Communications in Mathematics **33** (2025), no. 3, Paper no. 1.
DOI: https://doi.org/10.46298/cm.12613

Roots and right factors of polynomials and left eigenvalues of matrices over Cayley–Dickson algebras

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Abstract. Over a composition algebra A, a polynomial $f(x) \in A[x]$ has a root α if and only $f(x) = g(x) \cdot (x - \alpha)$ for some $g(x) \in A[x]$. We examine whether this is true for general Cayley–Dickson algebras. The conclusion is that it is when $f(x)$ is linear or monic quadratic, but it is false in general. Similar questions about the connections between f and its companion $C_f(x) = f(x) \cdot \overline{f(x)}$ are studied. Finally, we compute the left eigenvalues of 2×2 octonion matrices.

1 Introduction

Given an algebra A (in this article an algebra is unital but is not necessarily commutative or associative) with involution τ over a field F, and a choice of $\gamma \in F^\times$, the Cayley–Dickson double $B = A\{\gamma\}$ of A is defined to be $A \oplus A\ell$ with the product

$$(q + r\ell)(s + t\ell) = qs + \gamma\bar{t}r + (tq + r\bar{s})\ell$$

for any $q, r, s, t \in A$, and where $\overline{x} = \tau(x)$ for any $x \in A$. The involution τ extends to B by defining $\overline{q + r\ell} = \overline{q} - r\ell$. A Cayley–Dickson algebra is obtained by repeating this process several times, starting with a quadratic separable extension K of F, with τ being the unique non-trivial automorphism of order 2 of K acting trivially on F. We denote such an algebra A by $K\{\gamma_2, \dots, \gamma_m\}$ where $\gamma_2, \dots, \gamma_m$ are the chosen elements of $F^\times$, in this order. When $\mathrm{char}(F) \neq 2$, K is actually $F\{\gamma_1\}$ for the right choice of $\gamma_1 \in F^\times$, and thus $A = F\{\gamma_1, \gamma_2, \dots, \gamma_m\}$. Such algebras (regardless of the characteristic of the base field) are

MSC 2020: primary 17A75; secondary 17A45, 17A35, 17D05, 37P35, 37C25
Keywords: Cayley–Dickson Algebras; Polynomials; Roots; Decomposition
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endowed with a linear trace form $\mathrm{Tr}(q) = q + \overline{q}$ and a quadratic norm form $\mathrm{Norm}(q) = q \cdot \overline{q}$ satisfying $q^2 - \mathrm{Tr}(q)q + \mathrm{Norm}(q) = 0$ for any $q \in A$.

When $\dim A = 4$, A is called a quaternion F-algebra, and when $\dim A = 8$ an octonion algebra, and when $\dim A = 16$ a sedenion algebra. Special attention is given in the literature to the Cayley–Dickson algebras constructed over the real numbers by repeatedly choosing $\gamma = -1$, i.e., $\mathbb{R}\{-1, -1, \dots, -1\}$. The first four of those are $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and $\mathbb{S}$ (resp. the complex field, Hamilton's quaternions, the real octonion division algebra and the real sedenion algebra). These algebras can be described as real Cayley–Dickson algebras with anisotropic norm form, or as locally-complex Cayley–Dickson algebras (a real unital algebra is called locally-complex if any nonscalar element generates a subalgebra isomorphic to $\mathbb{C}$; c.f. [1]).

The norm form of a Cayley–Dickson algebra A is multiplicative as long as $\dim A \leq 8$. When $\dim A \geq 16$, it stops being multiplicative. For example, when $A = \mathbb{S}$ with its standard basis $e_0, e_1, \dots, e_{15}$ (for the definition of the standard basis in terms of the generators of $\mathbb{S}$ and its multiplication table, see for example [5, Section 2.2] and [2, Section 2]), the elements $\alpha = e_1 + e_{10}$ and $\beta = e_7 + e_{12}$ are of norm 2, so having a multiplicative norm form would mean $\mathrm{Norm}(\alpha\beta) = 4$, but in fact $\alpha\beta = 0$.

In this article we address several questions that came up in the writing of the articles [5, 7]. In Section 2 we discuss the decomposition of the form $f(x) = g(x) \cdot (x - \lambda)$ for roots λ of polynomials with coefficients in a Cayley–Dickson algebra. We show that such a decomposition exists for linear and monic quadratic polynomials but does not exist in general for Cayley–Dickson algebras of dimension ≥ 16. We also show that in the latter algebras there exist polynomials which have no roots. Finally, we give some counterexamples to questions regarding roots and critical points of Cayley–Dickson polynomials.

In Section 3 we prove that every octonion matrix has a left eigenvalue, and provide a method for their computation. This provides a direct generalization of Huang and So's results for quaternion matrices in [12].

2 Polynomials over Cayley–Dickson Algebras

Given a Cayley–Dickson algebra A over F, we define the polynomial algebra $A[x]$ to be $A \otimes_F F[x]$; i.e., the indeterminate x behaves like a central element, and in particular, every commutator or associator involving x is trivial. Every polynomial $f(x) \in A[x]$ can thus be written as $f(x) = c_n x^n + \dots + c_1 x + c_0$ for some $c_0, c_1, \dots, c_n \in A$. The substitution of $\lambda \in A$ in $f(x)$ is defined by $f(\lambda) = c_n(\lambda^n) + \dots + c_1\lambda + c_0$. This expression is well-defined as Cayley–Dickson algebras are always power-associative, see [15]. We say that $\lambda \in A$ is a root of $f(x) \in A[x]$ if $f(\lambda) = 0$.

For $f(x) \in A[x]$, the companion polynomial of $f(x)$ is defined by

$$C_f(x) = \mathrm{Norm}(f(x)) = f(x) \cdot \overline{f(x)} \in F[x],$$

where $\overline{f(x)} = \overline{c_n} x^n + \dots + \overline{c_1} x + \overline{c_0}$. It is known that when $\dim A \leq 8$, every root of $f(x)$ is a root of $C_f(x)$, and every root of $C_f(x)$ has an element quadratically equivalent to it, i.e., an element with the same trace and norm, which is a root of $f(x)$; see [3].

2.1 Roots and linear factors of Cayley–Dickson Polynomials

It is known that when A is a Cayley–Dickson algebra of $\dim A \leq 8$, then $\lambda \in A$ is a root of $f(x) \in A[x]$ if and only if $x - \lambda$ is a right factor of $f(x)$, i.e., $f(x) = g(x) \cdot (x - \lambda)$ for some $g(x) \in A[x]$, see [3].

Question 2.1. *Given a polynomial $f(x)$ over a Cayley–Dickson algebra A and $\lambda \in A$, is it true that λ is a root of $f(x)$ if and only if there exists $g(x) \in A[x]$ such that*

$$f(x) = g(x) \cdot (x - \lambda)?$$

We begin by settling Question 2.1 for linear and monic quadratic polynomials:

Proposition 2.2. *When $f(x) \in A[x]$ is either linear or monic quadratic, an element $\lambda \in A$ is a root of $f(x)$ if and only if $f(x) = g(x) \cdot (x - \lambda)$ for some $g(x) \in A[x]$.*

Proof. When $f(x)$ is linear, $f(x) = ax + b$ for some $a, b \in A$. If λ is a root, then

$$f(\lambda) = a\lambda + b = 0,$$

and so $b = -a\lambda$, which means $f(x) = ax - a\lambda = a(x - \lambda)$. In the opposite direction, when $(x - \lambda)$ is a right factor of $f(x)$, there exists $a \in A$ such that $f(x) = a(x - \lambda) = ax - a\lambda$, and then clearly $f(\lambda) = 0$.

When $f(x)$ is monic quadratic, $f(x) = x^2 + ax + b$. If λ is a root, then $f(\lambda) = \lambda^2 + a\lambda + b = 0$, and so $b = -\lambda^2 - a\lambda$, which means

$$f(x) = x^2 + ax - \lambda^2 - a\lambda = (x + \lambda)(x - \lambda) + a(x - \lambda) = (x + \lambda + a)(x - \lambda).$$

In the opposite direction, if $x - \lambda$ is a right factor, then $f(x) = (x + c)(x - \lambda)$ for some $c \in A$, and so $f(x) = x^2 + cx - \lambda x - c\lambda$, which means $f(\lambda) = \lambda^2 + c\lambda - \lambda^2 - c\lambda = 0$. □

We now provide negative answers to Question 2.1 in general, both for the implication that having a root λ means having a right factor $x - \lambda$, and its converse statement.

Example 2.3. Consider $A = \mathbb{S}$ with its standard basis $e_0, \dots, e_{15}$, and denote $\alpha = e_1 + e_{10}$ and $\beta = e_7 + e_{12}$. The elements α and β are zero divisors, and we have $\alpha\beta = \beta\alpha = 0$. We remark that anisotropic Cayley–Dickson algebras over fields of characteristic different from 2 are reversible, i.e. $\alpha\beta = 0$ implies $\beta\alpha = 0$; see [8, Theorem 2.3].

Now let $f(x) = \frac{1}{2}\beta x^2 + \beta$. Since

$$f(\alpha) = \frac{1}{2}\beta \cdot (\alpha^2) + \beta = \frac{1}{2}\beta \cdot (-2) + \beta = 0,$$

α is a root of $f(x)$. However, if $x - \alpha$ were a right factor of $f(x)$, there would exist $c \in A$ for which

$$f(x) = (\frac{1}{2}\beta x + c)(x - \alpha) = \frac{1}{2}\beta x^2 + cx - c\alpha,$$

implying both $c = 0$ and $c\alpha = -\beta$, a contradiction.

Example 2.4. Using the notations from the previous example, consider $f(x) = \beta x^2 + \beta x$. This polynomial decomposes as $f(x) = (\beta x + \beta)(x - \alpha)$, but $f(\alpha) = -2\beta$, so α is not a root of $f(x)$.

The question of real Cayley–Dickson algebras being "algebraically closed" is classic. In this context, "algebraically closed" means that every polynomial of degree at least 1 has a root in the algebra. This was proved in the wider setting of general real quaternion polynomials for polynomials with one leading monomial by Eilenberg and Niven (See [10]). This result was subsequently generalized to the real octonions by Jou [13], and to more complicated polynomials over any finite-dimensional real composition division algebra in [16]. For the special case of left or one-sided polynomials over $\mathbb{O}$ as discussed in this article, we propose the following short proof.

Proposition 2.5. *Every polynomial $f(x) \in \mathbb{O}[x]$ of degree at least 1 has a root in $\mathbb{O}$.*

Proof. The companion polynomial of $f(x)$ is defined by $C_f(x) = \mathrm{Norm}(f(x)) = f(x) \cdot \overline{f(x)}$. It is a polynomial with *real* coefficients of degree at least 2, and thus it admits a complex root α. Since the complex numbers embed into $\mathbb{O}$, we can assume that $\alpha \in \mathbb{O}$. By [4, Theorem 3.4], the root α of $C_f(x)$ is quadratically equivalent to some root of $f(x)$ in $\mathbb{O}$ (i.e., there exists a root of $f(x)$ with the same trace and norm as α). ☐

Corollary 2.6. *Every polynomial $f(x) \in \mathbb{O}[x]$ decomposes as a product of linear factors of the form*

$$f(x) = ((...(c(x - \lambda_n)) \dots (x - \lambda_3))(x - \lambda_2))(x - \lambda_1).$$

Proof. This follows from the former proposition and the correspondence between linear right factors and roots of octonion polynomials described at the beginning of the section and proven in [3]. ☐

We remark that the λ_i for $2 \leq i \leq n$ in Corollary 2.6 are not necessarily roots of $f(x)$. A natural question to ask is whether this type of decomposition extends to higher dimensional Cayley–Dickson algebras.

Question 2.7. *Are all real Cayley–Dickson algebras with anisotropic norm form algebraically closed?*

The answer to Question 2.7 is evidently negative, as real anisotropic Cayley–Dickson algebras of dimension ≥ 16 contain zero divisors. Each such zero divisor α defines a linear map $q \mapsto \alpha q$ on A which is not surjective, implying that there exists some $\beta \in A$ such that there is no solution in A to the equation $\alpha x = \beta$. This proves that the linear polynomial $f(x) = \alpha x - \beta$ has no roots in A. We now provide a concrete counterexample in $\mathbb{S}$:

Example 2.8. Let α and β be the elements of $A = \mathbb{S}$ from Example 2.3, and let $f(x) = \alpha x - \beta$. We show that this linear polynomial has no roots in $\mathbb{S}$.

Consider the real-valued symmetric bilinear form $\langle a, b \rangle$ associated with the quadratic form $\mathrm{Norm}(a)$ on an arbitrary Cayley–Dickson algebra A over $\mathbb{R}$, with $\langle a, a \rangle = \mathrm{Norm}(a)$

for all $a \in A$. The bilinear form satisfies the identities $\langle a, bc\rangle = \langle a\bar{c}, b\rangle = \langle \bar{b}a, c\rangle$ for all $a, b, c \in A$ (see [15, Lemma 6]). Consequently, if $\alpha x = \beta$, it follows that

$$2 = \langle \beta, \beta\rangle = \langle \alpha x, \beta\rangle = -\langle x, \alpha\beta\rangle = -\langle x, 0\rangle = 0,$$

a contradiction.

2.2 The Companion Polynomial

Let A be a Cayley–Dickson algebra over F, and $f(x) \in A[x]$. Recall that when $\dim A \leq 8$, every root of $f(x)$ is a root of $C_f(x)$, and every root of $C_f(x)$ has an element quadratically equivalent to it. In [5] it was pointed out that $f(x) = \alpha x \in \mathbb{S}[x]$ has β as a root even though β is not a root of $C_f(x) = 2x^2$ (where α and β are as in Example 2.3). The second part of the property stated in the first line of this paragraph has not been dealt with though, which brings up the following question.

Question 2.9. *Is every root of $C_f(x)$ quadratically equivalent to some root of $f(x)$, even when $\dim A \geq 16$?*

The answer is in general no:

Example 2.10. In example 2.8 we showed that the polynomial $f(x) = \alpha x - \beta \in \mathbb{S}[x]$ has no roots, but its companion $C_f(x) = 2x^2 + 2$ certainly does (all the elements of trace 0 and norm 1).

2.3 Isolated and Spherical Roots

Another property of $f(x) \in A[x]$ when $\dim A \leq 8$ and its norm form is anisotropic is that every root of $f(x)$ is either isolated (i.e., it is the only element in the quadratic equivalence class that is a root of $f(x)$), or spherical (i.e., every element in the quadratic equivalence class of the root is also a root of $f(x)$).

Question 2.11. *When the norm form of A is anisotropic, is every root of a polynomial $f(x)$ in $A[x]$ either spherical or isolated?*

This is not true when $\dim A \geq 16$:

Example 2.12. The polynomial $f(x) = \alpha x \in \mathbb{S}[x]$ has both β and $-\beta$ as quadratically equivalent roots, but α is not a root, even though α is quadratically equivalent to β.

2.4 Critical Points

The Gauss–Lucas theorem states that the critical points of $f(x) \in \mathbb{C}[x]$ (i.e. the roots of the derivative $f'(x)$) are contained in the convex hull of the roots of $f(x)$. In [11] it was shown that this does not extend to $f(x) \in \mathbb{H}[x]$ (here the derivative $f'(x)$ is defined formally). In [5, Theorem 4.2] it was proven that the *spherical* critical points of $f(x)$ (again, $f'(x)$ is defined formally) are contained in the convex hull of the roots of $C_f(x)$ for any $f(x) \in A[x]$ where A is any real anisotropic Cayley–Dickson algebra.

Question 2.13. *Are the spherical roots of $f'(x)$ contained in the convex hull of the roots of $f(x)$?*

We hereby show that this does not hold:

Example 2.14. The polynomial $f(x) = \frac{1}{3}x^3 + x + i \in \mathbb{H}[x]$ has three isolated roots, none of which are spherical. Indeed, the polynomial has three complex roots, but no real quadratic factors, i.e., no two roots are complex-conjugate. If $f(x)$ had a spherical root λ, it would have had exactly two distinct roots quadratically equivalent to λ in $\mathbb{C}$ (see e.g. [18, Theorem 2.2]; or [5, Corollary 3.8] for general locally-complex Cayley–Dickson algebras), and these must be complex-conjugate, leading to a contradiction. Note however, that $f'(x) = x^2 + 1$ does have spherical roots, most of which are not contained in the convex hull of the roots of $f(x)$, which are all complex.

3 Roots and Left Eigenvalues

When A is a non-commutative algebra, the notion of eigenvalues of a matrix B with coefficients in A divides into two categories: left eigenvalues λ which satisfy $B\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$, and right eigenvalues which satisfy $B\vec{v} = \vec{v}\lambda$ (in either case we say that $\vec{v}$ is the eigenvector associated to the eigenvalue λ). We focus here on left eigenvalues. The motivating observation is that λ is a left eigenvalue of $B \in M_n(A)$ if and only if there exists a nonzero column vector $\vec{v}$ in A^n (the direct product of n copies of A) for which $(B - \lambda I)\vec{v} = 0$, which means $B - \lambda I$ defines (by multiplication from the left) a singular linear endomorphism of A^n as an F-vector space.

In the associative case, right eigenvalues are well-understood. In [14] it was shown that over $\mathbb{H}$ they can be easily found through the embedding of $M_n(\mathbb{H})$ into $M_{2n}(\mathbb{C})$, and this method was generalized in [6] to any associative central division algebra. When A is a quaternion algebra, it was shown in [12] that the left eigenvalues of 2×2 matrices over A can be found by solving a quadratic equation. The goal of this section is to do the analogous thing for octonion division algebras. Note that the special case of 2×2 and 3×3 Hermitian matrices over $\mathbb{O}$ was studied in [9].

Let A be an octonion division algebra over a field F. Consider a matrix $B \in M_n(A)$. Let $\sigma_L(B)$ denote the set of left eigenvalues of B, and let

$$\Sigma_L(B) = \{(\lambda, \vec{v}) : \lambda \in A, \vec{0} \neq \vec{v} \in A^n, B\vec{v} = \lambda\vec{v}\}.$$

Lemma 3.1. *Let λ be a left eigenvalue of a matrix $B \in M_n(A)$ and $\vec{v} \in A^n$ an associated eigenvector, where A is an octonion division algebra over a field F. Then for any $0 \neq e \in A$, the element $e\lambda$ is an eigenvalue of the matrix eB with an associated eigenvector $\vec{v}e$.*

Proof. The algebra A satisfies the Moufang identity $(xy)(zx) = x(yz)x$. Therefore for any $e \in A$ we have $(eB)(\vec{v}e) = e(B\vec{v})e = e(\lambda\vec{v})e = (e\lambda)(\vec{v}e)$. □

Lemma 3.2. *Let $B \in M_n(A)$ be a lower (upper) triangular matrix with diagonal $(a_1, \dots, a_n)$, where A is any (not necessarily associative) division algebra over a field F. Then $\sigma_L(B) = \{a_1, \dots, a_n\}$.*

Proof. A direct elementary proof when A is a field does not use associativity or commutativity, so transfers to our case. To summarize, it is clear that $\sigma_L(B) \supseteq \{a_1, \dots, a_n\}$, since it is easy to see that for each a_i for $1 \leq i \leq n$, an associated eigenvector is $\vec{e_i} = (0, \dots, 1, \dots 0)^T$ with 1 in the i-th position (Here and in the rest of the article $\vec{v}^T$ denotes the transpose of the vector $\vec{v}$). In the other direction, let $\lambda \in \sigma_L(B)$, and $\vec{v} = (v_1, \dots, v_n)^T \in A^n$ be an associated eigenvector. The first row of B produces the equation $a_1 v_1 = \lambda v_1$ so either $\lambda = a_1$ or $v_1 = 0$. If $v_1 = 0$ we continue to the second row to obtain $a_2 v_2 = \lambda v_2$ implying either $\lambda = a_2$ or $v_2 = 0$. Continuing by induction we prove that either $\lambda \in \{a_1, \dots, a_n\}$ or $\vec{v} = 0$, where the latter is a contradiction. □

For a given $f(x) \in A[x]$, let $R(f(x))$ be the set of roots of $f(x)$ in A. We denote by $LMR(f(x))$ the union of $R(c \cdot f(x))$ where c ranges over all nonzero elements of A.

Lemma 3.3. *Let* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a matrix defined over an alternative algebra A. A vector of the form* $\vec{v} = \begin{pmatrix} 1 \\ s \end{pmatrix} \in A^2$ *is an eigenvector associated to an eigenvalue $\lambda \in \sigma_L(B)$ if and only if s is a root of the quadratic polynomial $f(x) = bx^2 + (a-d)x - c \in A[x]$ and $\lambda = a + bs$.*

Proof. The equation $B\vec{v} = \lambda\vec{v}$ induces the following system of equations:

$$\begin{cases} a + bs = \lambda \\ c + ds = \lambda s \end{cases} \tag{1}$$

Substituting $\lambda = a + bs$ in the second equation, and using alternative identity $(bs)s = bs^2$ we obtain $f(s) = bs^2 + (a-d)s - c = 0$, as required. The other direction follows readily by taking a root s of $f(x)$, setting $\lambda = a + bs$ and comparing $B\vec{v}$ and $\lambda\vec{v}$. □

Lemma 3.4. *Let M be a Moufang loop, and $x, y, z \in M$. Then $x((x^{-1}y)z) = (y(zx))x^{-1}$.*

Proof. Let $t = x^{-1}y$. Then $y = xt$ and

$$(y(zx))x^{-1} = ((xt)(zx))x^{-1} = (x(tz)x)x^{-1} = x((tz)(xx^{-1})) = x(tz) = x((x^{-1}y)z).$$

The second equality follows from the Moufang identity $(xt)(zx) = x(tz)x$, and the third by the Moufang identity $(zxz)y = z(x(zy))$ (or the fact that x and tz form a subgroup of M). □

Theorem 3.5. *Given a matrix* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *over a division octonion algebra A, if $b = 0$ then $\sigma_L(B) = \{a, d\}$, and if $b \neq 0$ then*

$$\Sigma_L(B) = \left\{ \left(\lambda, \begin{pmatrix} 1 \\ s \end{pmatrix} \cdot t \right) : t \in A^\times, s \in R(t^{-1}f(x)), \lambda = a + t((t^{-1}b)s) \right\},$$

where $f(x) = bx^2 + (a-d)x - c$*. In particular, when* $b \in F^\times$*, we have*

$$\sigma_L(B) = \{\lambda : b^{-1}(\lambda - a) \in LMR(f(x))\}.$$

Proof. When $b = 0$, the result follows from Lemma 3.2, so we assume that $b \neq 0$. Now let $\lambda \in \sigma_L(B)$, and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in A^2$ be an associated eigenvector. If $v_1 = 0$ then from $B\vec{v} = \lambda\vec{v}$ we get $bv_2 = 0$, so $v_2 = 0$, a contradiction. Therefore, if $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector, then $v_1 \neq 0$. By Lemma 3.1, we get that $v_1^{-1}\lambda$ is a left eigenvalue of the matrix $v_1^{-1}B$ with associated eigenvector $\vec{v}v_1^{-1} = \begin{pmatrix} 1 \\ v_2v_1^{-1} \end{pmatrix}$.

By Lemma 3.3 we get that $v_2v_1^{-1}$ is a root of $v_1^{-1}f(x)$, where $f(x) = bx^2 + (a-d)x - c$, with $v_1^{-1}\lambda = v_1^{-1}a + (v_1^{-1}b)(v_2v_1^{-1})$. Using the Moufang identity $(xy)(zx) = x(yz)x$ on the last equality, we get $v_1^{-1}\lambda = v_1^{-1}a + v_1^{-1}(bv_2)v_1^{-1}$. Multiplying by v_1 on the left, we get $\lambda = a + (bv_2)v_1^{-1}$ (Here we used the inverse identity $x^{-1}(xy) = y$ for $x \neq 0$). Summing up, we know that $v_2v_1^{-1} \in LMR(f(x))$, and so

$$\Sigma_L(B) \subseteq \left\{ \left(a + (bv_2)v_1^{-1}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) : v_1 \neq 0, v_2v_1^{-1} \in R(v_1^{-1}f(x)) \right\}. \tag{2}$$

Now let $\left(a + (bv_2)v_1^{-1}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$ be a pair in the set on the right hand side of (2). By $v_2v_1^{-1} \in R(v_1^{-1}f(x))$ and Lemma 3.3 we get that $\begin{pmatrix} 1 \\ v_2v_1^{-1} \end{pmatrix}$ is an eigenvector of the matrix $v_1^{-1}B$ associated to the eigenvalue $v_1^{-1}a + (v_1^{-1}b)(v_2v_1^{-1})$. Thus by Lemma 3.1 we know $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of the matrix B associated to the eigenvalue $a + (bv_2)v_1^{-1}$ (we used the Moufang identity $(xy)(zx) = x(yz)x$ to simplify the eigenvalue to this form), so $\left(a + (bv_2)v_1^{-1}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \in \Sigma_L(B)$. We have thus proved that

$$\Sigma_L(B) = \left\{ \left(a + (bv_2)v_1^{-1}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) : v_1 \neq 0, v_2v_1^{-1} \in R(v_1^{-1}f(x)) \right\}.$$

Setting $t = v_1$ and $s = v_2v_1^{-1}$, we obtain (here we used the identity $x((x^{-1}y)z) = (y(zx))x^{-1}$ from Lemma 3.4)

$$\Sigma_L(B) = \left\{ \left(\lambda, \begin{pmatrix} 1 \\ s \end{pmatrix} \cdot t \right) : t \in A^\times, s \in R(t^{-1}f(x)), \lambda = a + t((t^{-1}b)s) \right\}.$$

When $b \in F^\times$, λ depends only on the value of s and not of t, and thus $\sigma_L(B) = \{\lambda : b^{-1}(\lambda - a) \in LMR(f(x))\}$. □

It is known that any matrix with quaternion coefficients has a left eigenvalue, see [17]. At least for 2×2 matrices we can say the same for octonion matrices.

Corollary 3.6. *Let $B \in M_2(\mathbb{O})$. Then $\sigma_L(B) \neq \emptyset$, i.e. the 2×2 matrix B always has a left eigenvalue.*

Proof. Setting $t = 1$ in the theorem, and using the fact that $\mathbb{O}$ is "algebraically closed" (see Proposition 2.5), there always exists an eigenvector of the form $\begin{pmatrix} 1 \\ s \end{pmatrix}$ for some $s \in \mathbb{O}$ associated to the eigenvalue $a + bs$. □

Corollary 3.7. *Suppose $b \neq 0$. When $d = 0$, we have $0 \in \sigma_L(B)$ if and only if $c = 0$. When $d \neq 0$, we have $0 \in \sigma_L(B)$ if and only if there exists a nonzero $t \in A$ satisfying $d^{-1}(ct) - b^{-1}(at) = 0$.*

Proof. Let $\left(\lambda, \begin{pmatrix} 1 \\ s \end{pmatrix} \cdot t\right) \in \Sigma_L(B)$. By Theorem 3.5 we get the two conditions $s \in R(t^{-1}f(x))$ and $\lambda = a + t((t^{-1}b)s)$.

Setting $a' = t^{-1}a, b' = t^{-1}b, c' = t^{-1}c, d' = t^{-1}d, \lambda' = t^{-1}\lambda$, the conditions become $b's^2 + (a' - d')s - c' = 0$ and $s = b'^{-1}(\lambda' - a')$. Plugging in $s = b'^{-1}(\lambda' - a')$ into the equation $b's^2 + (a' - d')s - c' = 0$ we obtain a two-sided (non-standard) polynomial equation in λ' whose constant term (i.e., the coefficient obtained by setting $\lambda' = 0$) is $d'(b'^{-1}a') - c'$.

Assume $d = 0$. Then the constant term described above is equal to $-c'$, which is zero if and only if $c = 0$. Therefore $0 \in \sigma_L(B)$ if and only if $c = 0$.

When $d \neq 0$, the constant term is 0 if and only if $b'^{-1}a' - d'^{-1}c' = 0$. Therefore, $0 \in \sigma_L(B)$ if and only if $(b^{-1}t)(t^{-1}a) - (d^{-1}t)(t^{-1}c) = 0$. By multiplying from the right by t and applying the Moufang identity $((xy)z)y = x(yzy)$, the last equation becomes $d^{-1}(ct) - b^{-1}(at) = 0$, as required. □

Example 3.8. Consider the matrix $B = \begin{pmatrix} i & 1 \\ ij & j \end{pmatrix}$ over the real octonion division algebra $\mathbb{O}$ with standard generators i, j, ℓ. This matrix is defined over the quaternion subalgebra $\mathbb{H}$, and it is invertible as a quaternion matrix, with inverse $B^{-1} = -\frac{1}{2}\begin{pmatrix} i & ij \\ -1 & j \end{pmatrix}$ (as in the complex case, if a quaternion matrix has a one sided inverse, then it has a two-sided inverse; see [18, Proposition 4.1]), and therefore 0 is not in its spectrum. However, for $t = \ell$ we have $d^{-1}(ct) - b^{-1}(at) = j^{-1}((ij)\ell) - 1^{-1}(i\ell) = 0$, so by Corollary 3.7 we get $0 \in \sigma_L(B)$. A suitable eigenvector is $\vec{w} = \begin{pmatrix} -\ell \\ i\ell \end{pmatrix}$. Note that its right multiple $\vec{w} \cdot \ell^{-1} = \begin{pmatrix} -1 \\ i \end{pmatrix}$ is not an eigenvector of B anymore, unlike the quaternionic case (where the set of eigenvectors associated to a left eigenvalue is closed under multiplication by a scalar to the right). Therefore, this matrix, as a matrix in $M_2(\mathbb{O})$ is both invertible and a zero divisor. Moreover, it is a product of two matrices $B = \begin{pmatrix} i & 0 \\ 0 & ij \end{pmatrix} \cdot \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ where neither is a zero divisor. In fact, the left matrix only has i, ij in its spectrum by Lemma 3.2 and the right one does not have 0 in its spectrum by Corollary 3.7.

In order to compute the left eigenvalues of B, we shall consider the polynomial $f(x) = x^2 + (i-j)x - ij$, and then $\sigma_L(B) = LMR(f(x)) + i$. By [7, Remark 3.5],

$$\begin{aligned} LMR(f(x)) &= \{\xi \cdot (i-j)^{-1}(ij+1) + (1-\xi)(ij+1)(i-j)^{-1} + z\ell : 0 \leq \xi \leq 1, \\ &\qquad \mathrm{Norm}(z) = \xi \cdot (1-\xi) \cdot \mathrm{Norm}([ij-1,(i-j)^{-1}]\} \\ &= \{\xi j + (1-\xi)\cdot(-i) + z\ell : 0 \leq \xi \leq 1, \mathrm{Norm}(z) = 2\xi(1-\xi)\}. \end{aligned}$$

In particular, for $\xi = 0$, we get $-i \in LMR(f(x))$. Thus $0 \in LMR(f(x)) + i = \sigma_L(B)$.

The arguments presented in the proof of Theorem 3.5 can also be applied to an associative division algebra D to describe the left eigenvalues of any 2×2 matrix over D. Indeed, since the nonzero elements of an associative division algebra form a Moufang loop with respect to its product, Theorem 3.5 and the preceding lemmas are applicable. Since the algebra is associative, the expression $a + t(t^{-1}b)s$ appearing in Theorem 3.5 becomes $a + s$, and thus:

Corollary 3.9. *Given an associative division algebra D and a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over D, if $b = 0$ then $\sigma_L(B) = \{a, d\}$ and if $b \neq 0$, then*

$$\sigma_L(B) = \{\lambda \in D : b^{-1}(\lambda - a) \in R(bx^2 + (a-d)x - c)\}.$$

We should mention here that Theorem 2.3 in [12] for quaternion matrices is of the form of the corollary, and the proof there works for associative division algebras as well.

Acknowledgements

The authors wish to express their sincere gratitude to the three anonymous referees who read the manuscript carefully and provided many helpful comments and suggestions.

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Received: November 29, 2023
Accepted for publication: February 9, 2024
Communicated by: Ivan Kaygorodov and David Towers