

Non-associative algebraic structures: classification and structure*

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Abstract. These are detailed notes for a lecture on “Non-associative Algebraic Structures: Classification and Structure” which I presented as part of my Agregação em Matemática e Aplicações (University of Beira Interior, Covilhã, Portugal, 13-14/03/2023).

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1 Introduction

In this lecture, we aim to summarize our research interests and achievements as well as motivate what drives our work. The present work is based on various papers written together with Abror Khudoyberdiyev, Alexandre Pozhidaev, Aloberdi Sattarov, Amir Fernández Ouaridi, Antonio Jesús Calderón, Artem Lopatin, Bakhrom Omirov, Bruno Ferreira, Crislaine Kuster, Doston Jumaniyozov, Elisabete Barreiro, Farukh Mashurov, Feng Wei, Henrique Guzzo, Gulkhayo Solijanov, Iqboljon Karimjanov, Ilya Gorshkov, Inomjon Yuldashev, Isabel Hernández, Isamiddin Rakhimov, Ivan Shestakov, Jobir Adashev, José María Sánchez, Karimbergan Kudaybergenov, Luisa Camacho, Manat Mustafa, Manuel Ladra, María Alejandra Alvarez, Mikhail Ignatyev, Mohamed Salim, Mykola Khrypchenko, Nurlan Ismailov, Pasha Zusmanovich, Patrícia Beites, Paulo Saraiva, Pilar Páez-Guillán, Renato Fehlberg Júnior, Samuel Lopes, Sharifah Kartini Said Husain, Thiago Castilho de Mello, Ualbai Umirbaev, Vasily Voronin, Viktor Lopatkin, Yury Popov and Yury Volkov [1, 3, 4, 7–9, 22, 24, 28, 44, 46–48, 50, 56–59, 87–93, 108–110, 119–123, 128–131, 137–142, 144–146, 150–178, 180].

The main direction of our research is concentrated on non-associative algebraic structures, their classifications, and the study of their properties. Namely, we work with many

types of non-associative algebras and superalgebras, n -ary algebras and also algebraic structures with two binary multiplications (known as algebras of Poisson type). We consider small-dimensional algebras, simple and semisimple algebras, PI-algebras, and free algebras.

1.1 Non-associative algebras

Unless otherwise noted, all vector spaces and algebras are considered over the complex field. In our work, “algebra” will mean “non-associative algebra”, i.e., a vector space A equipped with a bilinear product which need not be associative or unital. In the case of associative algebras, we work with algebras defined by the following identity: $(xy)z = x(yz)$. On the other hand, in non-associative cases, there are algebras defined by some special conditions (not identities!), such as axial algebras, evolution algebras, composition algebras, conservative algebras, etc., and also algebras defined by polynomial identities. It seems that the first example of non-associative algebras defined by some family of polynomial identities are Lie algebras. All these algebras are closely related to associative algebras. For example, each associative algebra under the commutator product $[x, y] := xy - yx$ gives a Lie algebra (the variety of algebras satisfying this property is called Lie-admissible algebras).

Commutative associative algebras and Lie algebras admit many generalizations, which have triggered some interest. So, the varieties of Jordan algebras, alternative algebras, right alternative algebras, assosymmetric algebras, bicommutative algebras, Novikov algebras, left-symmetric algebras, right commutative algebras, \mathfrak{CD} -algebras, weakly associative algebras, noncommutative Jordan algebras or terminal algebras, are generalizations of commutative associative algebras. On the other hand, the varieties of Malcev, symmetric Leibniz, Leibniz, binary Lie, \mathfrak{CD} -algebras, and also noncommutative Jordan algebras and terminal algebras can be seen as generalizations of Lie algebras. Some types of algebras appeared in another way: Zinbiel algebras are Koszul dual to Leibniz algebras, Tortkara algebras are defined by identities that are found in Zinbiel algebras under the commutator multiplication; mock Lie algebras are commutative analogs of Lie algebras; and dual mock Lie algebras are Koszul dual to mock Lie algebras. There are many other relations between cited varieties, such as the following. Lie algebras and Jordan algebras are related by Kantor–Koecher–Tits construction. Alternative and noncommutative Jordan algebras under the symmetric product $x \bullet y = xy + yx$ give Jordan algebras, assosymmetric algebras under the symmetric product give commutative \mathfrak{CD} -algebras; Novikov, bicommutative, assosymmetric, and left-symmetric algebras are Lie-admissible algebras (i.e., under the commutator product they give Lie algebras) and so on.

Below, we provide the list of identities that defined the varieties of algebras that are under consideration in this manuscript.

$(-1, 1)$ -	:	$(xy)y = xy^2, (x, y, z) + (z, x, y) + (y, z, x) = 0$
Alternative	:	$(x, y, z) = -(y, x, z), (x, y, z) = -(x, z, y)$
Antiassociative	:	$(xy)z = -x(yz)$

Assosymmetric	: $(x, y, z) = (y, x, z), (x, y, z) = (x, z, y)$
Bicommutative	: $x(yz) = y(xz), (xy)z = (xz)y$
Binary Lie	: $xy = -yx, ((xy)y)x = ((xy)x)y$
\mathfrak{CD} -	: $((xy)a)b - ((xy)b)a = ((xa)b - (xb)a)y + x((ya)b - (yb)a),$ $(a(xy))b - a((xy)b) = ((ax)b - a(xb))y + x((ay)b - a(yb)),$ $a(b(xy)) - b(a(xy)) = (a(bx) - b(ax))y + x(a(by) - b(ay))$
Dual mock-Lie	: $xy = -yx, (xy)z = -x(yz)$
Jordan	: $xy = yx, (x^2y)x = x^2(yx)$
Left-symmetric	: $(x, y, z) = (y, x, z)$
Leibniz	: $(xy)z = (xz)y + x(yz)$
Malcev	: $xy = -yx, (xy)(xz) + (y(xz))x +$ $+((xz)x)y = ((xy)z + (yz)x + (zx)y)x$
Noncommutative Jordan	: $(xy)x = x(yx), (x^2y)x = x^2(yx)$
Novikov	: $(xy)z = (xz)y, (x, y, z) = (y, x, z)$
Right alternative	: $(x, y, z) = -(x, z, y)$
Right commutative	: $(xy)z = (xz)y$
Symmetric Leibniz	: $x(yz) = (xy)z + y(xz), (xy)z = (xz)y + x(yz)$
Terminal	: $3(b(a(xy) - (ax)y - x(ay)) - a((bx)y) + (a(bx))y +$ $(bx)(ay) - a(x(by)) + (ax)(by) + x(a(by))) =$ $= -(2ab + ba)(xy) + ((2ab + ba)x)y + x((2ab + ba)y)$
Tortkara	: $xy = -yx, (xy)(zy) = ((xy)z + (yz)x + (zx)y)y$
Weakly associative	: $(xy)z - x(yz) + (yz)x - y(zx) = (yx)z - y(xz)$
Zinbiel	: $(xy)z = x(yz + zy)$

Superalgebras. Let \mathbb{F} be a field and let G be the Grassmann algebra over \mathbb{F} given by the generators $1, \xi_1, \dots, \xi_n, \dots$ and the defining relations $\xi_i^2 = 0$ and $\xi_i \xi_j = -\xi_j \xi_i$. The elements $1, \xi_{i_1} \xi_{i_2} \dots \xi_{i_k}, i_1 < i_2 < \dots < i_k$, form a basis of the algebra G over \mathbb{F} . Denote by G_0 and G_1 the subspaces spanned by the products of even and odd lengths, respectively; then G can be represented as the direct sum of these subspaces, $G = G_0 \oplus G_1$. Here the relations

$$G_i G_j \subseteq G_{i+j \pmod{2}}, \quad i, j = 0, 1,$$

hold. In other words, G is a \mathbb{Z}_2 -graded algebra (or a superalgebra) over \mathbb{F} . Suppose now that $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1$ is an arbitrary superalgebra over \mathbb{F} . Consider the tensor product $G \otimes \mathbf{A}$ of \mathbb{F} -algebras. The subalgebra

$$G(\mathbf{A}) = G_0 \otimes \mathbf{A}_0 + G_1 \otimes \mathbf{A}_1$$

of $G \otimes \mathbf{A}$ is referred to as the Grassmann envelope of the superalgebra \mathbf{A} . Let Ω be a variety of algebras over \mathbb{F} . A superalgebra $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1$ is referred to as an Ω -superalgebra if its Grassmann envelope $G(\mathbf{A})$ is an algebra in Ω . In particular, $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1$ is referred to as a (noncommutative) Jordan superalgebra if its Grassmann envelope $G(\mathbf{A})$ is a (noncommutative) Jordan algebra.

1.2 Poisson algebras

The notion of a Poisson bracket has its origin in the works of S.D. Poisson on celestial mechanics at the beginning of the XIX century. Since then, Poisson algebras have appeared in several areas of mathematics, such as: Poisson manifolds [198], algebraic geometry [29, 105], noncommutative geometry [239], operads [204], quantization theory [118, 191], quantum groups [78], and classical and quantum mechanics. The study of Poisson algebras also led to other algebraic structures, such as generic Poisson algebras [174, 190], algebras of Jordan brackets and generalized Poisson algebras [60, 179, 188, 206], Gerstenhaber algebras [192], Novikov-Poisson algebras [248], Malcev-Poisson-Jordan algebras, transposed Poisson algebras [20, 93], n -ary Poisson algebras [62], etc.

Definition 1. A Poisson algebra is a vector space A with two bilinear operations \cdot and $\{\cdot, \cdot\}$, such that (A, \cdot) is a commutative associative algebra, $(A, \{\cdot, \cdot\})$ is a Lie algebra and

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

Poisson algebras are at the intersection of Lie and commutative associative algebras. They inherit many common properties of these algebras, and they can be used to solve some well-known problems in the theory of Lie or associative algebras. So, Poisson algebras have been used to prove the Nagata conjecture about wild automorphisms of the polynomial ring with three generators [234] and to describe simple Jordan [207] and noncommutative Jordan [225, 226] algebras and superalgebras. Namely, from each simple Poisson algebra, it is possible to obtain simple Jordan superalgebra by a process named as “Kantor double”; but on the other hand, each noncommutative Jordan superalgebra under the commutator product gives a structure of Poisson bracket. The systematic study of free Poisson algebras began in the paper by Shestakov [233], where he constructed a basis for the free Poisson algebra. Later, in a series papers by Makar-Limanov and Umirbaev (see [200, 201] and references therein), many analogs of classical results for free Poisson algebras were obtained. For example, they proved that there are no wild automorphisms in the free Poisson algebra with two generators, proved the freedom theorem (The Freiheitssatz), and described universal multiplicative envelopings of the free Poisson fields. The systematic study of noncommutative Poisson algebra structures began in the paper by Kubo [195]. He obtained a description of all the Poisson structures on the full and upper triangular matrix algebras, which was later generalized to prime associative algebras in [86]. Namely, it was proved in [86] that any Poisson bracket on a prime noncommutative associative algebra is the commutator bracket multiplied by an element of the extended centroid of the algebra. On the other hand, in his next paper, Kubo studied noncommutative Poisson algebra structures on affine Kac-Moody algebras. The investigation of Poisson structures on associative algebras continued in some papers of Yao, Ye and Zhang [246]; Mroczyńska, Jaworska-Pastuszek, and Pogorzały [125, 215], where Poisson structures on finite-dimensional path algebras and on canonical algebras were studied. Crawley-Boevey introduced a noncommutative Poisson structure, called an H_0 -Poisson structure, on the 0-th cyclic homology of a noncommutative associative algebra [69] and showed that an H_0 -Poisson structure can

be induced on the affine moduli space of (semisimple) representations of an associative algebra from a suitable Lie algebra structure on the 0-th Hochschild homology of the algebra [69]. The derived noncommutative Poisson bracket on Koszul Calabi-Yau algebras has been studied in [65]. Van den Bergh introduced double Poisson algebras in [239], and Van de Weyer described all the double Poisson structures on finite-dimensional semisimple algebras. Recently, the notion of a noncommutative Poisson bialgebra appeared in [199].

1.3 n -Ary algebras

Following Kurosh, an Ω -algebra over a field \mathbb{F} is a linear space over \mathbb{F} equipped with a system of multilinear algebraic operations $\Omega = \{\omega_i : |\omega_i| = n_i, n_i \in \mathbb{N}\}$, where $|\omega_i|$ denotes the arity of ω_i . The present notion of Ω -algebras is very wide. It includes associative and non-associative algebras (with one binary multiplication); dialgebras and Poisson algebras and their generalizations (with two binary multiplications); trialgebras (with three binary multiplications); quadri-algebras (with four binary multiplications), ennea-algebras (with nine binary multiplications); pluriassociative algebras (with 2γ binary multiplications); Jordan triple disystems and comtrans algebras (with two ternary multiplications); Akivis algebras (with one binary and one ternary multiplications); Bol algebras, Lie-Yamaguti algebras, Poisson n -Lie algebras, Sabinin algebras and so on. A particular case of Ω -algebras, that will be interesting for us is n -ary algebras.

Let \mathbf{A} be an Ω -algebra equipped with one n -ary operation $[\cdot, \dots, \cdot] : \mathbf{A} \times \dots \times \mathbf{A} \rightarrow \mathbf{A}$, which is n -linear. This operation is n -ary commutative if,

$$[x_1, \dots, x_n] = [x_{\sigma(1)}, \dots, x_{\sigma(n)}], \text{ for all } \sigma \in \mathbb{S}_n.$$

There are notions of partial commutativity and partial associativity for n -ary algebras. A generalization of a variety of binary algebras defined by a family of polynomial identities to the n -ary case is not an easy task. So, there are many types of generalizations of associative and Lie algebras to n -ary case. For example, there are partially associative and totally associative n -ary algebras, which give associative algebras in the binary case. In the case of Lie algebras, the known generalizations are the following: reduced Lie ternary algebras introduced by Pojidaev and n -Lie algebras introduced by Filippov in [95]. Recently, the theory of Filippov algebras has attracted much attention due to its close connection with Nambu mechanics, proposed by Nambu as a generalization of classical Hamiltonian mechanics. In his article, Filippov introduced the notion of a n -Lie algebra and proved some of its properties. He also obtained the classification of anticommutative n -ary algebras of dimension n and $n + 1$.

Let us give the definition of n -Lie (Filippov) algebras. Let \mathbf{A} be an algebra equipped with one n -ary operation $[\cdot, \dots, \cdot] : \mathbf{A} \times \dots \times \mathbf{A} \rightarrow \mathbf{A}$, which is n -linear. This algebra is n -Lie if,

$$[x_1, \dots, x_n] = (-1)^\sigma [x_{\sigma(1)}, \dots, x_{\sigma(n)}], \text{ for all } \sigma \in \mathbb{S}_n;$$

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n].$$

The first identity gives an n -ary analog of anticommutative identity, and the second gives the n -ary version of the Jacobi identity. An algebra A is called n -ary Leibniz algebra if it satisfies only the second identity from the identities mentioned above.

2 Classifications of non-associative algebras

The present part is based on the papers written together with Abror Khudoyberdiyev, Alexandre Pozhidaev, Aloverdi Sattarov, Amir Fernández Ouaridi, Antonio Jesús Calderón, Crislaine Kuster, Doston Jumaniyozov, Farukh Mashurov, Iqboljon Karimjanov, Ilya Gorshkov, Isabel Hernández, Isamidin Rakhimov, Jobir Adashev, Luisa Camacho, Manat Mustafa, Manuel Ladra, María Alejandra Alvarez, Mikhail Ignatyev, Mohamed Salim, Mykola Khrypchenko, Nurlan Ismailov, Pilar Páez-Guillán, Samuel Lopes, Sharifah Kartini Said Husain, Thiago Castilho de Mello, Vasily Voronin, Yury Popov and Yury Volkov [1, 3, 4, 7–9, 46–48, 56–58, 88, 89, 108–110, 119–123, 128–131, 137, 138, 144, 146, 150–153, 155, 160–164, 170, 171, 173, 176–178].

2.1 The algebraic classification of algebras

The algebraic classification (up to isomorphism) of algebras of dimension n from a certain variety defined by a certain family of polynomial identities is a classic problem in the theory of non-associative algebras. There are many results related to the algebraic classification of small-dimensional algebras in many varieties of non-associative algebras [2, 63, 67, 70, 72, 177]. So, algebraic classifications of 2-dimensional algebras [177, 222], 3-dimensional evolution algebras [41], 3-dimensional anticommutative algebras [189], 4-dimensional division algebras [71, 75], have been given.

Our method for classifying nilpotent commutative algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety (first, this method has been developed by Skjelbred and Sund for the Lie algebra case in [235]).

Our big program in the classification of complex small-dimensional nilpotent algebras results in three big classifications of the following algebras:

- 4-dimensional nilpotent algebras [151];
- 5-dimensional commutative nilpotent algebras [129];
- 6-dimensional anticommutative nilpotent algebras [150].

On the other hand, during the realization of our classification program, we classified small-dimensional algebras in the following varieties:

- I. 4-dimensional nilpotent:
 - assosymmetric algebras [120];
 - bicommutative algebras [163];

- \mathfrak{CD} -algebras [146];
- commutative algebras [89];
- left-symmetric algebras [3];
- noncommutative Jordan algebras [128];
- Novikov algebras [137];
- right commutative algebras [4];
- right alternative algebras [121];
- terminal algebras [153];
- weakly associative algebras [9];

II. 5-dimensional nilpotent:

- antiassociative algebras [88];
- anticommutative algebras [89];
- commutative associative algebras [173];
- commutative \mathfrak{CD} -algebras [130];
- symmetric Leibniz algebras [9];
- one-generated assosymmetric algebras [162];
- one-generated bicommutative algebras [164];
- one-generated Novikov algebras [57];
- one-generated terminal algebras [155];

III. 6-dimensional nilpotent:

- anticommutative \mathfrak{CD} -algebras [1];
- binary Lie algebras [1];
- Tortkara algebras [108];
- one-generated assosymmetric algebras [162];
- one-generated bicommutative algebras [164];
- one-generated Novikov algebras [57].

The classification of non-nilpotent algebras is given by other methods, which were developed in our papers. Let us now summarize the rest of the results obtained in the algebraic classification of algebras.

- 2-dimensional terminal, conservative and rigid algebras [47];
- 2-dimensional algebras with left quasi-units [175];

- 2-dimensional algebras [177];
- 3-dimensional anticommutative algebras [123];
- 8-dimensional dual Mock Lie algebras [58];
- central extensions of n -dimensional filiform associative algebras [138];
- central extensions of n -dimensional filiform Zinbiel algebras [56];
- non-associative central extensions of n -dimensional null-filiform associative algebras [161];
- n -dimensional algebras with $(n - 2)$ -dimensional annihilator [48];
- n -dimensional anticommutative algebras with $(n - 3)$ -dimensional annihilator [46];
- n -dimensional Zinbiel algebras with $(n - 3)$ -dimensional annihilator [7].

2.2 The geometric classification of algebras

Given a complex n -dimensional vector space \mathbb{V} , the set

$$\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V}) \cong \mathbb{V}^* \otimes \mathbb{V}^* \otimes \mathbb{V}$$

is a complex vector space of dimension n^3 . This space has the structure of the affine variety \mathbb{C}^{n^3} . Indeed, if we fix a basis $\{e_1, \dots, e_n\}$ of \mathbb{V} , then any $\mu \in \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is *Zariski-closed* if it is the set of solutions of a system of polynomial equations in the variables c_{ij}^k ($1 \leq i, j, k \leq n$).

Let T be a set of polynomial identities. Every algebra structure on \mathbb{V} satisfying polynomial identities from T forms a Zariski-closed subset of the variety $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugation:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbb{V}$, $\mu \in \mathbb{L}(T) \subset \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ and $g \in \mathrm{GL}(\mathbb{V})$. Thus, $\mathbb{L}(T)$ decomposes into $\mathrm{GL}(\mathbb{V})$ -orbits that correspond to the isomorphism classes of the algebras. We shall denote by $O(\mu)$ the orbit of $\mu \in \mathbb{L}(T)$ under the action of $\mathrm{GL}(\mathbb{V})$ and by $\overline{O(\mu)}$ its Zariski closure.

One of the main problems of the geometric classification of a variety of algebras is a description of its irreducible components. In [102], Gabriel described the irreducible components of the variety of 4-dimensional unital associative algebras, and the variety of 5-dimensional unital associative algebras were classified algebraically and geometrically by Mazzola [212, 213]. Later, Cibils [66] considered rigid associative algebras with 2-step nilpotent radical. Goze and Ancochea-Bermúdez proved that the varieties of 7 and 8-dimensional nilpotent Lie algebras are reducible [13]. The irreducible components of 2-step nilpotent commutative associative algebras were described in [228]. Often, the

irreducible components of the variety are determined by the rigid algebras, although this is not always the case. Indeed, Flanigan showed in [98] that the variety of 3-dimensional nilpotent associative algebras has an irreducible component which does not contain any rigid algebras — it is instead defined by the closure of a union of a one-parameter family of algebras.

Let A_1 and A_2 be two n -dimensional algebras satisfying the identities from T , and let $\mu, \lambda \in \mathbb{L}(T)$ represent A_1 and A_2 , respectively. We say that A_1 *degenerates to* A_2 , and write $A_1 \rightarrow A_2$, if $\lambda \in \overline{O(\mu)}$. Note that this implies $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of degeneration does not depend on the choice of μ and λ . If $A_1 \not\cong A_2$, then the assertion $A_1 \rightarrow A_2$ is called a *proper degeneration*. Following Gorbatsevich [106], we say that A has *level* m if there exists a chain of proper degenerations of length m starting in A and there is no such chain of length $m + 1$. Also, in [107] it was introduced the notion of *infinite level* of an algebra A as the limit of the usual levels of $A \oplus \mathbb{C}^m$.

Let A be represented by $\mu \in \mathbb{L}(T)$. Then A is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called *irreducible* if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be uniquely represented as a finite union of its irreducible components. Note that the algebra A is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

In our work, we discuss the following problems:

Problem 1 (Geometric classification). Let \mathfrak{V}^n be a variety of n -dimensional algebras defined by a family of identities T . Which are the irreducible components of \mathfrak{V}^n ?

Problem 2 (Degeneration classification). Let \mathfrak{V}^n be a variety of n -dimensional algebras defined by a family of identities T . Is there a degeneration from A_1 to A_2 for each pair of algebras from \mathfrak{V}^n ?

Problem 3 (Level classification). Let \mathfrak{V}^n be a variety of n -dimensional algebras defined by a family of identities T . Which algebras from \mathfrak{V}^n have level m ?

In the following list, we summarize our main results related to the geometric classification of algebras. Namely, we gave the geometric classification of the following varieties of algebras:

I. 4-dimensional nilpotent:

- assosymmetric algebras [120];
- bicommutative algebras [163];
- \mathcal{CD} -algebras [144];
- commutative algebras [89];
- left-symmetric algebras [3];
- Leibniz algebras [170];

- noncommutative Jordan algebras [128];
- Novikov algebras [137];
- right commutative algebras [4];
- right alternative algebras [121];
- terminal algebras [153];
- weakly associative algebras [9];

II. 5-dimensional nilpotent:

- antiassociative algebras [88];
- associative algebras [119];
- commutative \mathfrak{CD} -algebras [131];
- symmetric Leibniz algebras [9];

III. 6-dimensional nilpotent:

- anticommutative \mathfrak{CD} -algebras [1];
- binary Lie algebras [1];
- Malcev algebras [171];
- Tortkara algebras [109];

IV. 4-dimensional:

- binary Lie algebras [171];
- Leibniz algebras [122];
- Zinbiel algebras [170];

V. 7-dimensional:

- dual Mock Lie algebras [58].

In [119, 152] we completely solve the geometric classification problem for nilpotent and 2-step nilpotent, commutative nilpotent and anticommutative nilpotent algebras of arbitrary dimension.

Theorem 1 (Theorem A, [152]). *For any $n \geq 2$, the variety of all n -dimensional nilpotent algebras is irreducible and has dimension $\frac{n(n-1)(n+1)}{3}$.*

Moreover, we show that the family \mathcal{R}_n given in [152, Definition 10] is generic in the variety of n -dimensional nilpotent algebras and inductively gives an algorithmic procedure to obtain any n -dimensional nilpotent algebra as a degeneration from \mathcal{R}_n .

Theorem 2 (Theorem B, [152]). *For any $n \geq 2$, the variety of all n -dimensional commutative nilpotent algebras is irreducible and has dimension $\frac{n(n-1)(n+4)}{6}$.*

As above, we show that the family \mathcal{S}_n given in [152, Definition 15] is generic in the variety of n -dimensional commutative nilpotent algebras and inductively give an algorithmic procedure to obtain any n -dimensional nilpotent commutative algebra as a degeneration from \mathcal{S}_n .

Theorem 3 (Theorem C, [152]). *For any $n \geq 2$, the variety of all n -dimensional anticommutative nilpotent algebras is irreducible and has dimension $\frac{(n-2)(n^2+2n+3)}{6}$.*

We show also that the family \mathcal{T}_n given in [152, Definition 33], is generic in the variety of n -dimensional anticommutative nilpotent algebras and inductively give an algorithmic procedure to obtain any n -dimensional nilpotent anticommutative algebra as a degeneration from \mathcal{T}_n .

The notion of length for non-associative algebras has been recently introduced in [113], generalizing the corresponding notion for associative algebras. Using the above result, we show in [152, Corollary 39] that the length of an arbitrary (i.e., not necessarily nilpotent) n -dimensional anticommutative algebra is bounded above by the n^{th} Fibonacci number, and prove that our bound is sharp.

For $k \leq n$ consider the (algebraic) subset $\mathfrak{Nil}_{n,k}^2$ of the variety \mathfrak{Nil}_n^2 of 2-step nilpotent n -dimensional algebras defined by

$$\mathfrak{Nil}_{n,k}^2 = \{A \in \mathfrak{Nil}_n^2 : \dim A^2 \leq k, \dim \text{ann } A \geq k\}.$$

It is easy to see that $\mathfrak{Nil}_n^2 = \bigcup_{k=1}^n \mathfrak{Nil}_{n,k}^2$. Analogously, for the varieties $\mathfrak{Nil}_n^{2c}, \mathfrak{Nil}_n^{2ac}$ of commutative and anticommutative 2-step nilpotent algebras we define the subsets $\mathfrak{Nil}_{n,k}^{2c}$ and $\mathfrak{Nil}_{n,k}^{2ac}$, respectively.

Theorem 4 (Theorem A, [119]). *The sets $\mathfrak{Nil}_{n,k}^2$ are irreducible and*

$$\begin{aligned} \mathfrak{Nil}_n^2 &= \bigcup_k \mathfrak{Nil}_{n,k}^2, \quad \text{for } 1 \leq k \leq \left\lfloor n + \frac{1-\sqrt{4n+1}}{2} \right\rfloor \\ \mathfrak{Nil}_n^{2c} &= \bigcup_k \mathfrak{Nil}_{n,k}^{2c}, \quad \text{for } 1 \leq k \leq \left\lfloor n + \frac{3-\sqrt{8n+9}}{2} \right\rfloor, \\ \mathfrak{Nil}_n^{2ac} &= \bigcup_k \mathfrak{Nil}_{n,k}^{2ac}, \quad \text{for } 1 + (n+1) \bmod 2 \leq k \leq \left\lfloor n + \frac{1-\sqrt{8n+1}}{2} \right\rfloor \text{ for } n \geq 3. \end{aligned}$$

Moreover,

$$\begin{aligned} \dim \mathfrak{Nil}_{n,k}^2 &= (n-k)^2 k + (n-k)k, \\ \dim \mathfrak{Nil}_{n,k}^{2c} &= \frac{(n-k)(n-k+1)}{2} k + (n-k)k, \\ \dim \mathfrak{Nil}_{n,k}^{2ac} &= \frac{(n-k)(n-k-1)}{2} k + (n-k)k. \end{aligned}$$

2.3 Degenerations of algebras

The present part is dedicated to the results obtained regarding Problem 2 (Degeneration classification) and Problem 3 (Level classification). Typically, the results of degeneration classification are given by construction of the graph of primary degenerations

of the algebras from the variety. So, Grunewald and O'Halloran constructed the graph of primary degenerations for 5-dimensional nilpotent Lie algebras [112]; Seeley obtained a similar graph for 6-dimensional nilpotent Lie algebras in [227]. The degenerations graphs for 2-dimensional Jordan algebras and 4-dimensional nilpotent Jordan algebras are presented in [11, 12]. Other graphs of degenerations were constructed by Burde and co-authors for 4-dimensional Lie algebras [40], 2-dimensional Pre-Lie algebras [30] and 3-dimensional Novikov algebras [31].

In the following list, we summarize our main results related to the degeneration classification of algebras. Namely, we constructed graphs of primary degenerations of the following varieties of algebras:

- 2-dimensional algebras [177];
- 3-dimensional anticommutative algebras [123];
- 3-dimensional Jordan algebras [110];
- 3-dimensional Jordan superalgebras [8];
- 3-dimensional Leibniz algebras [123];
- 3-dimensional nilpotent algebras [89];
- 4-dimensional binary Lie algebras [171];
- 4-dimensional nilpotent commutative algebras [89];
- 4-dimensional nilpotent Leibniz algebras [170];
- 4-dimensional Zinbiel algebras [170].
- 5-dimensional nilpotent anticommutative algebras [89];
- 5-dimensional nilpotent commutative associative algebras [160].
- 6-dimensional nilpotent Malcev algebras [171];
- 7-dimensional dual Mock Lie algebras [58];
- $(n + 1)$ -dimensional n -Lie (Filippov) algebras [178].

The first attempt at classification of algebras of level one is given by Gorbatsevich [106], but he omitted some main cases. Later, his classification was completed in a paper by Khudoyberdiyev and Omirov [185]. After this, the classification of algebras of level two for some particular cases, such that associative algebras, Jordan algebras, Lie algebras, Leibniz algebras and nilpotent algebras has been given in [101, 184]. In [176] we completely solve the level classification problem for all algebras in the case of level two, generalizing all the cited results of Khudoyberdiyev and co-authors. Following our ideas on the level

classification of algebras, Volkov recently obtained some results on the classifications of n -ary algebras of level one and on classification of anticommutative algebras of levels 3, 4 and 5 [240, 241].

3 Conservative algebras and superalgebras

The present part is based on the papers written together with Alexandre Pozhidaev, Artem Lopatin, Renato Fehlbeg Júnior, Yury Popov, and Yury Volkov [87, 142, 157, 169, 175].

3.1 Kantor product

The idea of obtaining new objects from old ones by using derivative operations has long been known in algebra [5]. In its most general form, the idea was realized by Malcev [203]. Let M_n be an associative algebra of matrices of order n over a field \mathbb{F} . Assume that some finite collection $\Lambda = (a_{ij}, b_{ij}, c_{ij})$ of matrices in M_n is given. Denote by $M_n^{(\Lambda)}$ the algebra defined on a space of matrices in M_n with respect to new multiplication

$$x \cdot_{\Lambda} y = \sum_{i,j} a_{ij} x b_{ij} y c_{ij}.$$

It was proved that every n -dimensional algebra over \mathbb{F} is isomorphic to a subalgebra of $M_n^{(\Lambda)}$ [203]. Other interesting ways to derive the initial multiplication are isotopes, homotopes, and mutations [6, 84, 220]. The concept of an isotope was introduced by Albert [5]. Let A and A_0 be algebras with a common underlying linear space on which right multiplication operators R_x and $R_x^{(0)}$ are defined (for A and A_0 , resp.). We say that A_0 and A are isotopic if there exist invertible linear operators ϕ, ψ, ξ such that $R_x^{(0)} = \phi R_{x\psi} \xi$. We call A_0 an isotope of A . Let A be an arbitrary associative algebra, and let p, q be two fixed elements of A . Then a new algebra is derived from A by using the same vector space structure of A but defining a new multiplication $x * y = xpy - yqx$. The resulting algebra is called the (p, q) -mutation of the algebra A .

The definition of the Kantor product of multiplications comes from the study of a certain class of algebras. In 1972, Kantor introduced the class of conservative algebras [133], which contains many important classes of algebras (see [157]), for example, associative, Lie, Jordan, and Leibniz algebras. To define what will be called the Kantor product, we need to introduce the algebras $U(n)$ (see, for more details, [135, 157]). Consider the space $U(n)$ of all bilinear multiplications on the n -dimensional vector space V_n . Now, fix a vector $u \in V_n$. For $A, B \in U(n)$ (two multiplications) and $x, y \in V_n$, we set

$$x * y = \llbracket A, B \rrbracket(x, y) = A(u, B(x, y)) - B(A(u, x), y) - B(x, A(u, y)).$$

This new multiplication is called the (left) Kantor product of the multiplications A and B (it is also possible to define the right Kantor product). The Kantor product of a multiplication “ \cdot ” by itself will be the Kantor square of “ \cdot ”:

$$x * y = u \cdot (x \cdot y) - (u \cdot x) \cdot y - x \cdot (u \cdot y).$$

It is easy to see that the Kantor square of multiplication is a particular case of the Malcev construction in a non-associative sense. On the other hand,

- in the commutative associative case it coincides with a mutation;
- in the left commutative and left-symmetric cases it coincides with an isotope.

As in [142], we will assume that the Kantor product is always the left Kantor product. In [142], we studied the Kantor product and Kantor square of many well-known algebras, for example, associative, (anti)-commutative, Lie, Leibniz, Novikov, Poisson algebras, and dialgebras.

We summarize the main results of the present part in the following theorems.

Theorem 5 (Theorem 29, [142]). *Let (A, \cdot) be a finite-dimensional associative algebra. Then (A, \cdot) is isomorphic to $(A, \llbracket \cdot, \cdot \rrbracket)$, if and only if A is a skew field.*

Theorem 6 (Theorem 10, [142]). *Let $(A; \cdot)$ be an alternative algebra. Then $(A; \llbracket \cdot, \cdot \rrbracket)$ is a flexible algebra. Furthermore,*

- $(A; \llbracket \cdot, \cdot \rrbracket)$ is an alternative algebra if and only if A satisfies the identity

$$(x, u, (x, u, y)) = 0;$$

- $(A; \llbracket \cdot, \cdot \rrbracket)$ is a noncommutative Jordan algebra if and only if A satisfies the identity

$$[L_u L_x L_u L_x, R_u R_x] = [L_{xuxu}, R_{ux}];$$

- $(A; \llbracket \cdot, \cdot \rrbracket)$ is a Jordan algebra if (A, \cdot) is a commutative alternative algebra;
- $(\mathbf{C}; \llbracket \cdot, \cdot \rrbracket)$ is alternative for a Cayley — Dickson algebra \mathbf{C} , if and only if $u = u_0 \cdot 1$.

Theorem 7 (Theorem 19, [142]). *Let $(A; \cdot, \{\cdot, \cdot\})$ be a generalized Poisson algebra. Then $(A; \llbracket \{\cdot, \cdot\}, \cdot \rrbracket)$ is an associative-commutative algebra, and $(A; \llbracket \cdot, \{\cdot, \cdot\} \rrbracket)$ is a Lie algebra.*

Theorem 8 (Proposition 18, [87]). *Let $(A; \cdot, \{\cdot, \cdot\})$ be a Poisson algebra. Then*

$$(A; \llbracket \cdot + \{\cdot, \cdot\}, \cdot + \{\cdot, \cdot\} \rrbracket)$$

is a noncommutative Jordan algebra.

Theorem 9 (Corollary 31, [87]). *Let $(A; \cdot, \circ)$ be a left Novikov-Poisson algebra. Then $(A, \llbracket \circ, \cdot \rrbracket, \llbracket \cdot, \circ \rrbracket)$ is a left Novikov-Poisson algebra.*

Also, we give constructive methods for obtaining new transposed Poisson algebras and Poisson-Novikov algebras; and for classifying Poisson structures and commutative post-Lie structures on a given algebra and construct some new transposed Poisson algebras.

Let us now consider a special example of transposed Poisson algebras constructed in [93, Theorem 25]. The transposed Poisson algebra $(\mathcal{W}, \cdot, [\cdot, \cdot])$ is spanned by the generators $\{L_i, I_j\}_{i,j \in \mathbb{Z}}$. These generators satisfy

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n}, & [L_m, I_n] &= (m - n - a)I_{m+n}, \\ L_m \cdot L_n &= wL_{m+n}, & L_m \cdot I_n &= wI_{m+n}, \end{aligned}$$

where w is a fixed element from the vector space generated by $\{L_i, I_j\}_{i,j \in \mathbb{Z}}$ and the multiplication given by juxtaposition satisfies $L_i L_j = L_{i+j}$ and $L_i I_j = I_{i+j}$.

Theorem 10 (Proposition 27, [87]). *Let $\star = \llbracket [\cdot, \cdot], \cdot \rrbracket$ and $\{\cdot, \cdot\} = \llbracket \cdot, [\cdot, \cdot] \rrbracket$ be new multiplications defined on multiplications of the transposed Poisson algebra $(\mathcal{W}, \cdot, [\cdot, \cdot])$ defined above. Then $(\mathcal{W}, \star, \{\cdot, \cdot\})$ is a transposed Poisson algebra.*

3.2 Universal conservative superalgebras

Kantor introduced the class of conservative algebras in [133]. This class includes some well-known classes of algebras, such as associative, Jordan, Lie, Leibniz and Zinbiel [157]. To define conservative algebras we first introduce some notations. Let V be a vector space over a field \mathbb{F} , let φ be a linear map on V , and let B be a bilinear map on V (i. e., an algebra). Then we can consider a product of φ and B , which is the bilinear map $[\varphi, B]$ on V given by

$$[\varphi, B](x, y) = \varphi(B(x, y)) - B(\varphi(x), y) - B(x, \varphi(y)).$$

Note that this product measures how far is φ from being a derivation in the algebra B .

We say that an algebra A with a multiplication \cdot is called a (left) conservative algebra if there exists a (possibly different) algebra structure $*$ (called an associated algebra) on the underlying space of A such that

$$[L_a, [L_b, \cdot]] = -[L_{a*b}, \cdot].$$

By replacing the left multiplications with the right multiplications, we can define the right Kantor product and obtain a similar theory. In the theory of conservative algebras, the conservative algebra $U(n)$ introduced in Section 3.1 has high importance. One can verify that the algebra $U(n)$ is conservative with the associated multiplication $*$ given, for example, by $A * B(x, y) = -B(u, A(x, y))$ (there are other associated multiplications). In the theory of conservative algebras, the algebra $U(n)$ plays a role analogous to the role of \mathfrak{gl}_n in the theory of Lie algebras, that is, any conservative algebra (modulo its maximal Jacobian ideal) embeds in $U(n)$ for certain n . In 1989, Kantor defined one generalization of conservative algebras, which he called quasi-conservative algebras (now known as “rigid” algebras, see [61]). In 2010, Kac and Cantarini considered “rigid” (commutative and anticommutative) superalgebras and described simple superalgebras under some special conditions [61].

One may also use the general approach to define conservative superalgebras. Namely, let $\Gamma := \Gamma_0 \oplus \Gamma_1$ be the Grassmann superalgebra in generators $1, \xi_i, i \in \mathbb{N}$,

$$\Gamma_0 = \langle 1, \xi_{i_1} \dots \xi_{i_{2k}} : k \in \mathbb{N} \rangle, \Gamma_1 = \langle \xi_{i_1} \dots \xi_{i_{2k-1}} : k \in \mathbb{N} \rangle.$$

Let $\mathbf{A} := \mathbf{A}_0 \oplus \mathbf{A}_1$ be a superalgebra and \cdot and $*$ be two products on \mathbf{A} . Consider its *Grassmann envelope* $\Gamma(\mathbf{A}) := (\mathbf{A}_0 \otimes \Gamma_0) \oplus (\mathbf{A}_1 \otimes \Gamma_1)$ and extend the products \cdot and $*$ to $\Gamma(\mathbf{A})$ as follows:

$$\begin{aligned}(a \otimes f) \cdot (b \otimes g) &= (-1)^{ab} ab \otimes fg, \\ (a \otimes f) * (b \otimes g) &= (-1)^{ab} a * b \otimes fg\end{aligned}$$

for all homogeneous $a, b \in \mathbf{A}, f, g \in \Gamma$ ($p(a) = p(f), p(b) = p(g)$). Then (\mathbf{A}, \cdot) is conservative with an associated multiplication $*$ if and only if $(\Gamma(\mathbf{A}), \cdot)$ is a conservative algebra with an associated multiplication $*$.

The following theorem provides different examples of conservative superalgebras.

Theorem 11 (Theorem 2, [169]). *Let \mathcal{V} be a homogeneous variety of algebras. Assume that there exist $\alpha, \beta \in \mathbb{F}$ such that every \mathcal{V} -algebra is conservative with the associated multiplication is given by the rule $a * b = \alpha ab + \beta ba$. Then every \mathcal{V} -superalgebra is conservative with the associated multiplication $a * b = \alpha ab + (-1)^{ab} \beta ba$.*

It follows that associative, quasi-associative, Jordan, terminal, Lie, Leibniz, and Zinbiel superalgebras are conservative (see [157]). In particular, a superalgebra M is terminal if and only if it is conservative and the multiplication in the associated superalgebra M^* can be given by

$$M^*(x, y) = \frac{2}{3}xy + (-1)^{xy} \frac{1}{3}yx.$$

Theorem 12 (Proposition 4, [169]). *A conservative superalgebra \mathbf{A} with a unity is a non-commutative Jordan superalgebra.*

Let V be a superspace. The space of the superalgebra $U(V)$ is the superspace of all bilinear operations on V . Fix a nonzero homogeneous $a \in V$. Define the multiplication Δ_a in $U(V)$ by the rule

$$(A\Delta_a B)(x, y) = A(a, B(x, y)) - (-1)^{B(A+a)} B(A(a, x), y) - (-1)^{(A+a)(B+x)} B(x, A(a, y)).$$

Consider the natural action of the group $\text{GL}(V)$ of even automorphisms of V on $U(V)$:

$$\varphi(A)(x, y) = \varphi(A(\varphi^{-1}(x), \varphi^{-1}(y)))$$

(note that we denote an automorphism and its action by the same symbol φ). A direct computation shows that the mapping $A \mapsto \varphi(A)$ is an isomorphism between $(U(V), \Delta_a)$ and $(U(V), \Delta_{\varphi(a)})$. Therefore, different nonzero even (respectively, odd) vectors a give rise to isomorphic *even* (respectively, *odd*) superalgebras, which we denote by $U(V)^0$ and $U(V)^1$, respectively. Moreover, consider the opposite superspace V^Π given by $V_0^\Pi = V_1, V_1^\Pi = V_0$. Then the parity-reversing isomorphism $V \cong V^\Pi$ induces an isomorphism between $U(V^\Pi)^1$ and the odd superalgebra obtained from $U(V)^0$ by reversing the parity. Therefore, it suffices to consider only the superalgebras $U(V)^0$. For the sake of simplicity, we denote them by $U(V)$. If $V = V_{n,m}$ is a finite-dimensional superspace with $\dim V_0 = n$ and $\dim V_1 = m$ then we denote $U(V)$ by $U(n, m)$. Further, in this case, we say that V is of dimension $n + m$.

Theorem 13 (Theorem 8, [169]). *Let V be a superspace, and let $a \in V_{\bar{0}}$. The superalgebra $(U(V), \Delta_a)$ is conservative, and the associated multiplication can be given by*

$$A \nabla_a^1 B(x, y) = -(-1)^{AB} B(a, A(x, y))$$

or

$$A \nabla_a^2 B(x, y) = \frac{1}{3}(A^* \Delta_a B + (-1)^{AB} \tilde{B} \Delta_a A),$$

where $A^*(x, y) = A(x, y) + (-1)^{xy} A(y, x)$ and $\tilde{B}(x, y) = 2(-1)^{xy} B(y, x) - B(x, y)$.

An even element $e \in \mathbf{A}$ is said to be a *left quasiunity* if the equality

$$e(xy) = (ex)y + x(ey) - xy$$

holds.

An element a in a superalgebra \mathbf{A} is called a *Jacobi element* provided that

$$a(xy) = (ax)y + (-1)^{ax} x(ay).$$

Let \mathbf{A} be a conservative superalgebra on a space V with the Jacobi subspace J . Consider the space W , which we define as $W = V/J$ if \mathbf{A} has a left quasiunity, and $W = V/J \oplus E$ in the opposite case, where E is the one-dimensional even space with a basis element ϵ .

Assume that \mathbf{A} possesses a quasiunity. Define the *adjoint mapping* $\text{ad} : \mathbf{A} \rightarrow U(W)$ as follows:

$$\text{ad}(a)(\alpha, \beta) = (-1)^{\beta(a+\alpha)}((\beta * a) * \alpha + (-1)^{\alpha a} \beta * (\alpha a) - (-1)^{\alpha a} (\beta * \alpha) * a).$$

If \mathbf{A} does not have a quasiunity, we define the adjoint mapping $\text{ad} : \mathbf{A} \rightarrow U(W)$ by the equation above and the following equations:

$$\begin{aligned} \text{ad}(a)(\alpha, \epsilon) &= a * \alpha + (-1)^{\alpha a} \alpha a - (-1)^{\alpha a} \alpha * a, \\ \text{ad}(a)(\epsilon, \beta) &= (-1)^{\alpha \beta} \beta * a, \quad \text{ad}(a)(\epsilon, \epsilon) = a. \end{aligned}$$

Theorem 14 (Theorem 11, [169]). *Let \mathbf{A} be a conservative superalgebra on a vector space V with the Jacobi subspace J . Let either $W = V/J$ or $W = V/J \oplus \langle \epsilon \rangle$ as above. The adjoint mapping $\text{ad} : \mathbf{A} \rightarrow (U(W), \Delta_{-\epsilon})$ is a homomorphism whose kernel is the maximal Jacobi ideal. In particular, if V is finite-dimensional and J is of codimension $n + m$, then we have a homomorphism $\text{ad} : \mathbf{A} \rightarrow U(k, m)$, where $k = n$ if \mathbf{A} has a quasiunity and $k = n + 1$ otherwise.*

3.3 Conservative algebras of 2-dimensional algebras

Multiplication on a 2-dimensional vector space is defined by a $2 \times 2 \times 2$ matrix. The classification of algebras of dimension 2 was given in many papers (see, for example, [177]). Let us consider the space $U(2)$ of all multiplications on the 2-dimensional space V_2 with a basis v_1, v_2 . The definition of the multiplication on the algebra $U(2)$ can be found in Section 3.1. The algebra $U(2)$ is conservative. Let us consider the multiplications $\alpha_{i,j}^k$

$(i, j, k = 1, 2)$ on V_2 defined by the formula $\alpha_{i,j}^k(v_t, v_l) = \delta_{it}\delta_{jl}v_k$ for all t, l . It is easy to see that $\{\alpha_{i,j}^k | i, j, k = 1, 2\}$ is a basis of the algebra $U(2)$. The multiplication table of $U(2)$ in this basis is given in [157]. But in the next work [175], we found another useful basis that will be interesting to consider. Let's introduce the notation

$$\begin{aligned} e_1 &= \alpha_{11}^1 - \alpha_{12}^2 - \alpha_{21}^2, & e_2 &= \alpha_{11}^2, & e_3 &= \alpha_{22}^2 - \alpha_{12}^1 - \alpha_{21}^1, & e_4 &= \alpha_{22}^1, \\ e_5 &= 2\alpha_{11}^1 + \alpha_{12}^2 + \alpha_{21}^2, & e_6 &= 2\alpha_{22}^2 + \alpha_{12}^1 + \alpha_{21}^1, & e_7 &= \alpha_{12}^1 - \alpha_{21}^1, & e_8 &= \alpha_{12}^2 - \alpha_{21}^2. \end{aligned}$$

It is easy to see that the multiplication table of $U(2)$ in the basis e_1, \dots, e_8 is the following.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	$-e_1$	$-3e_2$	e_3	$3e_4$	$-e_5$	e_6	e_7	$-e_8$
e_2	$3e_2$	0	$2e_1$	e_3	0	$-e_5$	e_8	0
e_3	$-2e_3$	$-e_1$	$-3e_4$	0	e_6	0	0	$-e_7$
e_4	0	0	0	0	0	0	0	0
e_5	$-2e_1$	$-3e_2$	$-e_3$	0	$-2e_5$	$-e_6$	$-e_7$	$-2e_8$
e_6	$2e_3$	e_1	$3e_4$	0	$-e_6$	0	0	e_7
e_7	$2e_3$	e_1	$3e_4$	0	$-e_6$	0	0	e_7
e_8	0	e_2	$-e_3$	$-2e_4$	0	$-e_6$	$-e_7$	0

The subalgebra spanned by the elements e_1, \dots, e_6 is the terminal algebra W_2 of commutative 2-dimensional algebras. The subalgebra spanned by the elements e_1, \dots, e_4 is the conservative (and, moreover, terminal) algebra S_2 of all commutative 2-dimensional algebras with trace zero multiplication [157]. In a series of our papers devoted to the study of $U(2)$ [157, 175], we described one-sided ideals, subalgebras of codimension 1, derivations and automorphisms of $U(2)$, W_2 and S_2 .

We summarize all the main results of the present part in the following theorems.

Theorem 15 (Theorem 11, [175]). $\text{Aut}(U(2)) \cong \text{Aut}(W_2) \cong \text{Aut}(S_2)$. Also, this group is isomorphic to the matrix group $\begin{pmatrix} 1 & \mathbb{F} \\ 0 & \mathbb{F}^* \end{pmatrix}$ (the one-dimensional affine group over \mathbb{F}).

Theorem 16 (Theorem 11, [175]). The set of nonzero idempotents of the algebra $U(2)$ equals the disjoint union of the following sets (where $c, d \in \mathbb{F}$, $q \in \overline{\mathbb{F}}$):

$$\begin{aligned} &\mathcal{O}(e_8 + e_2 - e_1 + c(3e_8 + e_5 - 2e_1)) \\ &\mathcal{O}(-e_1 + c(e_5 - 2e_1) + de_8) \\ &\mathcal{O}(-e_1 - 2e_8 + 4e_3 + e_6 + 3e_7 + c(3e_8 - e_5 + 2e_1) + de_4) \\ &\mathcal{O}(-e_1 - 2e_8 + c(3e_8 - e_5 + 2e_1) + qe_4). \end{aligned}$$

4 Non-associative algebras and superalgebras

The present part is based on the papers written together with Antonio Jesús Calderón, Artem Lopatin, Bakhrom Omirov, Gulkhay Solijanova, Luisa Camacho, Pasha Zusmanovich and Yury Popov [44, 59, 159, 180].

4.1 Leibniz algebras

Gradings by abelian groups have played a key role in the study of Lie algebras and superalgebras, starting with the root space decomposition of the semisimple Lie algebras over the complex field, which is an essential ingredient in the Killing-Cartan classification of these algebras. Gradings by a cyclic group appear in the connection between Jordan algebras and Lie algebras through the Tits-Kantor-Koecher construction and in the theory of Kac-Moody Lie algebras. Gradings by the integers or the integers modulo 2 are ubiquitous in Geometry. In 1989, Patera and Zassenhaus [219] began a systematic study of gradings by abelian groups on Lie algebras. They raised the problem of classifying the fine gradings, up to equivalence, on the simple Lie algebras over the complex numbers. After that, gradings of simple alternative and simple Malcev algebras [80], the simple Kac Jordan superalgebra [45], countless simple Lie algebras [77, 81] and nilpotent Lie algebras [19] were described.

In the past years, Leibniz algebras have been under active research (see, for example, [74, 123, 170, 186, 187]). Recently, they have appeared in many geometric and physics applications (see, for example, [35, 74, 194, 238] and references therein). The main result on the structure of finite-dimensional Leibniz algebras asserts that a Leibniz algebra decomposes into a semidirect sum of a solvable radical and a semisimple Lie algebra. Therefore, the main problem of the description of finite-dimensional Leibniz algebras consists of the study of solvable Leibniz algebras. Similarly to the case of Lie algebras, the study of solvable Leibniz algebras is reduced to nilpotent ones. Since the description of all n -dimensional nilpotent Leibniz algebras is an unsolvable task (even in the case of Lie algebras), we have to study nilpotent Leibniz algebras under certain conditions (conditions on index of nilpotency, various types of grading, characteristic sequence etc.) [186, 187]. The well-known natural grading of nilpotent Lie and Leibniz algebras is very helpful when investigating the properties of those algebras without restrictions on the grading. Indeed, we can always choose a homogeneous basis and thus the grading allows us to obtain more explicit conditions for the structural constants. Moreover, such grading is useful for the investigation of cohomologies for the considered algebras, because it induces the corresponding grading of the group of cohomologies. Thus, it is crucial to know what kind of grading a nilpotent Leibniz algebra admits.

In our work [44] we begin the study of gradings on Leibniz algebras by classifying, up to equivalence, all abelian groups gradings of null-filiform and one-parametric filiform Leibniz algebras. Let us define all the considered algebras.

Definition 2. An arbitrary complex n -dimensional null-filiform Leibniz algebra is isomorphic to the algebra

$$e_i e_1 = e_{i+1}, \quad 1 \leq i \leq n-1.$$

Definition 3. A complex n -dimensional one-parametric filiform Leibniz algebra is isomorphic to the algebra of the following form:

$$e_1 e_2 = \theta e_n, \quad e_i e_1 = e_{i+1}, \quad 2 \leq i \leq n-1.$$

We described all abelian gradings of null-filiform Leibniz algebras and one-parametric filiform Leibniz algebras in [44, Theorems 13, 17 and 19].

A very powerful tool in the study of nilpotent algebras is the characteristic sequence, in which a priori gives the multiplication on one element of the basis.

Definition 4. For a nilpotent Leibniz (Lie) algebra \mathbf{A} and $x \in \mathbf{A} \setminus \mathbf{A}^2$ we consider the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ with respect to the lexicographical order of the dimensions Jordan's blocks of the operator R_x . The sequence $C(L) = \max_{x \in \mathbf{A} \setminus \mathbf{A}^2} C(x)$ is called the *characteristic sequence* of the Leibniz algebra \mathbf{A} .

Recently, in the paper [10] it was considered a finite-dimensional solvable Lie algebra \mathfrak{r}_c whose nilpotent radical \mathfrak{n}_c has the simplest structure with a given characteristic sequence $c = (n_1, n_2, \dots, n_k, 1)$. Using the Hochschild – Serre factorization theorem, the authors established that the low order cohomology groups of the algebra \mathfrak{r}_c with coefficients in the adjoint representation are trivial. In our paper [59], we study a family of nilpotent Leibniz algebras whose corresponding Lie algebra is \mathfrak{n}_c . Further, solvable Leibniz algebras with such nilpotent radicals and $(k + 1)$ -dimensional complementary subspaces to the nilpotent radicals are described. Namely, we prove that such solvable Leibniz algebra is unique and centerless.

$$\mathbf{R} : \begin{cases} [e_1, e_1] = h, & [h, x_1] = 2h, \\ [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n_1, \\ [e_{n_1+\dots+n_j+i}, e_1] = -[e_1, e_{n_1+\dots+n_j+i}] = e_{n_1+\dots+n_j+1+i}, & 2 \leq i \leq n_{j+1}, \\ [e_1, x_1] = -[x_1, e_1] = e_1, \\ [e_i, x_1] = -[x_1, e_i] = (i - 2)e_i, & 3 \leq i \leq n_1 + 1, \\ [e_{n_1+\dots+n_j+i}, x_1] = -[x_1, e_{n_1+\dots+n_j+i}] = (i - 2)e_{n_1+\dots+n_j+i} & 2 \leq i \leq n_{j+1}, \\ [e_i, x_2] = -[x_2, e_i] = e_i, & 2 \leq i \leq n_1 + 1, \\ [e_{n_1+\dots+n_j+i}, x_{j+2}] = -[x_{j+2}, e_{n_1+\dots+n_j+i}] = e_{n_1+\dots+n_j+i}, & 2 \leq i \leq n_{j+1}. \end{cases}$$

where $1 \leq j \leq k - 1$. For this Leibniz algebra, the triviality of the first and the second cohomology groups with coefficients in the adjoint representation is established as well.

Theorem 17 (Theorem 21, [59]). *The first and second groups of cohomologies of the solvable Leibniz algebra \mathbf{R} in coefficient itself are trivial.*

Theorem 18 (Corollary 22, [59]). *The solvable Leibniz algebra \mathbf{R} is rigid.*

Remark 1. Note that the structure of the rigid algebra \mathbf{R} depends on the given decreasing sequence (n_1, n_2, \dots, n_k) . Set $p(n)$ to be the number of such sequences, that is, $p(x)$ is the number of integer solutions of the equation

$$n_1 + n_2 + \dots + n_k = n \text{ with } n_1 \geq n_2 \geq \dots \geq n_k \geq 0.$$

The asymptotic value of $p(n)$, given by the expression

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad (\text{where } a(n) \approx b(n) \text{ means that } \lim_{n \rightarrow \infty} \frac{a(n)}{b(n)} = 1),$$

yields the existence of at least $p(n)$ irreducible components of the variety of Leibniz algebras of dimension $n + k + 3$.

4.2 Anticommutative \mathfrak{CD} -algebras

The idea of the definition of the variety of \mathfrak{CD} -algebras comes from the following property of Jordan and Lie algebras: *the commutator of any pair of multiplication operators is a derivation*. This gives three identities of degree four, which reduces to only one identity of degree four in the commutative or anticommutative case. Commutative and anticommutative \mathfrak{CD} -algebras are related to many interesting varieties of algebras. Thus, anticommutative \mathfrak{CD} -algebras are a generalization of Lie algebras, containing the intersection of Malcev and Sagle algebras as a proper subvariety. Moreover, the following intersections of varieties coincide: Malcev and Sagle algebras; Malcev and anticommutative \mathfrak{CD} -algebras; and Sagle and anticommutative \mathfrak{CD} -algebras. On the other hand, the variety of anticommutative \mathfrak{CD} -algebras is a proper subvariety of the varieties of binary Lie algebras and almost Lie algebras [180]. The variety of anticommutative \mathfrak{CD} -algebras coincides with the intersection of the varieties of binary Lie algebras and almost Lie algebras. Commutative \mathfrak{CD} -algebras are a generalization of Jordan algebras, which are a generalization of commutative associative algebras. On the other hand, the variety of commutative \mathfrak{CD} -algebras is also known as the variety of almost-Jordan algebras, which lies in the bigger variety of generalized almost-Jordan algebras [99]. The n -ary version of commutative \mathfrak{CD} -algebras was introduced in a recent paper by Kaygorodov, Pozhidaev and Saraiva [172]. The variety of almost-Jordan algebras is the variety of commutative algebras satisfying

$$2((yx)x)x + yx^3 = 3(yx^2)x.$$

This present identity appeared in 1965 in a paper by Osborn during the study of identities of degrees less than or equal to 4 of non-associative algebras. The identity is a linearized form of the Jordan identity. The systematic study of almost-Jordan algebras was initiated in the following paper by Osborn and it was continued in some papers by Petersson [221], Osborn [217], and Sidorov (sometimes, they were called Lie triple algebras). Hentzel and Peresi proved that every semiprime almost-Jordan ring is Jordan [116]. After that, Labra and Correa proved that a finite-dimensional almost-Jordan right-nilalgebra is nilpotent. Assosymmetric algebras under the symmetric product give almost-Jordan algebras [79].

An anticommutative algebra \mathbf{A} is called a \mathfrak{CD} -algebra if it satisfies the property that for any $a, b \in \mathbf{A}$, the commutator $[R_a, R_b]$ is a derivation of \mathbf{A} . This condition can be written as a homogeneous identity of degree 4 comprising 6 monomials:

$$((xy)a)b - ((xy)b)a - ((xa)b)y + ((xb)a)y + ((ya)b)x - ((yb)a)x = 0.$$

We introduced the notion of anticommutative \mathfrak{CD} -algebras and studied their properties in [180]. Firstly, we compare the variety of anticommutative \mathfrak{CD} -algebras with other

known varieties, namely Lie, binary Lie, Malcev, and Sagle. Secondly, we consider the cohomology theory of anticommutative $\mathfrak{C}\mathfrak{D}$ -algebras and obtain some relations with Lie algebras. Namely, we prove the following results.

Theorem 19 (Proposition 3, [180]). *For any $\mathfrak{C}\mathfrak{D}$ -algebra A , the quotient $A/Z(A)$ is a Lie algebra. In particular, any centerless (and, in particular, simple) $\mathfrak{C}\mathfrak{D}$ -algebra is a Lie algebra.*

Theorem 20 (Theorem, [180]). *For any simple finite-dimensional Lie algebra A over a field of characteristic zero, and any finite-dimensional A -module M , $H_{\mathfrak{C}\mathfrak{D}}^2(A, M) = 0$.*

4.3 Noncommutative Jordan superalgebras

The class of noncommutative Jordan algebras is vast. As an example, it includes alternative algebras, Jordan algebras, quasiassociative algebras, quadratic flexible algebras, and anticommutative algebras. Schafer proved that a simple noncommutative Jordan algebra is either a simple Jordan algebra, a simple quasiassociative algebra, or a simple flexible algebra of degree 2. Oehmke proved the analog of Schafer's classification for simple flexible algebras with strictly associative powers in the case of characteristic different from 2, 3, McCrimmon classified simple noncommutative Jordan algebras of degree greater than 2 in the case of characteristic different from 2, and Smith described such algebras of degree 2. The case of nodal simple algebras of positive characteristic was considered in the papers by Kokoris. The case of simple finite-dimensional Jordan superalgebras over algebraically closed fields of characteristic zero was studied by Kac and Kantor. Racine and Zelmanov classified the finite-dimensional Jordan superalgebras of characteristic different from 2 with a semisimple even part. The case when the even part is not semisimple was considered by Martínez and Zelmanov in [207] and Cantarini and Kac described all linearly compact simple Jordan superalgebras [60]. Simple noncommutative Jordan superalgebras were described by Pozhidaev and Shestakov in [225, 226]. Representations of simple noncommutative superalgebras were described by Popov [223]. Nowadays, the study of properties of simple non-associative algebras and superalgebras has aroused a strong interest. For example, Popov determined the structure of differentially simple Jordan algebras, and Barreiro, Elduque, and Martínez described the derivations of the Cheng-Kac Jordan superalgebra [23]. Moreover, Kaygorodov, Shestakov, and Zhelyabin studied generalized derivations of Jordan algebras and superalgebras [231, 232]. Another interesting problem in the study of Jordan algebras and superalgebras is a description of maximal subalgebras and automorphisms [82, 83].

A superalgebra A is called a noncommutative Jordan superalgebra if it satisfies the following operator identities:

$$[R_{xoy}, L_z] + (-1)^{|x|(|y|+|z|)}[R_{yoz}, L_x] + (-1)^{|z|(|x|+|y|)}[R_{zox}, L_y] = 0,$$

$$[R_x, L_y] = [L_x, R_y].$$

A binary linear operation $\{\cdot, \cdot\}$ is called a generic Poisson bracket [174] on a superalgebra (\mathbf{A}, \cdot) if for arbitrary homogeneous $a, b, c \in \mathbf{A}$ we have

$$\{a \cdot b, c\} = (-1)^{|b||c|} \{a, c\} \cdot b + a \cdot \{b, c\}.$$

We notice that there is a one-to-one correspondence between noncommutative Jordan superalgebras and superanticommutative Poisson brackets on adjoint Jordan superalgebras. Let (\mathbf{A}, \bullet) be a Jordan superalgebra and $[\cdot, \cdot]$ be a generic Poisson bracket on \mathbf{A} . Then the operation $ab = \frac{1}{2}(a \bullet b + [a, b])$ turns $(\mathbf{A}, \bullet, [\cdot, \cdot])$ into a noncommutative Jordan superalgebra. Conversely, if \mathbf{A} is a noncommutative Jordan superalgebra, then the supercommutator $[\cdot, \cdot]$ is a generic Poisson bracket on a Jordan superalgebra $\mathbf{A}^{(+)}$. Moreover, the multiplication in \mathbf{A} can be recovered by the Jordan multiplication in $\mathbf{A}^{(+)}$ and the generic Poisson bracket $[\cdot, \cdot] : ab = \frac{1}{2}(a \bullet b + [a, b])$.

In our paper with Lopatin and Popov, we study simple noncommutative Jordan superalgebras constructed in some papers by Pozhidaev and Shestakov [225]. We describe all subalgebras and automorphisms of simple noncommutative Jordan superalgebras $K_3(\alpha, \beta, \gamma)$ and $D_t(\alpha, \beta, \gamma)$ [159]. We also compute the derivations of the nontrivial simple finite-dimensional noncommutative Jordan superalgebras [159, Theorems 9 and 10]].

5 Maps on associative and non-associative algebras

The present part is based on the papers written together with Artem Lopatin, Bruno Ferreira, Feng Wei, Henrique Guzzo, Mykola Khrypchenko and Yury Popov [90, 91, 154, 158, 165].

5.1 Alternative and Jordan algebras with invertible derivations

In 1983, Bergen, Herstein, and Lanski initiated a study whose purpose was to relate the structure of a ring to the special behavior of one of its derivations. Namely, in their article [34] they described associative rings admitting derivations with invertible values, i.e., there is a derivation d such that $d(a)$ is an invertible element or zero for every a from our algebra. They proved that such ring must be either a division ring, or the ring of 2×2 matrices over a division ring, or a factor of a polynomial ring over a division ring of characteristic 2. They also characterized the division rings such that the 2×2 matrix ring over them has an inner derivation with invertible values. Further, associative rings with derivations with invertible values (and also their generalizations) were discussed in a variety of works (see, for instance, [104]). So, in [104] semiprime associative rings with involution, allowing a derivation with invertible values on the set of symmetric elements, were examined. In their work, Bergen and Carini studied associative rings, admitting a derivation with invertible values on some non-central Lie ideal. Also, in the papers by Chang, Hongan and Komatsu, the structure of associative rings that admit α -derivations with invertible values and their natural generalizations — (σ, τ) -derivations with invertible values, was described. Komatsu and Nakajima described associative rings that allow generalized derivations with invertible values. The case of associative superalgebras with

derivations with invertible values was studied in the paper by Demir, Albaş, Argaç, and Fosner.

The description of non-associative algebras admitting derivations with invertible values began in our paper with Popov [165], where it was proved that every alternative (non-associative) algebra admitting derivation with invertible values is a Cayley — Dickson division algebra over their center or a factor-algebra of a polynomial algebra $C[x]/(x^2)$ over a Cayley — Dickson division algebra. Our research was continued in a joint paper with Lopatin and Popov [158], where we describe Jordan algebras admitting derivations with invertible values. Let us summarize the main results from [158].

Theorem 21 (Theorem 3, [158]). *Let A be a Jordan algebra admitting a derivation with invertible values d . Then A is an extension of a simple Jordan algebra with derivation with invertible values by an ideal M such that $M^2 = 0$, $d(M) = 0$ and M is the largest ideal of A .*

Theorem 22 (Theorem 11, [158]). *Let A be a Jordan algebra of characteristic not 2 admitting a derivation with invertible values d . Then one of the following holds:*

- (1) A is an algebra $A^{(+)}$, where $A = D$ or D_2 , and D is an associative division algebra;
- (2) A is an algebra $H(A, *)$, where $A = D$ or D_2 , and D is an associative division algebra;
- (3) A is an algebra of symmetric nondegenerate bilinear form $J(V, f)$;
- (4) A is a division algebra of Albert type;
- (5) A is an extension of cases (1) – (4) by $M = \mathbf{P}(A)$ (where $\mathbf{P}(A)$ stands for the prime radical of A), where $M \subseteq \ker d$ and M is the largest ideal of A .

Theorem 23 (Theorem 20, [158]). *Let A be a finite-dimensional Jordan algebra admitting a derivation with invertible values. Then A is either simple or an algebra of a symmetric bilinear form (possibly degenerated).*

5.2 Maps on alternative algebras

The study of Lie isomorphisms of rings was originally inspired by Herstein's generalization of generalizing classical theorems on the Lie structure of total matrix rings to the Lie structure of arbitrary simple rings [117]. In this paper, Herstein formulated some open questions that gave rise to big research in Lie and Jordan maps on various structures. The main questions from Herstein's program were solved for associative algebras in papers by Beidar, Brešar, Chebotar and Martindale [26, 27]. In [205], Martindale studied Lie isomorphisms between primitive rings A, A' , where he assumed that the characteristic of A is different from 2 and 3 and that A contains three nonzero orthogonal idempotents whose sum is the identity. A few years later, he studied Lie isomorphisms between two simple rings A, A' . Alternative algebras, as well as associative algebras, admit a Peirce

decomposition. The recently introduced axial algebras reflect an active interest in the study of algebras admitting a Peirce-type decomposition [181]. In our works together with Ferreira and Guzzo, we continue the study of the Herstein program for some special types of alternative algebras admitting Peirce decomposition.

In the following theorems, we summarize the main results of the present part.

Theorem 24 (Theorem 2.1, [90]). *Let \mathbf{A} be a unital prime (simple, simple associative) alternative algebra, let \mathbf{A}' be another prime (simple, simple associative) alternative algebra and let $\varphi : \mathbf{A} \rightarrow \mathbf{A}'$ be a surjective Lie multiplicative map that preserves idempotents. Assume that \mathbf{A} has a nontrivial idempotent e_1 with associated Peirce decomposition $\mathbf{A} = \mathbf{A}_{11} \oplus \mathbf{A}_{12} \oplus \mathbf{A}_{21} \oplus \mathbf{A}_{22}$, such that*

- (1) *if $x_{ij}\mathbf{A}_{ji} = 0$, then $x_{ij} = 0$ ($i \neq j$);*
- (2) *if $x_{11}\mathbf{A}_{12} = 0$ or $\mathbf{A}_{21}x_{11} = 0$, then $x_{11} = 0$;*
- (3) *if $\mathbf{A}_{12}x_{22} = 0$ or $x_{22}\mathbf{A}_{21} = 0$, then $x_{22} = 0$;*
- (4) *if $z \in Z(\mathbf{A})$ with $z \neq 0$, then $z\mathbf{A} = \mathbf{A}$.*

Then, the following holds.

- *If $f_i\varphi(\mathbf{A}_{jj})f_i \subseteq Z(\mathbf{A}')f_i$, then φ is of the form $\psi + \tau$, where ψ is an additive isomorphism between \mathbf{A} and \mathbf{A}' and τ is a map from \mathbf{A} to $Z(\mathbf{A}')$, which maps commutators into zero.*
- *If $f_i\varphi(\mathbf{A}_{ii})f_i \subseteq Z(\mathbf{A}')f_i$, then φ is of the form $\psi + \tau$, where $-\psi$ is additive anti-isomorphism between \mathbf{A} and \mathbf{A}' and τ is a map from \mathbf{A} to $Z(\mathbf{A}')$, which maps commutators into zero. Observe $f_i = \varphi(e_i)$ and $f_j = 1_{\mathbf{A}'} - f_i$, $i \neq j$.*

Definition 5. Let be φ an additive map on \mathbf{A} . We call φ a commuting map if $[\varphi(x), x] = 0$ for all $x \in \mathbf{A}$.

Theorem 25 (Theorem 4, [91]). *Let \mathbf{A} be a unital alternative algebra. Assume that \mathbf{A} has a nontrivial idempotent e_1 with associated Peirce decomposition $\mathbf{A} = \mathbf{A}_{11} \oplus \mathbf{A}_{12} \oplus \mathbf{A}_{21} \oplus \mathbf{A}_{22}$, such that $(x\mathbf{A})e_i = 0$ implies $x = 0$ ($i = 1, 2$). Let $\varphi : \mathbf{A} \rightarrow \mathbf{A}$ be an additive map. Then the following statements are equivalent:*

- (♠) *φ is commuting;*
- (♣) *there exists $z \in Z(\mathbf{A})$ and an additive map $\Xi : \mathbf{A} \rightarrow Z(\mathbf{A})$ such that $\varphi(x) = zx + \Xi(x)$ for all $x \in \mathbf{A}$.*

5.3 Higher derivations of incidence algebras

Let (P, \leq) be a preordered set and R be a commutative unital ring. Assume that P is locally finite, i.e. for any $x \leq y$ in P there are only finitely many $z \in P$ such that $x \leq z \leq y$. The incidence algebra $I(P, R)$ of P over R is the set of functions

$$\{f : P \times P \rightarrow R \mid f(x, y) = 0 \text{ if } x \not\leq y\}$$

with the natural structure of an R -module and multiplication given by the convolution

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

for all $f, g \in I(P, R)$ and $x, y \in P$. It would be helpful to point out that the full matrix algebra $M_n(R)$, as well as the upper triangular matrix algebra $T_n(R)$, are particular cases of incidence algebras. In addition, in the theory of operator algebras, the incidence algebra $I(P, R)$ of a finite poset P is referred to as a bigraph algebra or a finite-dimensional commutative subspace lattice algebra. Incidence algebras appeared in the early work by Ward [243] as generalized algebras of arithmetic functions. Later, they were extensively used as the fundamental tool of enumerative combinatorics in the series of works ‘‘On the foundations of combinatorial theory’’. The study of algebraic mappings on incidence algebras was initiated by Stanley [237]. Since then, automorphisms, involutions, and derivations (and their generalizations) on incidence algebras have been actively investigated, see [100, 103, 218] and the references therein.

Let P be a poset and R a commutative unital ring. Recall from [182] that a finitary series is a formal sum of the form

$$\alpha = \sum_{x \leq y} \alpha_{xy} e_{xy},$$

where $x, y \in P$, $\alpha_{xy} \in R$ and e_{xy} is a symbol, such that for any pair $x < y$ there exists only a finite number of $x \leq u < v \leq y$ with $\alpha_{uv} \neq 0$. The set of finitary series, denoted by $FI(P, R)$, possesses a natural structure of an R -module. Moreover, it is closed under the convolution

$$\alpha\beta = \sum_{x \leq y} \left(\sum_{x \leq z \leq y} \alpha_{xz} \beta_{zy} \right) e_{xy}.$$

Thus, $FI(P, R)$ is an algebra, called the finitary incidence algebra of P over R . The identity element of $FI(P, R)$ is the series $\delta = \sum_{x \in P} 1_R e_{xx}$. Here and in what follows we adopt the following convention: if in the formal sum the indices run through a subset X of the ordered pairs (x, y) , $x, y \in P$, $x \leq y$, then α_{xy} is meant to be zero for $(x, y) \notin X$.

Higher derivations are closely related to derivations. It should be remarked that the first component d_1 of each higher derivation $D = \{d_n\}_{n=0}^\infty$ of an algebra A is itself a derivation of A . Conversely, let $d : A \rightarrow A$ be an ordinary derivation of an algebra A over

a field of characteristic zero. Then, $D = \{\frac{1}{n!}d^n\}_{n=0}^\infty$ is a higher derivation of A . Heerema, Mirzavaziri and Saymeh independently proved that each higher derivation of an algebra A over a field of characteristic zero is a combination of compositions of derivations, and hence one can characterize all higher derivations on A in terms of the derivations on A . Ribenboim systemically studied higher derivations of arbitrary rings and those of arbitrary modules, where some familiar properties of derivations are generalized to the case of higher derivations. Ferrero and Haetinger found the conditions under which Jordan higher derivations (or Jordan triple higher derivations) of a 2-torsion-free (semi-)prime ring are higher derivations, and the same authors studied higher derivations on (semi-)prime rings satisfying linear relations. Wei and Xiao [244] described higher derivations of triangular algebras and related mappings, such as inner higher derivations, Jordan higher derivations, Jordan triple higher derivations, and their generalizations.

Definition 6. A sequence $d = \{d_n\}_{n=0}^\infty$ of additive maps $R \rightarrow R$ is a *higher derivation of R (of infinite order)* if it satisfies the conditions

$$1. d_0 = \text{id}_R; \quad 2. d_n(rs) = \sum_{i+j=n} d_i(r)d_j(s)$$

for all $n \in \mathbb{N} = \{1, 2, \dots\}$ and all $r, s \in R$. If definition 6 holds for all $1 \leq n \leq N$, then the sequence $\{d_n\}_{n=0}^N$ is called a *higher derivation of order N* . Evidently, $\{d_n\}_{n=0}^\infty$ is a higher derivation if and only if $\{d_n\}_{n=0}^N$ is a higher derivation of order N for all $N \in \mathbb{N}$. In particular, d_1 is always a usual derivation of R .

Denote by $\text{HDer } R$ the set of higher derivations of R and consider the following operation on $\text{HDer } R$

$$(d' * d'')_n = \sum_{i+j=n} d'_i \circ d''_j.$$

In particular,

$$(d' * d'')_1 = d'_1 + d''_1.$$

It was proved in [115] that $\text{HDer } R$ forms a group with respect to $*$, whose identity is the sequence $\{\epsilon_n\}_{n=0}^\infty$ with $\epsilon_0 = \text{id}_R$ and $\epsilon_n = 0$ for $n \in \mathbb{N}$.

Given $r \in R$ and $k \in \mathbb{N}$, define

$$[r, k]_0 = \text{id}_R, \\ [r, k]_n(x) = \begin{cases} 0, & k \nmid n, \\ r^l x - r^{l-1} x r, & n = kl, \end{cases}$$

for all $n \in \mathbb{N}$ and $x \in R$. It was proved in [216] that $\{[r, k]_n\}_{n=0}^\infty \in \text{HDer } R$, so that for any sequence $\{r_n\}_{n=1}^\infty \subseteq R$ one may define $\{(\Delta_r)_n\}_{n=0}^\infty$ by means of $(\Delta_r)_0 = \text{id}_R$ and $(\Delta_r)_n = ([r_1, 1] * \dots * [r_n, n])_n$, where $n \in \mathbb{N}$. Higher derivations of the form Δ_r are called *inner*. By [216, Corollary 3.3] the set of inner higher derivations forms a normal subgroup in $\text{HDer } R$, which will be denoted by $\text{IHDer } R$. In particular,

$$(\Delta_r)_1(x) = [r_1, 1]_1(x) = r_1 x - x r_1$$

is the usual inner derivation of R associated with $r_1 \in R$, which we denote by ad_{r_1} .

Definition 7. A sequence $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ of maps on $I = \{(x, y) \in P \times P \mid x \leq y\}$ with values in R is called a *higher transitive map* if

- $\sigma_0(x, y) = 1_R$ for all $x \leq y$;
- $\sigma_n(x, y) = \sum_{i+j=n} \sigma_i(x, z)\sigma_j(z, y)$ for all $x \leq z \leq y$.

Theorem 26 (Lemma 2.6, [154]). *Given a higher transitive map σ , denote by $\tilde{\sigma}$ the following sequence of maps $FI(P, R) \rightarrow FI(P, R)$:*

$$\tilde{\sigma}_n(\alpha) = \sum_{x \leq y} \sigma_n(x, y) \alpha_{xy} e_{xy},$$

where $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in FI(P, R)$. Then, $\tilde{\sigma} \in \text{HDer } FI(P, R)$.

Theorem 27 (Theorem 2.8, [154]). *Every R -linear higher derivation of $FI(P, R)$ is of the form $\Delta_\rho * \tilde{\sigma}$ for some $\rho = \{\rho_n\}_{n=1}^{\infty} \subseteq FI(P, R)$ and some higher transitive map σ .*

6 Generalized derivations of non-associative algebras

The present part is based on the papers written together with Bruno Ferreira, Inomjon Yuldashev, Karimbergan Kudaybergenov, Patrícia Beites, and Yury Popov [28, 92, 139, 140, 156, 165–167].

6.1 Local derivations of n -ary algebras

The study of local derivations started in 1990 with Kadison's article [132]. A similar notion, which characterizes non-linear generalizations of derivations, was introduced by Šemrl as 2-local derivations. In his paper [242], it was proved that a 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H is a derivation. After these works, numerous new results related to the description of local and 2-local derivations of associative algebras appeared (see, for example, [183]). The study of local and 2-local derivations of non-associative algebras was initiated in some papers by Ayupov and Kudaybergenov (for the case of Lie algebras, see [16, 17]). In particular, they proved that there are no pure local and 2-local derivations on semisimple finite-dimensional Lie algebras. Also, [18] provides examples of 2-local derivations on nilpotent Lie algebras that are not derivations. After the cited works, the study of local and 2-local derivations was continued for Leibniz algebras and Jordan algebras [15]. Local automorphisms and 2-local automorphisms were also studied in many cases; for example, they were studied on Lie algebras [16, 68].

The description of local and 2-local derivations of n -ary algebras started in our paper with Ferreira and Kudaybergenov [92]. Namely, we gave the first example of a complex simple finite-dimensional (ternary) algebra with non-trivial local derivations. The idea of introducing a generalization of Filippov algebras comes from binary Malcev algebras and it

was carried out in a paper by Pozhidaev [224]. He defined n -ary Malcev algebras, generalizing Malcev algebras and n -ary Filippov algebras. Let us summarize the construction of the most important example of n -ary Malcev (non-Filippov). We denote by \mathbf{A} a composition 8-dimensional algebra with an involution $\bar{\cdot} : a \mapsto \bar{a}$ and unity 1. The symmetric bilinear form $\langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$ defined on \mathbf{A} is assumed to be nonsingular. If \mathbf{A} is equipped with a ternary multiplication $[\cdot, \cdot, \cdot]$ by the rule

$$[x, y, z] = (x\bar{y})z - \langle y, z \rangle x + \langle x, z \rangle y - \langle x, y \rangle z,$$

then \mathbf{A} becomes a ternary Malcev algebra [224], which will be denoted by M_8 .

Let us give two main definitions necessary for this part.

Definition 8. Let \mathbf{A} be an n -ary algebra. A linear map $\nabla : \mathbf{A} \rightarrow \mathbf{A}$ is called a local derivation if for any element $x \in \mathbf{A}$ there exists a derivation $\mathfrak{D}_x : \mathbf{A} \rightarrow \mathbf{A}$ such that $\nabla(x) = \mathfrak{D}_x(x)$.

Definition 9. A (not necessary linear) map $\Delta : \mathbf{A} \rightarrow \mathbf{A}$ is called a 2-local derivation, if for any two elements $x, y \in \mathbf{A}$ there exists an derivation $\mathfrak{D}_{x,y} : \mathbf{A} \rightarrow \mathbf{A}$ such that $\Delta(x) = \mathfrak{D}_{x,y}(x)$, $\Delta(y) = \mathfrak{D}_{x,y}(y)$.

In the following theorems, we summarize all the main results of the present part.

Theorem 28 (Theorem 3, [156]). *Every local derivation of a complex finite-dimensional semisimple Leibniz algebra is a derivation.*

Theorem 29 (Theorems 4 and 5, [92]). *Every local (and 2-local) derivation of a complex finite-dimensional simple n -Lie algebra is a derivation.*

Theorem 30 (Theorem 8, [92]). *A linear mapping ∇ on M_8 is a local derivation if and only if its matrix is antisymmetric. In particular, the dimension of the space $LocDerM_8$ of all local derivations of M_8 is equal to 28.*

6.2 Leibniz-derivations of non-associative algebras

The theory of Lie algebras having a nonsingular derivation has a rich history and is still an active research area. Such Lie algebras appear in many different situations, such as in the studies of pro- p groups of finite coclass by Shalev and Zelmanov [229, 230] and in the problems concerning the existence of left-invariant affine structures on Lie groups (see Burde's survey [38] for details).

In 1955, Jacobson [124] proved that a finite-dimensional Lie algebra over a field of characteristic zero admitting a nonsingular (invertible) derivation is nilpotent. The problem of whether the inverse of this statement is correct remained open until the work [76], where an example of a nilpotent Lie algebra in which all derivations are nilpotent (and hence, singular) was constructed. For Lie algebras in prime characteristic, the situation is more complicated. In that case, there exist non-nilpotent Lie algebras, even simple ones, which admit nonsingular derivations [32]. The main examples of nonsingular derivations are periodic derivations. Kostrikin and Kuznetsov [193] noted that a Lie algebra admitting

a nondegenerate derivation admits a periodic derivation, that is, a derivation d such that $d^N = \text{id}$ for some N , and proved that a Lie algebra admitting a derivation of period N is abelian provided that $N \not\equiv 0 \pmod{6}$. Burde and Moens proved that a finite-dimensional complex Lie algebra \mathbf{A} admits a periodic derivation if and only if \mathbf{A} admits a nonsingular derivation whose inverse is again a derivation if and only if \mathbf{A} is hexagonally graded [39]. In the case of positive characteristic p , Shalev proved that if a Lie algebra \mathbf{A} admits a nonsingular derivation of order $n = p^s m$, where $(m, p) = 1$ and $m < p^2 - 1$, then \mathbf{A} is nilpotent. The study of periodic derivations was continued by Mattarei [209–211].

The study of derivations of Lie algebras led to the appearance of the notion of their natural generalization — a pre-derivation of a Lie algebra, which is a derivation of a Lie triple system induced by that algebra. In [21] it was proved that Jacobson’s result remains true in terms of pre-derivations. Several examples of nilpotent Lie algebras whose pre-derivations are nilpotent were presented in [21].

A generalization of derivations and pre-derivations of Lie algebras is defined in the paper [214] as a Leibniz-derivation of order k . Moens proved that a finite-dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible Leibniz-derivation. After that, Fialowski, Khudoyberdiyev and Omirov [94] showed that with the definition of Leibniz-derivations from [214], the analogous result for non-Lie Leibniz algebras is not true. Namely, they gave an example of non-nilpotent Leibniz algebra, which admits an invertible Leibniz-derivation. To extend the results of the paper [214] for Leibniz algebras they introduced the definition of Leibniz-derivation of a Leibniz algebra which is coherent with the definition of Leibniz-derivation of a Lie algebra and proved that a finite-dimensional Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. In the paper [165], the authors showed that the same result holds for alternative algebras (particularly, for associative algebras). Also, in this article an example of a nilpotent alternative (non-associative) algebra over a field of positive characteristic possessing only singular derivations was provided.

It is well known that the radicals of finite-dimensional algebras belonging to the classical varieties (such as varieties of Jordan algebras, Lie algebras, alternative algebras, and many others) are invariant under their derivations [236]. Therefore, it is natural to state another interesting problem: is the radical of an algebra invariant under its Leibniz-derivations? Moens proved that the solvable radical of a Lie algebra is invariant under all its Leibniz-derivations, Fialowski, Khudoyberdiyev and Omirov showed the invariance of solvable and nilpotent radicals of a Leibniz algebra, and the authors of [165] proved analogous results for alternative algebras. Another interesting task is to describe the Leibniz-derivations of algebras belonging to certain “nice” classes, such as semisimple and perfect algebras. Moens described all Leibniz-derivations of semisimple Lie algebras [214] and Zhou described all pre-derivations of perfect centerless Lie algebras of characteristic $\neq 2$.

Definition 10. Let \mathbf{A} be an algebra, n be a natural number ≥ 2 , and f be an arrangement of brackets on a product of length n . A linear mapping d on \mathbf{A} is called an f -Leibniz

derivation of \mathbf{A} , if for any $a_1, \dots, a_n \in \mathbf{A}$ we have

$$d([a_1, \dots, a_n]_f) = \sum_{i=1}^n [a_1, a_2, \dots, d(a_i), \dots, a_n]_f.$$

Particularly, if $f = l(n)$ ($f = r(n)$) is the left (right) arrangement of brackets of length n , that is,

$$[x_1, \dots, x_n]_{l(n)} = ((\dots (x_1 x_2) \dots) x_{n-1}) x_n, \quad [x_1, \dots, x_n]_{r(n)} = (x_1 (x_2 \dots (x_{n-1} x_n) \dots)),$$

then an $l(n)$ -Leibniz derivation ($r(n)$ -Leibniz derivation) of \mathbf{A} will be called a *left (right) Leibniz-derivation* of \mathbf{A} of order n . If d is an f -derivation for any arrangement f of length n , then d will be called a *Leibniz-derivation of \mathbf{A} of order n* . One can see that in our terms a Leibniz-derivation by Moens is a right Leibniz-derivation of a Lie algebra \mathbf{A} , and a Leibniz-derivation by Fialowski-Khudoyberdiev-Omirov is a left Leibniz-derivation of a Leibniz algebra \mathbf{A} . It is easy to see that for

- I. (anti)commutative algebras, the notion of a left Leibniz-derivation coincides with the notion of a right Leibniz derivation;
- II. left Leibniz algebras, every right Leibniz-derivation is a left Leibniz-derivation;
- III. left Zinbiel algebras, every left Leibniz-derivation is a right Leibniz-derivation;
- IV. associative and Lie algebras, the notion of a Leibniz-derivation coincides with the notion of an f -Leibniz derivation for any f .

We summarize the main results of the present part in the following theorems.

Theorem 31 (Theorem 18, [166]). *Let \mathbf{A} be a Malcev algebra over a field of characteristic zero. Then $\text{Rad}(\mathbf{A})$ is invariant under all the left Leibniz-derivations of \mathbf{A} .*

Theorem 32 (Theorem 18, [166]). *Let \mathbf{A} be a semisimple Malcev algebra over a field of characteristic 0. Then $\mathfrak{Der}(\mathbf{A}) = \text{LDer}_l(\mathbf{A})$.*

Theorem 33 (Theorem 28, [166]). *A Malcev algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible left Leibniz-derivation.*

Theorem 34 (Theorem 30, [166]). *A Jordan algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible left Leibniz-derivation.*

Theorem 35 (Theorem 32, [166]). *A $(-1, 1)$ -algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible left Leibniz-derivation.*

Theorem 36 (Theorem 33, [166]). *A right alternative algebra over a field of characteristic zero admitting an invertible Leibniz-derivation is right nilpotent.*

6.3 Generalized derivations of non-associative algebras

In 1998, the notion of δ -derivations appeared in the paper by Filippov [96]. He studied δ -derivations of prime Lie algebras [97]. After that, δ -derivations of structurable algebras, n -ary algebras, and Jordan and Lie superalgebras were studied (see [250] and references therein). The notion of $\frac{1}{2}$ -derivation plays an important role in the description of transposed Poisson structures on a certain Lie algebra [93]. The notion of generalized derivations is a generalization of δ -derivations and (α, β, γ) -derivations. The most important and systematic research on the generalized derivations algebras of a Lie algebra and their subalgebras was in a paper by Leger and Luks [197]. In their article, they studied the properties of generalized derivation algebras and their subalgebras, for example, the quasiderivation algebras. They determined the structure of algebras of quasiderivations and generalized derivations and proved that the quasiderivation algebra of a Lie algebra can be embedded into the derivation algebra of a larger Lie algebra. Their results were generalized by many authors. For example, Zhang and Zhang [249] generalized the above results to the case of Lie superalgebras; Chen, Ma, Ni and Zhou considered the generalized derivations of color Lie algebras, Hom-Lie superalgebras and Lie triple systems [64]. Generalized derivations of simple algebras and superalgebras were investigated in [126, 231, 232]. Pérez-Izquierdo and Jiménez-Gestal used the generalized derivations to study non-associative algebras [127]. Derivations and generalized derivations of n -ary algebras were considered in many papers. For example, Williams proved that, unlike the case of binary algebras, for any $n \geq 3$ there exists a non-nilpotent n -Lie algebra with invertible derivation [245].

The main purpose of the results contained in the present part is to generalize the results of Leger and Luks [197] to the case of color n -ary algebras and Hom-algebras. Particularly, we proved some properties of generalized derivations of color n -ary algebras and Hom-algebras, that the quasiderivation algebra of a color n -ary Ω -algebra (resp., Ω -Hom-algebra) can be embedded into the derivation algebra of a large color n -ary Ω -algebra (resp., Ω -Hom-algebra) [28, 167], and obtained the classification of all n -ary algebras with the property $\text{End} = \text{QDer}$. Let us give the main definitions of this part.

Definition 11. A linear mapping $\mathfrak{D} \in \text{End}(\mathbf{A})$ is called a generalized derivation of an n -ary algebra \mathbf{A} if there exists linear mappings $\mathfrak{D}', \mathfrak{D}'', \dots, \mathfrak{D}^{(n-1)}, \mathfrak{D}^{(n)} \in \text{End}(\mathbf{A})$ such that

$$\sum[x_1, \dots, \mathfrak{D}^{(i-1)}(x_i), \dots, x_n] = \mathfrak{D}^{(n)}([x_1, \dots, x_n]).$$

An $(n+1)$ -tuple $(\mathfrak{D}, \mathfrak{D}', \dots, \mathfrak{D}^{(i-1)}, \dots, \mathfrak{D}^{(n-1)}, \mathfrak{D}^{(n)})$ is called an $(n+1)$ -ary derivation. An $(n+1)$ -ary derivation is trivial if it is a sum of (d, \dots, d) and $(\phi_1, \dots, \phi_n, \sum \phi_i)$, where d is a derivation and ϕ_i are elements from the centroid.

Definition 12. A linear mapping $\mathfrak{D} \in \text{End}(\mathbf{A})$ is said to be a quasiderivation if there exists a $\mathfrak{D}' \in \text{End}(\mathbf{A})$ such that

$$\sum[x_1, \dots, \mathfrak{D}(x_i), \dots, x_n] = \mathfrak{D}'([x_1, \dots, x_n]).$$

Definition 13. A pair of linear mappings (d, f) satisfying the condition of the previous definition is called a pair of quasiderivations of \mathbf{A} . The image of the projection of $\text{QDer}(\mathbf{A})$

onto the first coordinate will be denoted as $\text{QDer}_{KS}(\mathbf{A})$ and will be called a space of quasiderivations in the sense of Kaygorodov and Shestakov [139, 231], and the image of the projection of $\text{QDer}(\mathbf{A})$ onto the second coordinate will be denoted as $\text{QDer}_{LL}(\mathbf{A})$ and will be called a space of quasiderivations in the sense of Leger and Luks [197].

In the following theorems, we summarize the main results of the present part.

Theorem 37 (Theorem 1, [140]). *There are no nontrivial ternary derivations of the simple Malcev algebra M_7 .*

Theorem 38 (Theorem 5, [140]). *There are no nontrivial 4-ary derivations of the simple ternary Malcev algebra M_8 .*

Theorem 39 (Theorem 7, [139]). *Let \mathbf{A} be a semisimple finite-dimensional n -ary Filippov algebra over an algebraically closed field of characteristic zero. Then, the algebra of generalized derivations of \mathbf{A} is isomorphic to $\oplus \mathfrak{sl}_{n+1}$.*

To understand the main result of this part, let us give some additional information about n -ary algebras (see [95]).

- Up to isomorphism, there is only one n -ary anticommutative algebra A_n in dimension n . The product of the basis elements e_1, \dots, e_n of A_n is given in the following way:

$$[e_1, \dots, e_n] = e_1.$$

- Up to isomorphism, there is only one perfect (i.e., $\mathbf{A}^2 = \mathbf{A}$) n -Lie algebra D_{n+1} of dimension $n + 1$. The product of the basis elements e_1, \dots, e_{n+1} of D_{n+1} is given in the following way:

$$[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = (-1)^{n+i+1} e_i.$$

- Let \mathbf{A} be an n -ary $(n + 1)$ -dimensional algebra with the basis e_1, \dots, e_{n+1} . Let

$$e^i = (-1)^{n+i+1} [e_1, \dots, \hat{e}_i, \dots, e_n], i = 1, \dots, n + 1. \quad (1)$$

Then, the multiplication in \mathbf{A} is defined by the matrix $B = (\beta_{ij})$ which is given by

$$e^i = \beta_{1i} e_1 + \dots + \beta_{n+1i} e_{n+1},$$

or in terms of matrices as $(e^1, \dots, e^{n+1}) = (e_1, \dots, e_{n+1})B$. The rank of B is equal to the dimension of $\mathbf{A}^2 = [\mathbf{A}, \dots, \mathbf{A}]$. An $(n + 1)$ -dimensional anticommutative algebra with multiplication defined by (1) will be denoted by \mathbf{A}_B . It is easy to see that D_{n+1} is an algebra \mathbf{A}_I , where I is an identity matrix of order $n + 1$.

Later in the discussion we will also need the description of 1-dimensional n -ary algebras. One can easily see that the multiplication in such algebras is completely defined by an element $\alpha \in \mathbb{F}$, where \mathbb{F} is the base field, for it is enough to determine $[v, v, \dots, v] = \alpha v$ for any nonzero $v \in \mathbf{A}$ and extend the multiplication linearly. We denote such algebras by \mathbf{A}_α . It is also easy to see that $\mathbf{A}_\alpha \cong \mathbf{A}_\beta$ for $\alpha, \beta \neq 0$ if and only if the polynomial $x^{n-1} - \frac{\alpha}{\beta}$ has a root in \mathbb{F} .

Theorem 40 (Theorem 5.6, [167]). *Let \mathbf{A} be an n -ary algebra such that $\text{QDer}_{LL}(\mathbf{A}) = \text{End}(\mathbf{A})$. Then*

- *If \mathbf{A} is commutative, then either \mathbf{A} has zero product or $\mathbf{A} \cong \mathbf{A}_\alpha$ for some α in base field \mathbb{F} . Moreover, if the base field \mathbb{F} of \mathbf{A} is algebraically closed, then one can define a binary multiplication \cdot on \mathbf{A} such that $(\mathbf{A}, \cdot) \cong \mathbb{F}$ and $[x_1, \dots, x_n] = x_1 \cdot \dots \cdot x_n$, where $[\cdot, \dots, \cdot]$ is multiplication in \mathbf{A} and $x_1, \dots, x_n \in \mathbf{A}$.*

- *If \mathbf{A} is anticommutative, then either \mathbf{A} has zero product or \mathbf{A} is isomorphic to either \mathbf{A}_n or \mathbf{A}_B for a nondegenerate $(n+1) \times (n+1)$ -matrix B . Moreover, in the last case if \mathbf{A} is a Filippov algebra and the base field is algebraically closed, then $\mathbf{A} \cong D_{n+1}$.*

Theorem 41 (Theorem 5.9, [167]). *Let \mathbf{A} be an n -ary algebra such that $\text{QDer}_{KS}(\mathbf{A}) = \text{End}(\mathbf{A})$. Then*

- *If \mathbf{A} is commutative, then $\mathbf{A} \cong \mathbf{A}_\alpha$ for some $\alpha \in \mathbb{F}$. Moreover, if the base field \mathbb{F} of \mathbf{A} is algebraically closed, then one can define a binary multiplication \cdot in \mathbf{A} such that $(\mathbf{A}, \cdot) \cong \mathbb{F}$ and $[x_1, \dots, x_n] = x_1 \cdot \dots \cdot x_n$, where $[\cdot, \dots, \cdot]$ is multiplication in \mathbf{A} and $x_1, \dots, x_n \in \mathbf{A}$.*

- *If \mathbf{A} is anticommutative, then $\mathbf{A} \cong \mathbf{A}_B$ for a nondegenerate $(n+1) \times (n+1)$ -matrix B or \mathbf{A} has zero multiplication. Moreover, if \mathbf{A} is a Filippov algebra and the base field is algebraically closed, then $\mathbf{A} \cong D_{n+1}$.*

7 Poisson type algebras and superalgebras

The present part is based on the papers written together with Bruno Ferreira, Ivan Shestakov, Mykola Khrypchenko, Ualbai Umirbaev and Viktor Lopatkin [93, 141, 145, 174].

7.1 Poisson structures on finitary incidence algebras

The systematic study of noncommutative Poisson algebra structures began in the paper by Kubo [195]. He obtained a description of all the Poisson structures on the full and upper triangular matrix algebras, which was later generalized to prime associative algebras in [86]. Namely, it was proved in [86] that any Poisson bracket on a prime noncommutative associative algebra is the commutator bracket multiplied by an element from the extended centroid of the algebra. On the other hand, in his next paper, Kubo studied noncommutative Poisson algebra structures on affine Kac-Moody algebras. The investigation of Poisson structures on associative algebras continued in some papers by Yao, Ye and Zhang [246]; Mroczyńska, Jaworska-Pastuszak, and Pogorzały [125, 215], where Poisson structures on finite-dimensional path algebras and on canonical algebras were studied. In our joint work with Khrypchenko we described Poisson structures on finitary incidence algebras [145].

Definition 14. Let (\mathbf{A}, \cdot) be an associative algebra and $\{\cdot, \cdot\}$ an additional multiplication on the same vector space. We call $\{\cdot, \cdot\}$ a Poisson structure on (\mathbf{A}, \cdot) if $(\mathbf{A}, \cdot, \{\cdot, \cdot\})$ is a Poisson algebra.

The main tool for the description of Poisson structures on associative algebra is the study of biderivations.

Definition 15. Let \mathbf{A} be an associative algebra and M an \mathbf{A} -bimodule. A bilinear map $B : \mathbf{A} \times \mathbf{A} \rightarrow M$ is called an *antisymmetric biderivation of \mathbf{A} with values in M* , if it is anticommutative and a derivation $\mathbf{A} \rightarrow M$ with respect to each of its two variables, i.e.

$$B(xy, z) = B(x, z)y + xB(y, z), \quad B(x, x) = 0.$$

For any $\lambda \in C(\mathbf{A})$, the map $B(x, y) = \lambda[x, y]$, where $[x, y]$ is the commutator $xy - yx$, is an antisymmetric biderivation of \mathbf{A} . Such biderivations will be called *inner*.

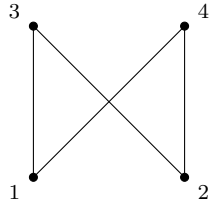
For any $\lambda \in C(\mathbf{A})$, the inner biderivation $\{a, b\} = \lambda[a, b]$ is clearly a Poisson structure on \mathbf{A} . Following [246], we call such Poisson structures *standard*. There is a generalization of this notion introduced in the same paper [246]. Namely, a Poisson structure on \mathbf{A} is said to be *piecewise standard* if \mathbf{A} decomposes into a direct sum $\bigoplus_{i=1}^m \mathbf{A}_i$ of indecomposable Lie ideals in such a way that $\{a, b\} = \lambda_i[a, b]$ for all $a \in \mathbf{A}_i$ and $b \in \mathbf{A}$, where $\lambda_i \in C(\mathbf{A})$.

It is well-known that the *upper triangular matrix algebra* $T_n(R)$ is the incidence algebra $I(C_n, R)$ of a chain C_n of cardinality n . There is a description of (not necessarily anti-symmetric) biderivations of $T_n(R)$ given by Benkovič in [33, Corollary 4.13] (case $n \geq 3$) and [33, Proposition 4.16] (case $n = 2$). A straightforward calculation, based on this description, shows that all the antisymmetric biderivations of $T_n(R)$ are inner. Consequently, we have the following (see also [195]).

Example 1. All the Poisson structures on $T_n(R)$ are standard.

This is not the case for a general incidence algebra, as the next easy example shows.

Example 2. Let R be a commutative ring and consider $P = \{1, 2, 3, 4\}$ with the following Hasse diagram (a crown).



Remark 2 (Remark 1.3, [145]). If R is a field, then the Poisson structure B from example 2 is not even piecewise standard.

Let us recall that the definition and some main properties of incidence algebras were considered in Section 5.3. For any pair $x \leq y$ we shall identify e_{xy} with $1_R e_{xy} \in FI(P, R)$ and denote by E the set $\{e_{xy} \mid x \leq y\} \subseteq FI(P, R)$. Denote by $\tilde{I}(P, R)$ the subalgebra of $FI(P, R)$ generated by E . Clearly, $\tilde{I}(P, R) = \text{span}\{E\}$ as an R -module. We will first deal with biderivations defined on $\tilde{I}(P, R)$.

Theorem 42 (Proposition 3.1, [145]). *For all $x \leq y$ and $u \leq v$ we have*

$$B(e_{xy}, e_{uv}) = \lambda(e_{xy}, e_{uv})[e_{xy}, e_{uv}]$$

for some map $\lambda : E \times E \rightarrow R$.

Theorem 43 (Proposition 3.5, [145]). *Let $\lambda : E \times E \rightarrow R$ be a symmetric map. Then the bilinear map B given by $B(e_{xy}, e_{uv}) = \lambda(e_{xy}, e_{uv})[e_{xy}, e_{uv}]$ is an antisymmetric biderivation of $\tilde{I}(P, R)$ with values in $FI(P, R)$ if and only if*

$$[e_{xy}, e_{x'y'}] \neq 0 \ \& \ [e_{uv}, e_{u'v'}] \neq 0 \ \Rightarrow \ \lambda(e_{xy}, e_{x'y'}) = \lambda(e_{uv}, e_{u'v'})$$

holds for any chain $C \subseteq P$.

Definition 16. A map $\sigma : (P \times P)_{<} \rightarrow R$ is said to be *constant on chains* if for any chain $C \subseteq P$ and for all $x < y, u < v$ from C one has $\sigma(x, y) = \sigma(u, v)$.

Theorem 44 (Theorem 3.7, [145]). *There is a one-to-one correspondence between the anti-symmetric biderivations B of $\tilde{I}(P, R)$ with values in $FI(P, R)$ and the maps $\sigma : (P \times P)_{<} \rightarrow R$, which are constant on chains. More precisely,*

$$B(f, g)(x, y) = \begin{cases} 0, & x = y, \\ \sigma(x, y)[f, g](x, y), & x < y, \end{cases} \quad (2)$$

where $f, g \in \tilde{I}(P, R)$.

Theorem 45 (Theorem 4.3, [145]). *The Poisson structures B on $FI(P, R)$ are in a one-to-one correspondence with the maps $\sigma : (P \times P)_{<} \rightarrow R$ which are constant on chains. The correspondence is given by eq. (2), in which $f, g \in FI(P, R)$.*

7.2 Generalized and generic Poisson algebras and superalgebras

A generalization of Poisson algebras known as generic Poisson algebras was studied in the papers by Kolesnikov, Makar-Limanov, Shestakov [190] and Kaygorodov, Shestakov, Umirbaev [174].

Definition 17. The triple $(A, \cdot, \{\cdot, \cdot\})$ is a generic Poisson algebra if (A, \cdot) is a commutative associative algebra, $(A, \{\cdot, \cdot\})$ is an anticommutative algebra and

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

In our paper written, together with Shestakov and Umirbaev [174], we establish that many results known true for Poisson algebras also hold for generic Poisson algebras. For example, we study the general properties of generic Poisson modules and of universal multiplicative enveloping algebras (Section 2, [174]); we determine the structure of free generic Poisson algebras and of free generic Poisson fields (Section 3, [174]); we prove an analog of Makar–Limanov–Shestakov’s Theorem, namely that two Poisson dependent elements in a free generic Poisson field are polynomial dependent (Section 4, [174]); we apply the obtained results to the study of automorphisms of the free generic Poisson

algebra $GP\{x, y\}$ with two generators and prove that the groups of automorphisms of the free generic Poisson algebra with two generators, of the free Poisson algebra with two generators, of the free associative algebra with two generators and of the free commutative associative algebra with two generators are isomorphic (Section 5, [174]).

Other generalizations of Poisson superalgebras include generalized Poisson superalgebras and superalgebras of Jordan brackets. Every Poisson superalgebra is a superalgebra of Jordan brackets [134, 136]. The identities defining unital superalgebras of Jordan brackets were described in [188]. Superalgebras of Jordan brackets are important in the classification of finite-dimensional simple Jordan superalgebras [207]. Kac and Cantarini studied linearly compact simple superalgebras of Jordan brackets [60], Kaygorodov and Zhelyabin studied δ -superderivations of simple superalgebras of Jordan brackets [179], Zelmanov, Shestakov, and Martínez studied the relation between Jordan brackets and Poisson brackets [206], and special superalgebras of Jordan brackets were considered in [179]. Kantor's construction gives interesting relations between Novikov–Poisson algebras and Jordan superalgebras [248].

A superalgebra \mathbf{A} is called a superalgebra of Jordan brackets whenever its Kantor double is a Jordan superalgebra [134, 136]. As [188] implies, a superalgebra \mathbf{A} whose associative–supercommutative multiplication \cdot and superanticommutative multiplication $\{\cdot, \cdot\}$ satisfy

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\} - D(a)bc,$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\} + D(a)\{b, c\} + (-1)^{|a||bc|}D(b)\{c, a\} + (-1)^{|c||ab|}D(c)\{a, b\},$$

with $D(a) = \{a, 1\}$, is a superalgebra of Jordan brackets. Observe that D is an even derivation of the superalgebra (\mathbf{A}, \cdot) . If $D = 0$, then $(\mathbf{A}, \cdot, \{\cdot, \cdot\})$ is a Poisson superalgebra [134]. As [60] implies, on the same vector space we can consider the new multiplication $\{a, b\}_D = \{a, b\} + \frac{1}{2}(aD(b) - D(a)b)$ and the new linear mapping $D_D = \frac{1}{2}D$. It is a new superalgebra with the identities

$$\{a, bc\}_D = \{a, b\}_D c + (-1)^{|a||b|}b\{a, c\}_D - D_D(a)bc,$$

$$\{a, \{b, c\}_D\}_D = \{\{a, b\}_D, c\}_D + (-1)^{|a||b|}\{b, \{a, c\}_D\}_D.$$

Every superalgebra satisfying the last two identities is called a generalized Poisson superalgebra (also known as a superalgebra of contact brackets).

In our paper, we construct bases for the n -generated free unital superalgebra of Jordan brackets and the n -generated free unital generalized Poisson superalgebra in [141, Theorems 2 and 5].

The celebrated Amitsur–Levitsky theorem states that the algebra $M_k(R)$ of $k \times k$ matrices over a commutative ring R satisfies the identity $s_{2k} = 0$. Furthermore, every associative PI algebra satisfies $(s_k)^l = 0$ by Amitsur's theorem. Farkas defined customary polynomials

$$g = \sum_{\sigma \in \mathbb{S}_m} c_\sigma \{x_{\sigma(1)}, x_{\sigma(2)}\} \cdots \{x_{\sigma(2i-1)}, x_{\sigma(2i)}\}$$

and proved that every Poisson PI algebra satisfies some customary identity [85]. Farkas' theorem was established for generic Poisson algebras in [190].

We have found an analog of customary identities for generalized Poisson algebras and algebras of Jordan brackets [141]:

$$g_* = \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{\sigma \in \mathbb{S}_m} c_{\sigma,i} \langle x_{\sigma(1)}, x_{\sigma(2)} \rangle \cdots \langle x_{\sigma(2i-1)}, x_{\sigma(2i)} \rangle D(x_{\sigma(2i+1)}) \cdots D(x_{\sigma(m)}), \quad (3)$$

where

$$\langle x, y \rangle := \{x, y\} - (D(x)y - xD(y)).$$

Theorem 46 (Theorem 13, [141]). *If a unital generalized Poisson algebra \mathbf{A} satisfies a polynomial identity g_0 , then \mathbf{A} satisfies a polynomial identity g_* of type (3).*

Consequently, we obtain the following statement.

Theorem 47 (Corollary 14, [141]). *Every PI generalized Poisson algebra satisfies an identity of the type*

$$f_* = \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{\sigma \in \mathbb{S}_m} c_{\sigma,i} \prod_{k=1}^i [x_{\sigma(2k-1)}; x_{\sigma(2k)}; z_{2k-1}; z_{2k}] \cdot \prod_{k=1}^{m-2i} \{x_{\sigma(2i+k)}; z_{2i+2k-1}; z_{2i+2k}\} \cdot \prod_{k=1}^{2i} z_{2m-2i+k}, \quad (4)$$

where

$$[u_1; u_2; w_1; w_2] = \{u_1, u_2\}w_1w_2 + \{u_1, w_1w_2\}u_2 + u_1\{w_1w_2, u_2\} + \sum_{\sigma_1, \sigma_2 \in \mathbb{S}_2} \{u_{\sigma_1(1)}, w_{\sigma_2(1)}\}u_{\sigma_1(2)}w_{\sigma_2(2)},$$

$$\{t_1; t_2; t_3\} = \{t_2t_3, t_1\} - \{t_2, t_1\}t_3 - \{t_3, t_1\}t_2.$$

Theorem 48 (Theorem 15, [141]). *If a unital algebra of Jordan brackets \mathbf{A} satisfies a polynomial identity g_0 , then \mathbf{A} satisfies polynomial identities of types (3) and (4).*

7.3 Transposed Poisson algebra structures

Recently, a dual notion of the Poisson algebra (transposed Poisson algebra) has been introduced in the paper by Bai, Bai, Guo, and Wu [20]. Roughly speaking, this new type of algebra is defined by exchanging the roles of the two binary operations in the Leibniz rule defining the Poisson algebra. The authors showed that the transposed Poisson algebras defined in this way not only share common properties with Poisson algebras, including the closure under taking tensor products and the Koszul self-duality as an operad but also admit a rich class of identities. More significantly, a transposed Poisson algebra naturally arises from a Novikov-Poisson algebra by taking the commutator Lie algebra of the Novikov algebra. Consequently, we find a parallelism between the classical construction of a Poisson algebra from a commutative associative algebra with a pair of commuting derivations and the construction of a transposed Poisson algebra when there is some non-zero derivation. Furthermore, the transposed Poisson algebras also capture the algebraic structure when the commutator is taken in pre-Lie Poisson algebras and two other algebras of Poisson type.

Definition 18. Let \mathbf{A} be a vector space equipped with two nonzero bilinear operations \cdot and $[\cdot, \cdot]$. The triple $(\mathbf{A}, \cdot, [\cdot, \cdot])$ is called a transposed Poisson algebra if (\mathbf{A}, \cdot) is a commutative associative algebra and $(\mathbf{A}, [\cdot, \cdot])$ is a Lie algebra that satisfies the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

The study of δ -derivations of Lie algebras was initiated by Filippov in 1998 [96, 97]. Let us define $\frac{1}{2}$ -derivations which will be very useful for our work.

Definition 19. Let $(\mathbf{A}, [\cdot, \cdot])$ be an algebra with multiplication $[\cdot, \cdot]$ and φ be a linear map. Then φ is a $\frac{1}{2}$ -derivation if it satisfies $\varphi[x, y] = \frac{1}{2}([\varphi(x), y] + [x, \varphi(y)])$.

Summarizing definitions 18 and 19, we have the following key statement.

Theorem 49 (Lemma 7, [93]). *Let $(\mathbf{A}, \cdot, [\cdot, \cdot])$ be a transposed Poisson algebra and z an arbitrary element from \mathbf{A} . Then the right multiplication R_z in the commutative associative algebra (\mathbf{A}, \cdot) gives a $\frac{1}{2}$ -derivation of the Lie algebra $(\mathbf{A}, [\cdot, \cdot])$.*

In our joint work with Ferreira and Lopatkin [93], we found a way to describe all transposed Poisson algebra structures with a certain Lie algebra. Our main tools are the description of the space of $\frac{1}{2}$ -derivations of Lie algebras and the connection between the space of $\frac{1}{2}$ -derivations of the Lie part of a transposed Poisson algebra and the space of right multiplications of the associative part of this transposed Poisson algebra. Namely, every right *associative* multiplication is a *Lie* $\frac{1}{2}$ -derivation. Using the known description of δ -derivations of semisimple finite-dimensional Lie algebras, we have found that there are no transposed Poisson algebras with a semisimple finite-dimensional Lie part. In the case of simple infinite-dimensional algebras, we have a different situation: it has been proved that the Witt algebra admits many nontrivial structures of transposed Poisson algebras. Later we studied structures of transposed Poisson algebras defined on one of the most interesting generalizations of the Witt algebra. Namely, we considered $\frac{1}{2}$ -derivations of the algebra $\mathcal{W}(a, b)$ and proved that the algebra $\mathcal{W}(a, b)$ does not admit structures of transposed Poisson algebras if and only if $b \neq -1$. All the transposed Poisson algebra structures defined on $\mathcal{W}(a, -1)$ have been described. In the following section, we considered another example of an algebra related to the Witt algebra. We proved that there are no transposed Poisson algebras defined on the Virasoro algebra. As for some corollaries, we proved that there are no transposed Poisson algebra structures defined on the $N = 1$ and $N = 2$ superconformal algebras. The rest of the paper is dedicated to a classification of $\frac{1}{2}$ -derivations and constructions of transposed Poisson algebras defined on the thin Lie algebra and the solvable Lie algebra with the abelian nilpotent radical of codimension 1. Other interesting examples of transposed Poisson algebras constructed in a series of papers together with Khrypchenko [143, 147–149].

8 n -Ary algebras

The present part is based on the papers written together with Alexandre Pozhidaev, Antonio Jesús Calderón, Elisabete Barreiro, José María Sánchez, Paulo Saraiva, and Yury Popov [22, 24, 50, 168, 172].

8.1 Ternary Jordan algebras

Based on the relation between the notions of Lie triple systems and Jordan algebras, we introduce the n -ary Jordan algebras [172], an n -ary generalization of Jordan algebras obtained via the generalization of the following property $[R_x, R_y] \in \mathfrak{Der}(\mathbb{A})$, where \mathbb{A} is an n -ary algebra. As we can see, the present generalization of algebras is an n -ary generalization of almost Jordan and commutative \mathfrak{CD} -algebras.

In [172], we give the first example of ternary Jordan algebras defined in this way and study some properties of this algebra. Namely, let \mathbb{V} be an n -dimensional vector space over a field \mathbb{F} , and denote the bilinear form by (\cdot, \cdot) . Consider the following ternary multiplication defined on \mathbb{V} :

$$[[x, y, z]] = (y, z)x + (x, z)y + (x, y)z.$$

Denote the obtained ternary algebra by \mathbb{A} . The following three theorems give us a characterization of \mathbb{A} .

Theorem 50 (Theorem 5, [172]). \mathbb{A} is a ternary Jordan algebra.

Theorem 51 (Theorem 6, [172]). The ternary Jordan algebra \mathbb{A} is simple, except if $\dim \mathbb{V} = 2$ and $\text{char}(\mathbb{F}) = 2$.

Theorem 52 (Theorem 11, [172]). $\mathfrak{Der}(\mathbb{A}) = \text{Inder}(\mathbb{A}) = \mathfrak{so}(n)$.

Remark 3 (Remark 12, [172]). In 1955 Jacobson proved that if a finite-dimensional Lie algebra over a field of characteristic zero has an invertible derivation, then it is a nilpotent algebra [124]. The same result was proved for Jordan algebras [166], but as we can see from Theorem 52, the Theorem of Jacobson is not true for ternary Jordan algebras. We can take the ternary Jordan algebra \mathbb{A} (as in Theorem 52) with dimension 4 and consider the map defined by the matrix $\sum_{1 \leq i < j \leq 4} (e_{ij} - e_{ji})$. As follows, there is a simple ternary Jordan algebra with an invertible derivation.

Later in [172], we study a ternary example of these algebras. Finally, based on the construction of a family of ternary algebras defined using the Cayley — Dickson algebras, we present an example of a ternary \mathfrak{CD} -algebra (n -ary \mathfrak{CD} -algebras are the n -ary version of (not necessarily commutative) \mathfrak{CD} -algebras) in [172, Section 5].

Let \mathcal{U}_2 be the generalized quaternions and \mathcal{U}_3 be the generalized octonions. Define on \mathcal{U}_t , for $t = 2, 3$, the ternary multiplication:

$$[[x, y, z]] = (x\bar{y})z,$$

and take $\mathcal{D}_t = (\mathcal{U}_t, \llbracket \cdot, \cdot, \cdot \rrbracket)$.

This ternary multiplication is not totally commutative, so these algebras are not ternary Jordan algebras.

Theorem 53 (Theorem 15, [172]). \mathcal{D}_2 is a simple ternary \mathfrak{CD} -algebra.

Theorem 54 (Lemma 15, [172]). \mathcal{D}_3 is not a ternary \mathfrak{CD} -algebra.

8.2 Split regular Hom-Leibniz color 3-algebras

The study of Hom-structures began in the paper by Hartwig, Larsson, and Silvestrov [114]. The notion of Hom-Lie triple systems was introduced in [202]. In the paper [247] Yau gave a general method for constructing Hom-type algebras starting from usual algebras and a twisting self-map.

In our joint work with Popov [168], we study the structure of split regular Hom-Leibniz 3-algebras of arbitrary dimension and over an arbitrary base field \mathbb{F} . Split structures first appeared in the classical theory of (finite-dimensional) Lie algebras, but have been extended to more general settings like, for example, Leibniz algebras [53], Poisson algebras, Leibniz superalgebras, regular Hom-Lie algebras, regular Hom-Lie superalgebras, regular Hom-Lie color algebras, regular Hom-Poisson algebras [14], regular Hom-Leibniz algebras, regular BiHom-Lie algebras [54], and regular BiHom-Lie superalgebras, among many others. As for the study of split ternary structures, see [49] for Lie triple systems, twisted inner derivation triple systems, Lie 3-algebras [49], Leibniz 3-algebras [55], and for Leibniz triple systems.

Definition 20. Let \mathbb{F} be a field and \mathbb{G} be an abelian group. A map $\epsilon : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{F}^\times$ is called a bicharacter on \mathbb{G} if the following relations hold for all $f, g, h \in \mathbb{G}$:

- (1) $\epsilon(f, g + h) = \epsilon(f, g)\epsilon(f, h)$;
- (2) $\epsilon(g + h, f) = \epsilon(g, f)\epsilon(h, f)$;
- (3) $\epsilon(g, h)\epsilon(h, g) = 1$.

Definition 21. A \mathbb{G} -graded color n -ary algebra T is a vector space $T = \bigoplus_{g \in \mathbb{G}} T_g$ with an n -linear map $[\cdot, \dots, \cdot] : T \times \dots \times T \rightarrow T$ satisfying

$$[T_{\theta_1}, \dots, T_{\theta_n}] \subseteq T_{\theta_1 + \dots + \theta_n}, \quad \theta_i \in \mathbb{G}.$$

Definition 22. A regular Hom-Leibniz color 3-algebra $(T, [\cdot, \cdot, \cdot], \epsilon, \phi)$ is a \mathbb{G} -graded vector space T with a bicharacter ϵ , an even trilinear map $[\cdot, \cdot, \cdot]$ and an even automorphism ϕ satisfying

$$\begin{aligned} & [\phi(x_1), \phi(x_2), [y_1, y_2, y_3]] = [[x_1, x_2, y_1], \phi(y_2), \phi(y_3)] + \\ & \epsilon(x_1 + x_2, y_1)[\phi(y_1), [x_1, x_2, y_2], \phi(y_3)] + \epsilon(x_1 + x_2, y_1 + y_2)[\phi(y_1), \phi(y_2), [x_1, x_2, y_3]]. \end{aligned}$$

Additionally, if the identities

$$[x_1, x_2, x_3] = -\epsilon(x_1, x_2)[x_2, x_1, x_3], \quad [x_1, x_2, x_3] = -\epsilon(x_2, x_3)[x_1, x_3, x_2]$$

hold in T , then T is called a regular Hom-Lie color 3-algebra, and if the identities

$$\begin{aligned} [x_1, x_2, x_3] &= -\epsilon(x_1, x_2)[x_2, x_1, x_3], \\ \epsilon(x_3, x_1)[x_1, x_2, x_3] + \epsilon(x_1, x_2)[x_2, x_3, x_1] + \epsilon(x_2, x_3)[x_3, x_1, x_2] &= 0 \end{aligned}$$

hold in T , then T is called a regular Hom-Lie triple color system.

Definition 23. Let $T = (T, [\cdot, \cdot, \cdot], \epsilon, \phi)$ be a Hom-Leibniz color 3-algebra. Consider the space $\mathfrak{L} = \text{span}\{\text{ad}(x, y) : x, y \in T\}$, where $\text{ad}(x, y)(z) := [x, y, z]$. The *standard embedding* of a regular Hom-Leibniz color 3-algebra $(T, [\cdot, \cdot, \cdot], \epsilon, \phi)$ is a color 2-graded algebra with an automorphism $(\mathbf{A}, \epsilon, \Phi)$, where $\mathbf{A}^0 := \mathfrak{L}$, $\mathbf{A}^1 := T$, the product is given by

$$\begin{aligned} &(\text{ad}(x, y), z) \cdot (\text{ad}(u, v), w) = \\ &(\text{ad}([x, y, u]^{\phi^{-1}}, v) + \epsilon(x+y, u)\text{ad}(u, [x, y, v]^{\phi^{-1}}) + \text{ad}(z, w) [x, y, w]^{\phi^{-1}} - \epsilon(z, u+v)[u, v, z]^{\phi^{-1}}), \end{aligned}$$

the automorphism Φ by

$$\Phi : \begin{cases} x & \mapsto \phi(x), \\ \text{ad}(x, y) & \mapsto \phi \text{ad}(x, y) \phi^{-1} = \text{ad}(\phi(x), \phi(y)), \end{cases} \quad \text{for } x, y \in T,$$

and the \mathbb{G} -grading is induced by the \mathbb{G} -gradings of T and \mathfrak{L} .

Definition 24. Let $T = (T, [\cdot, \cdot, \cdot], \epsilon, \phi)$ be a Hom-Leibniz color 3-algebra (or a Leibniz color 3-algebra with an automorphism) and let $\mathbf{A} = (\mathfrak{L} \oplus T, \cdot, \epsilon, \Phi)$ be its standard embedding. Let H be a maximal abelian subalgebra (shortly MASA) of \mathfrak{L}_0 , the zeroth \mathbb{G} -component of \mathfrak{L} . The root space of T with respect to H associated to a linear functional $\alpha \in H^*$ is the subspace

$$T_\alpha := \{v \in T : h \cdot v = \alpha(h)v \text{ for any } h \in H\}.$$

The elements $\alpha \in H^*$ such that $T_\alpha \neq 0$ are called roots of T (with respect to H), and we write $\Lambda^T := \{\alpha \in H^* \setminus \{0\} : T_\alpha \neq 0\}$. Analogously, by $\Lambda^{\mathfrak{L}}$ we denote the set of all non-zero $\alpha \in H^*$ such that $\mathfrak{L}_\alpha \neq 0$, where

$$\mathfrak{L}_\alpha := \{e \in \mathfrak{L} : [h, e] = \alpha(h)e \text{ for any } h \in H\}$$

are the root subspaces of \mathfrak{L} with respect to T .

Definition 25. Let T be a Hom-Leibniz color 3-algebra or a Leibniz color 3-algebra with an automorphism and $\mathbf{A} = \mathfrak{L} \oplus T$ its standard embedding. Then T is said to be a split algebra if there exists a MASA H of \mathfrak{L}_0 such that

$$T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha \right). \quad (5)$$

in the sense of Definition 24. The set Λ^T is called the root system of T . We refer to the decomposition (5) as the root spaces decomposition of T .

Definition 26. The root system Λ^T ($\Lambda^\mathfrak{L}$) is called symmetric if $\Lambda^T = -\Lambda^T$ ($\Lambda^\mathfrak{L} = -\Lambda^\mathfrak{L}$), where for $\emptyset \neq \Upsilon \subset H^*$ the set $-\Upsilon$ is just $\{-\alpha : \alpha \in \Upsilon\}$.

Theorem 55 (Theorem 3.16, [168]). *Let (T, ϕ) be a split Hom-Leibniz color 3-algebra with multiplication algebra \mathfrak{L} . Suppose that the root systems Λ^T and $\Lambda^\mathfrak{L}$ are symmetric. Then there exists an equivalence relation \sim on Λ^T and a subspace $\mathcal{U} \subseteq T_0$ such that $T = \mathcal{U} \oplus \sum_{[\alpha] \in \Lambda^T / \sim} T_{[\alpha]}$. Moreover,*

$$[T, T_{[\alpha]}, T_{[\beta]}] + [T_{[\alpha]}, T, T_{[\beta]}] + [T_{[\alpha]}, T_{[\beta]}, T] = 0$$

whenever $[\alpha] \neq [\beta]$.

8.3 n -Ary algebras with a multiplicative type basis

In the literature, it is usual to describe an algebra by exhibiting a multiplicative table among the elements of a fixed basis. There exist many classical examples of algebras admitting multiplicative bases in the setting of several varieties as associative algebras, Lie algebras, Malcev algebras, Leibniz algebras, Hom-Lie algebras, etc. For instance, in the class of associative algebras, we have the classes of full matrix algebras, group algebras, quiver algebras, etc. In the class of Lie algebras, we can consider the semisimple finite-dimensional Lie algebras over algebraically closed fields of characteristic 0, the Heisenberg algebras, the twisted Heisenberg algebras, and so on. In the class of Leibniz algebras, we have the classes of (complex) finite-dimensional naturally graded filiform Leibniz algebras and n -dimensional filiform graded filiform Leibniz algebras of length $n - 1$. By looking at the multiplication table of the non-Lie Malcev algebra M_7 (a 7-dimensional simple algebra), we have another example of an algebra with a multiplicative basis. For Zinbiel algebras we have, for instance, that any complex n -dimensional null-filiform Zinbiel algebra admits a multiplicative basis. We can also mention the simple finite-dimensional Filippov algebras in [95] as examples of n -ary algebras admitting a multiplicative basis ($n = 3, 4, \dots$). Many infinite-dimensional simple Filippov algebras admitting a multiplicative basis were constructed in papers by Bai, Ding, Pozhidaev, etc. The concept of multiplicative bases appears in a natural way in the study of different physical problems. One may expect that there are problems that are naturally and more simply formulated by exploiting multiplicative bases. There are many works about the study of algebras with a multiplicative basis [25, 36, 37, 42, 51, 52, 73, 111, 196, 208]. Calderón introduced the notion of quasi-multiplicative bases as a generalization of the notion of multiplicative bases in [42] and described the structure of associative algebras with a quasi-multiplicative basis.

Definition 27. A basis of homogeneous elements $\mathbb{B} = \{e_i\}_{i \in I}$ of a graded n -ary algebra A is multiplicative if for any $i_1, \dots, i_n \in I$ we have $[e_{i_1}, \dots, e_{i_n}] \in \mathbb{F}e_j$ for some $j \in I$.

In the paper [51] of Calderón, Navarro Izquierdo, and Sánchez Delgado, the authors considered a generalization of the multiplicative basis of an algebra. Let \mathbb{V} and \mathbb{W} be two vector spaces over a base field \mathbb{F} . It is said that \mathbb{V} is a module over \mathbb{W} if it is endowed with a bilinear map $\mathbb{V} \times \mathbb{W} \rightarrow \mathbb{V}$. A basis $\mathbb{B} = \{v_i\}_{i \in I}$ of \mathbb{V} is called multiplicative with respect

to another basis $\mathbb{B}' = \{w_j\}_{j \in J}$ of \mathbb{W} if for any $i \in I, j \in J$ we have $v_i w_j \in \mathbb{F}v_k$ for some $k \in I$. They showed that if \mathbb{V} admits a multiplicative basis in the above sense, then it decomposes as the direct sum $\mathbb{V} = \bigoplus V_\alpha$ of well-described submodules, admitting each one a multiplicative basis. Also, under a mild condition, the minimality of \mathbb{V} is characterized in terms of the multiplicative basis and it is shown that the above direct sum is using the family of its minimal submodules, admitting each one a multiplicative basis.

In our joint paper with Barreiro and Sánchez [24], we generalize the cited results to n -ary case. We study the structure of certain k -modules \mathbb{V} over linear spaces \mathbb{W} without restrictions neither on the dimensions of \mathbb{V} and \mathbb{W} nor on the base field \mathbb{F} . A basis $\mathbb{B} = \{v_i\}_{i \in I}$ of \mathbb{V} is called multiplicative with respect to another basis $\mathbb{B}' = \{w_j\}_{j \in J}$ of \mathbb{W} if for any $\sigma \in \mathbb{S}_n, i_1, \dots, i_k \in I$ and $j_{k+1}, \dots, j_n \in J$ we have

$$[v_{i_1}, \dots, v_{i_k}, w_{j_{k+1}}, \dots, w_{j_n}]_\sigma \in \mathbb{F}v_{r_\sigma}$$

for some $r_\sigma \in I$. We show that if \mathbb{V} admits a multiplicative basis then it decomposes as the direct sum $\mathbb{V} = \bigoplus_\alpha V_\alpha$ of well-described k -submodules V_α each one admitting a multiplicative basis. Also, the minimality of \mathbb{V} is characterized in terms of the multiplicative basis and it is shown that the above direct sum is using the family of its minimal k -submodules, admitting each one a multiplicative basis. Finally, we study an application of k -modules with a multiplicative basis over an arbitrary n -ary algebra with a multiplicative basis.

We continue our study of n -ary linear maps on a vector space in a joint paper together with Calderón and Saraiva [50]. Let \mathbb{V} be an arbitrary linear space and $f : \mathbb{V} \times \dots \times \mathbb{V} \rightarrow \mathbb{V}$ an n -linear map. It is proved that, for each choice of a basis \mathbb{B} of \mathbb{V} , the n -linear map f induces a (nontrivial) decomposition $\mathbb{V} = \bigoplus V_j$ as a direct sum of linear subspaces of \mathbb{V} , with respect to \mathbb{B} . It is shown that this decomposition is f -orthogonal in the sense that $f(\mathbb{V}, \dots, V_j, \dots, V_k, \dots, \mathbb{V}) = 0$ when $j \neq k$, and in such a way that any V_j is strongly f -invariant, meaning that $f(\mathbb{V}, \dots, V_j, \dots, \mathbb{V}) \subset V_j$. We deduce a sufficient condition for the existence of an isomorphism between two different decompositions of \mathbb{V} induced by an n -linear map f , concerning two different bases of \mathbb{V} . We also characterize the f -simplicity – an analog of the usual simplicity in the framework of n -linear maps – of any linear subspace V_j of a certain decomposition induced by f . Finally, an application to the structure theory of arbitrary n -ary algebras is provided. This work is a close generalization of the results obtained by Calderón (2018) [43].

The rest of our results in the present topic are dedicated to a generalization of the paper [42] of Calderón. Namely, in a joint work together with Barreiro, Calderón, and Sánchez [22], we obtained an analog of Calderón's results in the class of color generalized Lie type n -ary algebras.

Definition 28. A graded n -ary algebra \mathbf{A} admits a *quasi-multiplicative basis* if $\mathbf{A} = \mathbb{V} \oplus \mathbb{W}$, with \mathbb{V} and $0 \neq \mathbb{W}$ graded linear subspaces in such a way that there exists a basis of homogeneous elements $\mathbb{B} = \{e_i\}_{i \in I}$ of \mathbb{W} satisfying:

1. For $i_1, \dots, i_n \in I$ we have either $[e_{i_1}, \dots, e_{i_n}] \in \mathbb{F}e_j$ for some $j \in I$ or $[e_{i_1}, \dots, e_{i_n}] \in \mathbb{V}$.

2. Given $0 < k < n$, for $i_1, \dots, i_k \in I$ and $\sigma \in \mathbb{S}_n$ we have $[e_{i_1}, \dots, e_{i_k}, \mathbb{V}, \dots, \mathbb{V}]_\sigma \subset \mathbb{F}e_{j_\sigma}$ for some $j_\sigma \in I$.
3. We have either $[\mathbb{V}, \dots, \mathbb{V}] \subset \mathbb{F}e_j$ for some $j \in I$ or $[\mathbb{V}, \dots, \mathbb{V}] \subset \mathbb{V}$.

Definition 29. An n -ary algebra $(\mathbf{A}, [\cdot, \dots, \cdot])$ is called a *generalized Lie-type algebra* if it satisfies the following n identities:

$$[y_1, \dots, \underbrace{[x_1, \dots, x_n]}_{\text{pos } k}, \dots, y_{n-1}] = \sum_{\substack{1 \leq i, j \leq n \\ \sigma_1 \in \mathbb{S}_n \\ \sigma_2 \in \mathbb{S}_{n-1}}} \alpha_{i,j,k}^{\sigma_1, \sigma_2} [x_{\sigma_1(1)}, \dots, x_{\sigma_1(i-1)}, [y_{\sigma_2(1)}, \dots, \underbrace{x_{\sigma_1(i)}}_{\text{pos } j}, \dots, y_{\sigma_2(n-1)}], x_{\sigma_1(i+1)}, \dots, x_{\sigma_1(n)}],$$

for $k = 1, \dots, n$, being $\alpha_{i,j,k}^{\sigma_1, \sigma_2} \in \mathbb{F}$, and where *pos* j means that the element $x_{\sigma_1(i)}$ is in the position j in the inside n -product.

Observe that we can obtain several binary and n -ary algebras depending of the values of $\alpha_{i,j,k}^{\sigma_1, \sigma_2}$:

- Lie algebras, Leibniz algebras, Novikov algebras, associative algebras, alternative algebras, bicommutative algebras, commutative pre-Lie algebras, etc.;
- n -Lie (Filippov) algebras, commutative Leibniz n -ary algebras, totally associative-commutative n -ary algebras, etc.

Theorem 56 (Theorem 21, [22]). *A color generalized Lie type algebra $\mathbf{A} = \mathbb{V} \oplus \mathbb{W}$ admitting a quasi-multiplicative basis of $\mathbb{W} \neq 0$ decomposes as*

$$\mathbf{A} = \mathcal{U} \oplus \left(\sum_{[i] \in I/\sim} \mathfrak{J}_{[i]} \right),$$

where \mathcal{U} is a linear complement of $\sum_{[i] \in I/\sim} \mathbb{V}_{[i]}$ in \mathbb{V} and any $\mathfrak{J}_{[i]}$ is one of the color generalized Lie-type ideals, admitting a quasi-multiplicative basis. Furthermore

$$[\mathfrak{J}_{[i]}, \mathfrak{J}_{[h]}, \mathbf{A}, \dots, \mathbf{A}]_\sigma = 0$$

whenever $[i] \neq [h]$.

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