [22], algebra [20].

**Definition 3.1.** A  $F$ -manifold algebra is a triple  $(\cdot, [\cdot, \cdot], X)$ , where the multiplication  $\cdot$  is associative and commutative, and  $[\cdot, \cdot]$  is a Lie bracket with the additional identity

$$[a_1 \cdot a_2, a_3 \cdot a_4] = [a_1 \cdot a_2, a_3] \cdot a_4 + [a_1 \cdot a_2, a_4] \cdot a_3 + a_1 \cdot [a_2, a_3 \cdot a_4] + a_2 \cdot [a_1, a_3 \cdot a_4] - \\ (a_1 \cdot a_3) \cdot [a_2, a_4] - (a_2 \cdot a_3) \cdot [a_1, a_4] - (a_2 \cdot a_4) \cdot [a_1, a_3] - (a_1 \cdot a_4) \cdot [a_2, a_3]. \quad (20)$$

**Theorem 3.2.** Every transposed Poisson algebra is  $F$ -manifold.

*Proof.* Since both multiplications are the same, it is enough to prove that the identity (20) holds in the free transposed Poisson algebra. For that purpose, we use the rewriting system of Theorem 2.2 to the identity (20). For simplicity, we write  $ab$  for  $a \cdot b$  for all  $a$  and  $b$ . Initially, let us start from the monomials  $[a_1 a_2, a_3 a_4]$ ,  $[a_1 a_2, a_3] a_4$ ,  $[a_1 a_2, a_4] a_3$ ,  $[a_1, a_3 a_4] a_2$  and  $a_1 [a_2, a_3] a_4$ :

$$[a_1 a_2, a_3 a_4] \stackrel{(10)}{=} 2[a_1, a_3 a_4] a_2 - [a_1, a_2 a_3 a_4] \stackrel{(17)}{=} \\ [a_1, a_2 a_3 a_4] + 4a_1 [a_2, a_4] a_3 - 2a_1 [a_2, a_3 a_4] - 2a_1 a_2 [a_3, a_4], \\ - [a_1 a_2, a_3] a_4 \stackrel{(10)}{=} -2[a_1, a_3] a_2 a_4 + [a_1, a_2 a_3] a_4 \stackrel{(11), (15)}{=} -[a_1, a_3] a_2 a_4 \\ - [a_1, a_2 a_3 a_4] - 3a_1 [a_2, a_4] a_3 + 2a_1 [a_2, a_3 a_4] + 2a_1 a_2 [a_3, a_4] + [a_1, a_4] a_2 a_3 - a_1 [a_2 a_3, a_4] \\ \stackrel{(10)}{=} -[a_1, a_3] a_2 a_4 - [a_1, a_2 a_3 a_4] - 3a_1 [a_2, a_4] a_3 + 2a_1 [a_2, a_3 a_4] + 2a_1 a_2 [a_3, a_4] \\ + [a_1, a_4] a_2 a_3 - 2a_1 [a_2, a_4] a_3 + a_1 [a_2, a_3 a_4],$$

$$\begin{aligned}
& - [a_1 a_2, a_4] a_3 \stackrel{(10)}{=} -2[a_1, a_4] a_2 a_3 + [a_1, a_2 a_4] a_3 \stackrel{(11)}{=} -2[a_1, a_4] a_2 a_3 + [a_1, a_3] a_2 a_4 \\
& \quad - a_1 [a_2 a_4, a_3] \stackrel{(15),(13)}{=} -2[a_1, a_4] a_2 a_3 + [a_1, a_2 a_3 a_4] + a_1 [a_2, a_4] a_3 - a_1 [a_2, a_3 a_4], \\
& - [a_1, a_3 a_4] a_2 \stackrel{(17)}{=} -[a_1, a_2 a_3 a_4] - 2a_1 [a_2, a_4] a_3 + a_1 [a_2, a_3 a_4] + a_1 a_2 [a_3, a_4], \\
& \quad a_1 [a_2, a_3] a_4 \stackrel{(11)}{=} a_1 [a_2, a_4] a_3 - a_1 a_2 [a_3, a_4].
\end{aligned}$$

Substituting the monomials by given expressions we obtain that every TP-algebra satisfies (20) identity.  $\square$

## 4 Special identities of GD-algebras on transposed Poisson algebras

In [16] proved that the class of special GD-algebras forms a variety. Since every variety can be defined by a set of identities, all known special identities of GD-algebra are contained in this set. There is no special identity of degree 3, but there are two special identities (4) and (5) of degree 4. Furthermore, in [16], all special identities of degree 5 were computed, and it was demonstrated that they are a consequence of the identities (4) and (5).

**Theorem 4.1.** *Every TP-algebra satisfies the identities (4) and (5).*

*Proof.* Using the rewriting system of Theorem 2.2 let us first rewrite all monomials in the identity (4) and then consider their sum.

$$\begin{aligned}
& [a_1, a_2] a_3 a_4 \stackrel{(11)}{=} [a_1, a_3 a_4] a_2 - a_1 [a_2, a_3 a_4] \stackrel{(17)}{=} [a_1, a_2 a_3 a_4] \\
& \quad + 2a_1 [a_2, a_4] a_3 - a_1 [a_2, a_3 a_4] - a_1 a_2 [a_3, a_4] - a_1 [a_2, a_3 a_4], \\
& - [a_1, a_2 a_3] a_4 \stackrel{(11)}{=} -[a_1, a_4] a_2 a_3 + a_1 [a_2 a_3, a_4] \stackrel{(16),(10)}{=} -[a_1, a_2 a_3 a_4] \\
& \quad - 3a_1 [a_2, a_4] a_3 + 2a_1 [a_2, a_3 a_4] + a_1 a_2 [a_3, a_4] + 2a_1 [a_2, a_4] a_3 - a_1 [a_2, a_3 a_4], \\
& - [a_1, a_2 a_4] a_3 \stackrel{(11)}{=} -[a_1, a_3] a_2 a_4 + a_1 [a_2 a_4, a_3] \stackrel{(15),(13)}{=} -[a_1, a_2 a_3 a_4] \\
& - 3a_1 [a_2, a_4] a_3 + 2a_1 [a_2, a_3 a_4] + 2a_1 a_2 [a_3, a_4] + 2a_1 [a_2, a_4] a_3 - a_1 [a_2, a_3 a_4] - 2a_1 a_2 [a_3, a_4].
\end{aligned}$$

We note that the sum of the monomials gives zero and therefore, the identity (4) is an identity in a TP-algebra.

In a similar way, we show that the identity (5) is an identity in a TP-algebra.

$$\begin{aligned}
& [a_1 a_3, a_2 a_4] - [a_1 a_4, a_2 a_3] \stackrel{(13)}{=} 2[a_1, a_3] a_2 a_4 - 2a_1 [a_2 a_4, a_3] - 2[a_1, a_4] a_2 a_3 \\
& \quad + 2a_1 [a_2 a_3, a_4] \stackrel{(15),(16)}{=} -2a_1 a_2 [a_3, a_4] - 2a_1 [a_2 a_4, a_3] + 2a_1 [a_2 a_3, a_4], \\
& [a_1 a_4, a_3] a_2 \stackrel{(13)}{=} 2[a_1, a_4] a_3 a_2 - [a_1, a_3 a_4] a_2 - 2a_1 [a_3, a_4] a_2, \\
& \quad - [a_1 a_3, a_4] a_2 \stackrel{(10)}{=} -2[a_1, a_4] a_3 a_2 + [a_1, a_3 a_4] a_2.
\end{aligned}$$

$\square$

In [15] is given an example of exceptional GD-algebra. However, it was proved that Novikov commutator algebras are special, namely, every algebra from this class can be embedded into appropriate Poisson algebra with derivation. As we saw since the variety of transposed Poisson algebras contained in the variety of GD-algebras and every transposed Poisson algebra satisfy special identities (4) and (5), therefore, we have the following conjecture

**Conjecture 4.2.** *Every transposed Poisson algebra is special.*

In [14] was given an answer for a particular case of the conjecture. In a related study [18], the embedding of left-symmetric algebras into differential perm algebras was explored, and a criterion for such embedding was discovered. In the event of a negative answer, it may be worth exploring the possibility of establishing a similar test for the present case.

#### ACKNOWLEDGMENTS

The author is grateful to the referees for their valuable remarks that improved the exposition. The author is also grateful to Professor N. Ismailov for his comments and advice, thanks to which the article has acquired such a full-fledged look.

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP14871710).

#### References

- [1] C. M. Bai, R. P. Bai, L. Guo, and Y. Wu. Transposed Poisson algebras, Novikov-Poisson algebras and 3-Lie algebras. *Journal of Algebra*, 632:535–566, 2023.
- [2] P. D. Beites, A. F. Ouaridi, and I. Kaygorodov. The algebraic and geometric classification of transposed Poisson algebras. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A: Matemáticas. RACSAM*, 117(2, Paper No. 55):25 pp, 2023.
- [3] M. R. Bremner and V. Dotsenko. *Algebraic Operads An Algorithmic Companion*. Chapman Hall, 2016.
- [4] L. David and I. A. B. Strachan. Compatible metrics on a manifold and nonlocal bi-Hamiltonian structures. *International Mathematics Research Notices*, 66:3533–3557, 2004.
- [5] L. David and I. A. B. Strachan. Dubrovins duality for F-manifolds with eventual identities. *Advances in Mathematics*, 226(5):4031–4060, 2011.
- [6] V. Dotsenko and W. Heijltjes. Gröbner bases for operads, <http://irma.math.unistra.fr/dotsenko/operads.html>, 2019.
- [7] V. Dotsenko and A. Khoroshkin. Gröbner bases for operads. *Duke Mathematical Journal*, 153(2):363–396, 2010.
- [8] A. S. Dzhumadil'daev and C. Löfwall. Trees, free right-symmetric algebras, free Novikov algebras and identities. *Homology, Homotopy And Applications*, 4(2, part 1):165–190, 2002.

- [9] V. Gubarev and B. K. Sartayev. Free special Gelfand-Dorfman algebra. *Journal of Algebra and Its Applications*, doi.org/10.1142/S0219498825500057, 2023.
- [10] C. Hertling. *Frobenius Manifolds and Moduli Spaces for Singularities*. Cambridge Tracts in Mathematics. Cambridge University Press, 2002.
- [11] I. Kaygorodov and M. Khrypchenko. Transposed Poisson structures on Block Lie algebras and superalgebras. *Linear Algebra and Its Applications*, 656:167–197, 2023.
- [12] I. Kaygorodov and M. Khrypchenko. Transposed Poisson structures on Witt type algebras. *Linear Algebra and Its Applications*, 665:196–210, 2023.
- [13] I. Kaygorodov, V. Lopatkin, and Z. Zhang. Transposed Poisson structures on Galilean and solvable Lie algebras. *Journal of Geometry and Physics*, 187(Paper No. 104781):13 pp, 2023.
- [14] P. S. Kolesnikov and A. A. Nesterenko. Conformal envelopes of Novikov–Poisson algebras. *Siberian Mathematical Journal*, 64(3):598–610, 2023.
- [15] P. S. Kolesnikov and A. S. Panasenko. Novikov commutator algebras are special. *Algebra and Logic*, 58(6):538–539, 2020.
- [16] P. S. Kolesnikov and B. Sartayev. On the special identities of Gelfand-Dorfman algebras. *Experimental Mathematics*, DOI:10.1080/10586458.2022.2041134, 2022.
- [17] P. S. Kolesnikov, B. Sartayev, and A. Orazgaliev. Gelfand-Dorfman algebras, derived identities, and the Manin product of operads. *Journal of Algebra*, 539:260–284, 2019.
- [18] P. S. Kolesnikov and B. K. Sartayev. On the embedding of left-symmetric algebras into differential Perm algebras. *Communications in Algebra*, 50(8):3246–3260, 2022.
- [19] Y. P. Lee. Quantum K-theory, I: Foundations. *Duke Mathematical Journal*, 121(3):389–424, 2004.
- [20] J. Liu, Y. Sheng, and C. Bai. F-Manifold algebras and deformation quantization via pre-Lie algebras. *Journal of Algebra*, 559:467–495, 2020.
- [21] P. Lorenzoni, M. Pedroni, and A. Raimondo. F-manifolds and integrable systems of hydrodynamic type. *Archivum Mathematicum (Brno)*, 47(3):163–180, 2011.
- [22] S. A. Merkulov. Operads, deformation theory and F-manifolds. *Aspects of Mathematics*, 36:213–251, 2004.
- [23] A. I. Shirshov. On free Lie rings. *Matematicheskii Sbornik*, 42(2):113–122, 1958.
- [24] X. Xu. Quadratic Conformal Superalgebras. *Journal of Algebra*, 231(1):1–38, 2000.
- [25] X. Xu. Gel’fand-Dorfman bialgebras. *Southeast Asian Bulletin of Mathematics*, 27:561–574, 2003.

*Received:* May 23, 2023

*Accepted for publication:* September 20, 2023

*Communicated by:* Ivan Kaygorodov, Adam Chapman, Mohamed Elhamdadi