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Identities for subspaces of the Weyl algebra

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Abstract. In this paper we describe the polynomial identities of degree 4 for a certain subspace of the Weyl algebra over an infinite field of arbitrary characteristic.

1 Introduction

Assume that $\mathbb{F}$ is an infinite field of arbitrary characteristic $p = \operatorname{char}\mathbb{F} \geq 0$. All vector spaces and algebras are over $\mathbb{F}$ and all algebras are unital and associative, unless stated otherwise. We write $\mathbb{F}\langle x_1, \ldots, x_n\rangle$ for the free unital $\mathbb{F}$-algebra with free generators $x_1, \ldots, x_n$. In case the set of free generators is infinite and enumerable, and denoted by $X = \{x_1, x_2, \ldots\}$, the corresponding free algebra is denoted by $\mathbb{F}\langle X\rangle$.

1.1 Witt algebra W_1

The *Weyl algebra* A_1 is the unital associative algebra over $\mathbb{F}$ generated by letters x, y subject to the defining relation $yx = xy+1$ (equivalently, $[y, x] = 1$, where $[y, x] = yx-xy$), i.e.,

$$\mathsf{A}_1 = \mathbb{F}\langle x, y\rangle/\mathrm{id}\{yx - xy - 1\}.$$

For $s > 0$ define by $\mathsf{A}_1^{(-,s)}$ the $\mathbb{F}$-span of ay^s in A_1 for all $a \in \mathbb{F}[x]$. It is easy to see that the following two conditions hold:

- the space $\mathsf{A}_1^{(-,s)}$ is closed with respect to the Lie bracket $[\cdot, \cdot]$;

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- the Lie bracket $[\,\cdot,\cdot\,]$ is not trivially zero on $\mathsf{A}_1^{(-,s)}$ if and only if $s=1$ (for example, see Corollary 3.5 of [13]).

Note that in case $p=0$ the space $\mathsf{A}_1^{(-,1)}$ together with the multiplication given by the Lie bracket is the Witt algebra W_1, which is a simple infinite dimensional Lie algebra. Similarly, considering n pairs $\{x_i, y_i\}$ $(1 \leq i \leq n)$ instead of $\{x, y\}$ we can define the n^{th} Witt algebra W_n, which is also a simple infinite dimensional Lie algebra.

1.2 Polynomial identities

A polynomial identity for a unital $\mathbb{F}$-algebra $\mathcal{A}$ is an element $f(x_1,\ldots,x_m)$ of $\mathbb{F}\langle X\rangle$ such that $f(a_1,\ldots,a_m)=0$ in $\mathcal{A}$ for all $a_1,\ldots,a_m\in\mathcal{A}$. The set $\mathrm{Id}_{\mathbb{F}}(\mathcal{A})=\mathrm{Id}(\mathcal{A})$ of all polynomial identities for $\mathcal{A}$ is a T-ideal, that is, $\mathrm{Id}(\mathcal{A})$ is an ideal of $\mathbb{F}\langle X\rangle$ such that $\phi(\mathrm{Id}(\mathcal{A}))\subset\mathrm{Id}(\mathcal{A})$ for every endomorphism ϕ of $\mathbb{F}\langle X\rangle$. An algebra that satisfies a nontrivial polynomial identity is called a PI-algebra. A T-ideal I of $\mathbb{F}\langle X\rangle$ generated polynomials $f_1,\ldots,f_k$ in $\mathbb{F}\langle X\rangle$ is the minimal T-ideal of $\mathbb{F}\langle X\rangle$ that contains $f_1,\ldots,f_k$. Denote $I=\mathrm{Id}(f_1,\ldots,f_k)$. We say that $f\in\mathbb{F}\langle X\rangle$ is a consequence of $f_1,\ldots,f_k$ if $f\in I$. Given a monomial w in $\mathbb{F}\langle x_1,\ldots,x_m\rangle$, we write $\deg_{x_i}(w)$ for the number of letters x_i in w and $\mathrm{mdeg}(w)\in\mathbb{N}^m$ for the multidegree $(\deg_{x_1}(w),\ldots,\deg_{x_m}(w))$ of w, where $\mathbb{N}=\{0,1,2,\ldots\}$. An element $f\in\mathbb{F}\langle X\rangle$ is called (multi)homogeneous if it is a linear combination of monomials of the same (multi)degree. Given $f=f(x_1,\ldots,x_m)$ of $\mathbb{F}\langle X\rangle$, we write $f=\sum_{\underline{\delta}\in\mathbb{N}^m} f_{\underline{\delta}}$ for multihomogeneous components $f_{\underline{\delta}}$ of f with multidegree $\mathrm{mdeg}\, f_{\underline{\delta}}=\underline{\delta}$. For $\underline{\delta}=(\delta_1,\ldots,\delta_m)$ we denote $|\underline{\delta}|=\delta_1+\cdots+\delta_m$. We say that algebras $\mathcal{A}$, $\mathcal{B}$ are PI-equivalent and write $\mathcal{A}\sim_{\mathrm{PI}}\mathcal{B}$ if $\mathrm{Id}(\mathcal{A})=\mathrm{Id}(\mathcal{B})$.

Given an $\mathbb{F}$-subspace $\mathcal{V}\subset\mathcal{A}$, we write $\mathrm{Id}_{\mathbb{F}}(\mathcal{V})=\mathrm{Id}(\mathcal{V})$ for the ideal of all polynomial identities for $\mathcal{V}$. Note that $\phi(\mathrm{Id}(\mathcal{V}))\subset\mathrm{Id}(\mathcal{V})$ for every linear endomorphism ϕ of $\mathbb{F}\langle X\rangle$, but $\mathrm{Id}(\mathcal{V})$ is not a T-ideal in general.

Assume that $p=0$. It is well-known that the algebra A_1 does not have nontrivial polynomial identities. Namely, it follows from Kaplansky's Theorem [10] and the fact that A_1 is simple with $\mathsf{Z}(\mathsf{A}_1)=\mathbb{F}$. Nevertheless, some subspaces of A_1 satisfy certain polynomial identities. As an example, Dzhumadil'daev proved that the standard polynomial

$$\mathrm{St}_N(x_1,\ldots,x_N)=\sum_{\sigma\in S_N}(-1)^{\sigma}x_{\sigma(1)}\cdots x_{\sigma(N)}$$

is a polynomial identity for $\mathsf{A}_1^{(-,s)}$ if and only if $N>2s$ (Theorem 1 of [5]). More results on polynomial identities for some subspaces of n^{th} Weyl algebra were obtained in [4,6]. The polynomial Lie identities for the n^{th} Witt algebra W_n were studied by Mishchenko [15], Razmyslov [16] and others. The well-known open conjecture claims that all polynomial identities for W_1 follow from the standard Lie identity

$$\sum_{\sigma\in S_4}(-1)^{\sigma}[[[[x_0,x_{\sigma(1)}],x_{\sigma(2)}],x_{\sigma(3)}],x_{\sigma(4)}].$$

$\mathbb{Z}$-graded identities for W_1 were described by Freitas, Koshlukov and Krasilnikov [9]. Moreover, $\mathbb{Z}$-graded identities for the related Lie algebra of the derivations of the algebra of Laurent polynomials were described in [7,8].

The situation is drastically different in case $p > 0$. Namely, A_1 is PI-equivalent to the algebra M_p of all $p \times p$ matrices over $\mathbb{F}$. Moreover, the Weyl algebra A_1 over an arbitrary associative (but possible non-commutative) $\mathbb{F}$-algebra B is PI-equivalent to the algebra $M_p(\mathsf{B})$ of all $p \times p$ matrices over B (see Theorem 4.9 of [12] for more general result).

Over a field of an arbitrary characteristic, minimal polynomials identities for

- $\mathsf{A}_1^{(-,1)}$ for an arbitrary p,
- $\mathsf{A}_1^{(-,s)}$ for $p = 2$,

were described in [13]. Moreover, similar result was obtained in [13] for the so-called parametric Weyl algebras, which were introduced and studied by Benkart, Lopes, Ondrus [1, 2, 3].

1.3 Results

In this paper we described all polynomial identities for $\mathsf{A}_1^{(-,1)}$ of degree 4. Namely, polynomial identities of multidegree

- (3,1) were considered in Proposition 3.2;
- (2,2) were considered in Proposition 3.4;
- (2,1,1) were considered in Proposition 3.6;
- (1,1,1,1) were considered in Propositions 4.3 and 4.4.

It is clear that there is no polynomial identities of multidegree (4). See [11] for the computer program for Wolfram Mathematica to assist the proofs of Lemma 3.3, 4.2 and Propositions 4.3, 4.4.

2 Auxiliary notions

2.1 Properties of A_1

Given $a \in \mathbb{F}[x]$, we write a' for the usual derivative of the polynomial a with respect to the variable x. Using the linearity of derivative and induction on the degree of $a \in \mathbb{F}[x]$ it is easy to see that

$$[y, a] = a' \text{ holds in } \mathsf{A}_1 \text{ for all } a \in \mathbb{F}[x]. \tag{1}$$

Thus for all $i, j \geq 0$ the associative multiplication and the Lie bracket on $\mathsf{A}_1^{(-,1)}$ are given by

$$x^i y\, x^j y = x^{i+j} y^2 + j x^{i+j-1} y \quad \text{and} \quad [x^i y, x^j y] = (j - i) x^{i+j-1} y, \tag{2}$$

where we use notation that

$$x^i = 0 \text{ in } \mathsf{A}_1 \text{ in case } i \in \mathbb{Z} \text{ is negative.}$$

The following properties are well-known (for example, see [3]):

Proposition 2.1. *(a)* $\{x^i y^j \mid i,j \geq 0\}$ *and* $\{y^j x^i \mid i,j \geq 0\}$ *are* $\mathbb{F}$*-bases for* A_1*. In particular,* $\{x^i y \mid i \geq 0\}$ *is an* $\mathbb{F}$*-basis for* $\mathsf{A}_1^{(-,1)}$*.*

(b) If $p = 0$*, then the center* $\mathsf{Z}(\mathsf{A}_1)$ *of* A_1 *is* $\mathbb{F}$*; if* $p > 0$*, then* $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$*.*

(c) If $p > 0$*, then* A_1 *is a free module over* $\mathsf{Z}(\mathsf{A}_1)$ *and the set* $\{x^i y^j \mid 0 \leq i,j < p\}$ *is a basis.*

(d) The algebra A_1 *is simple if and only if* $p = 0$*.*

Theorem 5.4 of [13] implies the following statement:

Proposition 2.2. *(a) A minimal polynomial identity for* $\mathsf{A}_1^{(-,1)}$ *has degree* 3*.*

(b) Every homogeneous polynomial identity for $\mathsf{A}_1^{(-,1)}$ *of degree* 3 *is equal to* $\xi\,\mathrm{St}_3$ *for some* $\xi \in \mathbb{F}$*.*

2.2 Partial linearizations

Assume $f \in \mathbb{F}\langle X\rangle$ is multihomogeneous of multidegree $\underline{\delta} \in \mathbb{N}^m$. Given $1 \leq i \leq m$ and $\underline{\gamma} \in \mathbb{N}^k$ for some $k > 0$ with $|\underline{\gamma}| = \delta_i$, the *partial linearization* $\mathrm{lin}_{x_i}^{\underline{\gamma}}(f)$ of f of multidegree $\underline{\gamma}$ with respect to x_i is the multihomogeneous component of

$$f(x_1, \ldots, x_{i-1}, x_i + \cdots + x_{i+k-1}, x_{i+k}, \ldots, x_{m+k-1})$$

of multidegree $(\delta_1, \ldots, \delta_{i-1}, \gamma_1, \ldots, \gamma_k, \delta_{i+1}, \ldots, \delta_m)$. As an example,

$$\mathrm{lin}_{\mathrm{x}_2}^{(1,1)}(\mathrm{x}_1\mathrm{x}_2^2\mathrm{x}_3^3) = \mathrm{x}_1(\mathrm{x}_2\mathrm{x}_3 + \mathrm{x}_3\mathrm{x}_2)\mathrm{x}_4^3.$$

The result of subsequent applications of partial linearizations to f is also called a partial linearization of f. The *complete linearization* $\mathrm{lin}(f)$ of f is the result of subsequent applications of $\mathrm{lin}_{x_1}^{1^{\delta_1}}, \ldots, \mathrm{lin}_{x_m}^{1^{\delta_m}}$ to f, where 1^k stands for $(1, \ldots, 1)$ (k times).

Assume $\mathcal{A}$ is a unital $\mathbb{F}$-algebra and $\mathcal{V} \subset \mathcal{A}$ is an $\mathbb{F}$-subspace. Since $\mathbb{F}$ is infinite, it is well-known that if f is a polynomial identity for $\mathcal{V}$, then all partial linearizations of f are also polynomial identities for $\mathcal{V}$. Note that the above claim does not hold in general for a finite field (as an example, see [14] for the case of $f(x_1) = x_1^n$ and $\mathcal{V} = \mathcal{A}$). The following lemma is well-known.

Lemma 2.3. *Assume that all partial linearizations of a multihomogeneous element* f *of* $\mathbb{F}\langle X\rangle$ *are equal to zero over some basis of* $\mathcal{V}$*. Then* f *is a polynomial identity for* $\mathcal{V}$*.*

Proof. Let $\{v_j\}$ be a basis for $\mathcal{V}$. Note that $f(\sum_j \alpha_{1j} v_j, \ldots, \sum_j \alpha_{mj} v_j)$ is a linear combination of partial linearizations of f evaluated on the basis $\{v_j\}$, where $\alpha_{1j}, \ldots, \alpha_{mj} \in \mathbb{F}$ for all j. Therefore, the required is proven. ∎

Example 2.4. Assume that $p = 2$ and $\mathcal{A}$ is the unital associative commutative algebra generated by $e_1, \ldots, e_n$ with the ideal of relations generated by $e_1^2, \ldots, e_n^2$, where $n > 0$. Denote by $\mathcal{V}$ the maximal ideal of $\mathcal{A}$ generated by $e_1, \ldots, e_n$. Let us apply Lemma 2.3 to show that $f(x_1) = x_1^2$ is the polynomial identity for $\mathcal{V}$.

All partial linearizations of $f(x_1)$ are $f(x_1)$ and $f_{11}(x_1, x_2) = x_1x_2 + x_2x_1$. Since $B = \{e_{i_1} \cdots e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n,\ k \geq 1\}$ is a basis for $\mathcal{V}$ and

$$f(e_{i_1} \cdots e_{i_k}) = f_{11}(e_{i_1} \cdots e_{i_k}, e_{j_1} \cdots e_{j_r}) = 0$$

in $\mathcal{A}$ for all $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_r \leq n$ with $k, r > 0$, we obtain that $f(x_1) \in \mathrm{Id}(\mathcal{V})$ by Lemma 2.3. Note that $f(x_1)$ is not a polynomial identity for $\mathcal{A}$, since $f(1) = 1$, but $f_{11} \in \mathrm{Id}(\mathcal{A})$.

3 Non-multilinear identities for $\mathsf{A}_1^{(-,1)}$ of degree four

In this section we will denote $c_i = x^i y$ for $i \geq 0$. Note that in the presentation $c_ic_jc_kc_l = \sum_{r,t\geq 0} \beta_{rt} x^r y^t$ in A_1 the coefficient $\beta_{rt} \in \mathbb{F}$ is unique for all $r, t \geq 0$ by part (a) of Proposition 2.1.

Consider the following multihomogeneous elements of $\mathbb{F}\langle X\rangle$ of degree 4:

$$\Phi_{22} = x_1^2x_2^2 - 3x_1x_2x_1x_2 + 2x_1x_2^2x_1 + 2x_2x_1^2x_2 - 3x_2x_1x_2x_1 + x_2^2x_1^2,$$

$$\Psi = x_2[x_1, x_4]x_3 + x_3[x_1, x_4]x_2,$$

$$\Psi_{211} = \Psi(x_1, x_1, x_3, x_2) = x_1[x_1, x_2]x_3 + x_3[x_1, x_2]x_1.$$

Denote $\Phi_{211} = \mathrm{lin}_{\mathrm{x}_2}^{(1,1)}\Phi_{22}$, $\Phi_{\mathrm{lin}} = \mathrm{lin}(\Phi_{22})$ and

$$\Psi_{\mathrm{lin}} = \mathrm{lin}(\Psi_{211}) = \Psi(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_4, \mathrm{x}_3) + \Psi(\mathrm{x}_2, \mathrm{x}_1, \mathrm{x}_4, \mathrm{x}_3).$$

Lemma 3.1. *Given $i, j, k, l \geq 0$, we denote $e = i + j + k + l$. Then in the presentation of $c_ic_jc_kc_l$ as the linear combination of basis elements $\{x^r y^t \,|\, r, t \geq 0\}$ of A_1 the coefficient of*

(a) $x^e y^4$ *is* 1*;*

(b) $x^{e-1}y^3$ *is* $j + 2k + 3l$*, in case* $e \geq 1$*;*

(c) $x^{e-2}y^2$ *is* $(k + 2l)(j + k + l - 1) + l(k + l - 1)$*, in case* $e \geq 2$*;*

(d) $x^{e-3}y$ *is* $l(k + l - 1)(j + k + l - 2)$*, in case* $e \geq 3$*.*

The remaining coefficients are zeros. Moreover, we may apply parts (b), (c), (d) for every $e \geq 0$, since in case of negative degree of x the corresponding coefficient is zero.

Proof. Assume $i, j, k, l \geq 1$. We apply equality (1) to obtain that

$$\begin{aligned} c_j c_k c_l &= x^j y x^k (x^l y + l x^{l-1}) y \\ &= x^j (y x^{k+l}) y^2 + l x^j (y x^{k+l-1}) y \\ &= x^{j+k+l} y^3 + (k+2l) x^{j+k+l-1} y^2 + l(k+l-1) x^{j+k+l-2} y \quad \text{in} \quad \mathsf{A}_1. \end{aligned}$$

Applying the obtained formula to $c_i c_j c_k c_l$ we conclude the proof. Note that in these calculations x^k with negative $k \in \mathbb{Z}$ always has zero coefficient. Then these calculation are valid for $i, j, k, l \geq 0$ and the required is proven. □

Proposition 3.2. *There is no non-trivial multihomogeneous polynomial identities for* $\mathsf{A}_1^{(-,1)}$ *of multidegree* $(3,1)$.

Proof. Assume that $f(x_1, x_2) = \alpha_1 x_1^3 x_2 + \alpha_2 x_1^2 x_2 x_1 + \alpha_3 x_1 x_2 x_1^2 + \alpha_4 x_2 x_1^3$ is a polynomial identity for $\mathsf{A}_1^{(-,1)}$, where $\alpha_1, \ldots, \alpha_4 \in \mathbb{F}$. We will show that $f = 0$ is the trivial identity. For all $i, j \geq 1$ we have $f(c_i, c_j) = 0$ in A_1. Applying parts (a)–(d), respectively, of Lemma 3.1, we obtain that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \tag{3}$$

$$(3i + 3j)\alpha_1 + (4i + 2j)\alpha_2 + (5i + j)\alpha_3 + 6i\alpha_4 = 0, \tag{4}$$

$$\begin{gathered} (2i^2 + 6ij + 3j^2 - i - 3j)\alpha_1 + (5i^2 + 5ij + j^2 - 3i - j)\alpha_2 + \\ (8i^2 + 3ij - 4i)\alpha_3 + (11i^2 - 4i)\alpha_4 = 0, \end{gathered} \tag{5}$$

$$\begin{gathered} j(i + j - 1)(2i + j - 2)\alpha_1 + i(i + j - 1)(2i + j - 2)\alpha_2 + \\ i(2i - 1)(2i + j - 2)\alpha_3 + i(2i - 1)(3i - 2)\alpha_4 = 0, \end{gathered} \tag{6}$$

respectively. We can rewrite formula (4) as

$$(3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4)i + (3\alpha_1 + 2\alpha_2 + \alpha_3)j = 0. \tag{7}$$

We subtract equality (7) with $i = j = 1$ from equality (7) with $i = 1$, $j = 2$ to obtain that $3\alpha_1 + 2\alpha_2 + \alpha_3 = 0$. Thus it follows from equality (3) that for an arbitrary p we have $\alpha_3 = -3\alpha_1 - 2\alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$. Taking $i = 1$, $j = 2$ and $i = 1$, $j = 3$ in equality (6) we obtain that $\alpha_2 = -4\alpha_1 = 0$. Thus $f = 0$ in case $p \neq 2$.

Assume $p = 2$. We have $\alpha_2 = \alpha_4 = 0$ and $\alpha_3 = \alpha_1$. Considering $i = 1$, $j = 2$ in equality (5) we can see that $\alpha_1 = 0$. The required is proven. □

Lemma 3.3. *The elements* Φ_{22}, Φ_{211}, Φ_{lin} *are polynomial identities for* $\mathsf{A}_1^{(-,1)}$. *In case* $p = 2$ *the elements* Ψ_{211}, Ψ, *and* $[[x_1, x_2], [x_3, x_4]]$ *are polynomial identities for* $\mathsf{A}_1^{(-,1)}$.

Proof. Using Lemma 3.1 and straightforward calculations (by means of a computer program) we can see that Φ_{22} and its partial linearizations $\mathrm{lin}_{x_1}^{(1,1)}(\Phi_{22})$, $\Phi_{211} = \mathrm{lin}_{x_2}^{(1,1)}(\Phi_{22})$ and the complete linearization $\Phi_{\mathrm{lin}} = \mathrm{lin}(\Phi_{22})$ are equal to zero over the set $\{c_i \,|\, i \geq 0\}$. Since $\{c_i \,|\, i \geq 0\}$ is a basis of $\mathsf{A}_1^{(-,1)}$, Lemma 2.3 concludes the proof for Φ_{22}, Φ_{211}, Φ_{lin}.

Assume $p = 2$. Since Ψ_{211}, Ψ, $\Psi_{\rm lin}$, and $[[x_1, x_2], [x_3, x_4]]$ are zero over the set $\{c_i \mid i \geq 0\}$, Lemma 2.3 concludes the proof for Ψ_{211}, Ψ, and $[[x_1, x_2], [x_3, x_4]]$.

□

Proposition 3.4. *Assume f is a multihomogeneous polynomial identity for $\mathsf{A}_1^{(-,1)}$ of multidegree $(2, 2)$. Then $f = \alpha\,\Phi_{22}$ for some $\alpha \in \mathbb{F}$.*

Proof. Assume that

$$f(x_1, x_2) = \alpha_1 x_1^2 x_2^2 + \alpha_2 x_1 x_2 x_1 x_2 + \alpha_3 x_1 x_2^2 x_1 + \alpha_4 x_2 x_1^2 x_2 + \alpha_5 x_2 x_1 x_2 x_1 + \alpha_6 x_2^2 x_1^2$$

is a polynomial identity for $\mathsf{A}_1^{(-,1)}$, where $\alpha_1, \ldots, \alpha_6 \in \mathbb{F}$. Hence $f(c_i, c_j) = 0$ in A_1 for all $i, j \geq 0$. Applying parts (a)–(d), respectively, of Lemma 3.1, we obtain that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0, \tag{8}$$

$$\begin{gathered}(i + 5j)\alpha_1 + (2i + 4j)\alpha_2 + (3i + 3j)\alpha_3 + \\ (3i + 3j)\alpha_4 + (4i + 2j)\alpha_5 + (5i + j)\alpha_6 = 0, \quad \text{if} \quad i + j \geq 1,\end{gathered} \tag{9}$$

$$\begin{gathered}(3ij + 8j^2 - 4j)\alpha_1 + (i^2 + 5ij + 5j^2 - i - 3j)\alpha_2 + \\ (3i^2 + 6ij + 2j^2 - 3i - j)\alpha_3 + (2i^2 + 6ij + 3j^2 - i - 3j)\alpha_4 + \\ (5i^2 + 5ij + j^2 - 3i - j)\alpha_5 + (8i^2 + 3ij - 4i)\alpha_6 = 0, \quad \text{if} \quad i + j \geq 1,\end{gathered} \tag{10}$$

$$\begin{gathered}j(2j - 1)(i + 2j - 2)\alpha_1 + j(i + j - 1)(i + 2j - 2)\alpha_2 + \\ i(i + j - 1)(i + 2j - 2)\alpha_3 + j(i + j - 1)(2i + j - 2)\alpha_4 + \\ i(i + j - 1)(2i + j - 2)\alpha_5 + i(2i - 1)(2i + j - 2)\alpha_6 = 0, \quad \text{if} \quad i + j \geq 2.\end{gathered} \tag{11}$$

Taking $i = 0$, $j = 1$ in equality (9) we obtain

$$5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = 0.$$

Considering $i = 0$, $j = 1$ and $i = 1$, $j = 0$ in equality (10) we obtain that

$$4\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \quad \text{and} \quad \alpha_4 + 2\alpha_5 + 4\alpha_6 = 0, \tag{12}$$

respectively.

Let $p \neq 2$. Considering $i = 0$, $j = 2$ and $i = 2$, $j = 0$ in equality (11) we obtain that $3\alpha_1 + \alpha_2 = 0$ and $\alpha_5 + 3\alpha_6 = 0$, respectively. It it easy to see that the above five equalities imply that

$$\alpha_2 = \alpha_5 = -3\alpha_1, \;\; \alpha_3 = \alpha_4 = 2\alpha_1 \;\text{ and }\; \alpha_6 = \alpha_1. \tag{13}$$

Hence, $f = \alpha_1 \Phi_{22}$.

Let $p = 2$. Equality (9) implies that $(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6)(i + j) = 0$ for all $i, j \geq 0$ with $i + j \geq 1$. Considering $i = 1$, $j = 0$ we obtain that $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6 = 0$. Similarly, equality (10) implies that $ij(\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6) + j\alpha_3 + i\alpha_4 = 0$ for all $i, j \geq 0$ with $i + j \geq 1$. Equalities (12) imply that $\alpha_3 = \alpha_4 = 0$. Applying equality (8) we can see that

$$\alpha_3 = \alpha_4 = 0, \;\; \alpha_5 = \alpha_2 \;\text{ and }\; \alpha_6 = \alpha_1. \tag{14}$$

In other words, $f=\alpha_1[x_1^2,x_2^2]+\alpha_2(x_1x_2x_1x_2+x_2x_1x_2x_1)$. By Lemma 3.1 we can see that

$$0=\mathrm{lin}_{x_2}^{(1,1)}(f)(c_1,c_1,c_2)=(\alpha_1+\alpha_2)x^2y. \tag{15}$$

Thus $\alpha_1=\alpha_2$ and the proof is completed. □

Recall that for $i_1,\ldots,i_k,j_1,\ldots,j_k\in\mathbb{Z}$, two *multisets* $\{i_1,\ldots,i_k\}_{\rm m}$ and $\{j_1,\ldots,j_k\}_{\rm m}$ are equal if for every $l\in\mathbb{Z}$ we have $|\{1\le t\le k\mid i_t=l\}|=|\{1\le t\le k\mid j_t=l\}|$.

Lemma 3.5. *Assume $\mathcal{A}$ is an associative algebra and $\mathcal{V}\subset\mathcal{A}$ is an $\mathbb{F}$-subspace. Suppose*

(a) any polynomial identity of $\mathcal{V}$ of multidegree $(3,1)$ is trivial;

(b) any polynomial identity of $\mathcal{V}$ of multidegree $(2,2)$ is equal to $\xi\Phi_{22}$ for some $\xi\in\mathbb{F}$.

Then every polynomial identity f of $\mathcal{V}$ of multidegree $(2,1,1)$ is equal to

$$\alpha x_1\mathrm{St}_3(x_1,x_2,x_3)-\beta\,[[x_1,x_2],[x_1,x_3]]+\gamma\,\mathrm{St}_3(x_1,x_2,x_3)x_1+\xi h(x_1,x_2,x_3)$$

for some $\alpha,\beta,\gamma,\xi\in\mathbb{F}$ and $h(x_1,x_2,x_3)$ is given as

$$h(x_1,x_2,x_3)=x_1^2x_3x_2+2x_3x_1^2x_2+x_3x_2x_1^2-x_1x_2x_3x_1+3x_1x_3x_2x_1-3x_1x_3x_1x_2-3x_3x_1x_2x_1.$$

Proof. First, we have that $f(x_1,x_2,x_3)=\sum\alpha_{ijkl}x_ix_jx_kx_l$, where the sum ranges over all $1\le i,j,k,l\le 3$ with $\{i,j,k,l\}_{\rm m}=\{1,1,2,3\}_{\rm m}$ and $\alpha_{ijkl}\in\mathbb{F}$. For short, we write α_{1^223} for α_{1123}, etc. Applying part (a) to $f(x_1,x_1,x_2)=0$ we obtain that

$$\begin{array}{rcl}\alpha_{1^223}+\alpha_{21^23}+\alpha_{1213} &=& 0,\\ \alpha_{231^2}+\alpha_{1312}+\alpha_{1321} &=& 0,\\ \alpha_{31^22}+\alpha_{321^2}+\alpha_{3121} &=& 0.\end{array}$$

Similarly, applying part (b) to $f(x_1,x_2,x_2)=0$ we obtain that

$$\begin{array}{rcl}\alpha_{1^223}+\alpha_{1^232} &=& \xi,\\ \alpha_{231^2}+\alpha_{321^2} &=& \xi,\\ \alpha_{1213}+\alpha_{1312} &=& -3\xi,\\ \alpha_{2131}+\alpha_{3121} &=& -3\xi,\\ \alpha_{1231}+\alpha_{1321} &=& 2\xi,\\ \alpha_{21^23}+\alpha_{31^22} &=& 2\xi.\end{array}$$

These equations imply the required equality for f for $\alpha=\alpha_{1^223}$, $\beta=\alpha_{21^23}$, $\gamma=\alpha_{231^2}$. □

Proposition 3.6. *The following set is an $\mathbb{F}$-basis of the space of all polynomial identities for $\mathsf{A}_1^{(-,1)}$ of multidegree $(2,1,1)$:*

(a) $x_1\mathrm{St}_3(x_1,x_2,x_3)$, $\mathrm{St}_3(x_1,x_2,x_3)x_1$, Φ_{211}, in case $p\neq 2$;

(b) $x_1\mathrm{St}_3(x_1,x_2,x_3)$, $\mathrm{St}_3(x_1,x_2,x_3)x_1$, Ψ_{211}, $[[x_1,x_2],[x_1,x_3]]$, in case $p=2$.

Proof. By Proposition 2.2 and Lemma 3.3 all elements from the formulation of the proposition are identities for $\mathsf{A}_1^{(-,1)}$.

1. At first, we will show that any polynomial identity $f \in \mathbb{F}\langle X\rangle$ for $\mathsf{A}_1^{(-,1)}$ of multidegree $(2,1,1)$ is an $\mathbb{F}$-linear combination of elements from the formulation of the proposition. Since $x_1\mathrm{St}_3(x_1,x_2,x_3)$ and $\mathrm{St}_3(x_1,x_2,x_3)x_1$ are polynomial identities for $\mathsf{A}_1^{(-,1)}$ by Lemma 2.2, Lemma 3.5 implies that it is enough to complete the proof for

$$f(x_1,x_2,x_3) = -\beta\,[[x_1,x_2],[x_1,x_3]] + \xi h(x_1,x_2,x_3),$$

where $\beta,\xi \in \mathbb{F}$.

Assume $p \neq 2$. Since $0 = f(c_3,c_2,c_1) = 2(\beta-\xi)x^6y$ by Lemma 3.1, we obtain that $\xi=\beta$. By straightforward calculations we can see that

$$[[x_1,x_2],[x_1,x_3]] - h(x_1,x_2,x_3) = \frac{1}{2}\left(x_1\mathrm{St}_3(x_1,x_2,x_3) + \mathrm{St}_3(x_1,x_2,x_3)x_1 - \Phi_{211}\right).$$

The required is proven.

Assume that $p=2$. By straightforward calculations we can see that

$$h(x_1,x_2,x_3) = x_1\mathrm{St}_3(x_1,x_2,x_3) + \Psi_{211}(x_1,x_2,x_3).$$

Note that

$$\Phi_{211} = x_1\mathrm{St}_3(x_1,x_2,x_3) + \mathrm{St}_3(x_1,x_2,x_3)x_1.$$

The required is proven.

2. To show that elements from the formulation of the lemma are linearly independent in case $p \neq 2$, we consider

$$\alpha\, x_1\mathrm{St}_3(x_1,x_2,x_3) + \beta\,\mathrm{St}_3(x_1,x_2,x_3)x_1 + \gamma\,\Phi_{211}(x_1,x_2,x_3) = 0$$

for some $\alpha,\beta,\gamma \in \mathbb{F}$. Taking the coefficients of $x_1^2x_2x_3$ and $x_1^2x_3x_2$ we therefore obtain $\alpha+\gamma = -\alpha+\gamma = 0$. Thus $\alpha=\gamma=0$ and the required is proven.

Similarly, for $p=2$ we consider

$$\alpha\, x_1\mathrm{St}_3(x_1,x_2,x_3) + \beta\,\mathrm{St}_3(x_1,x_2,x_3)x_1 + \gamma\,\Psi_{211} + \delta[[x_1,x_2],[x_1,x_3]] = 0$$

for some $\alpha,\beta,\gamma,\delta \in \mathbb{F}$. Taking the coefficients of $x_1^2x_3x_2$, $x_2x_3x_1^2$, $x_1x_3x_2x_1$ we obtain $-\alpha=\beta=-\alpha-\beta+\delta=0$. Thus $\alpha=\beta=\delta=0$ and the required is proven. □

4 Multilinear identities for $\mathsf{A}_1^{(-,1)}$ of degree four

As in Section 3 we denote $c_i = x^iy$ for $i \geq 0$. Consider the following multilinear elements of $\mathbb{F}\langle X\rangle$ of degree 4:

$$\Gamma = -x_1x_2x_3x_4 + 2x_1x_2x_4x_3 + x_1x_3x_4x_2 - 2x_1x_4x_2x_3$$

$$+2x_2x_1x_3x_4 - 2x_2x_1x_4x_3 - 2x_2x_3x_1x_4 + x_2x_3x_4x_1 + x_2x_4x_1x_3$$
$$+x_3x_1x_2x_4 - 2x_3x_1x_4x_2 + x_3x_4x_1x_2 + x_4x_1x_2x_3 - x_4x_2x_3x_1,$$

$$\Lambda = -3x_1x_2x_3x_4 + 3x_1x_2x_4x_3 + 2x_1x_3x_2x_4 - 2x_1x_4x_2x_3$$
$$+3x_2x_1x_3x_4 - 3x_2x_1x_4x_3 - 2x_2x_3x_1x_4 + 2x_2x_4x_1x_3$$
$$-x_3x_1x_4x_2 + x_3x_2x_4x_1 + x_4x_1x_3x_2 - x_4x_2x_3x_1,$$

$$\Delta = x_2x_1x_3x_4 + x_2x_4x_1x_3 + x_3x_1x_2x_4 + x_3x_4x_1x_2 + x_4x_1x_2x_3 + x_4x_1x_3x_2.$$

A monomial from $\mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$ is called *reduced* if it does not belong to the following list:

$$x_1x_4x_3x_2,\ x_2x_4x_3x_1,\ x_3x_2x_1x_4,\ x_3x_4x_2x_1,\ x_4x_2x_1x_3,\ x_4x_3x_1x_2,\ x_4x_3x_2x_1. \tag{16}$$

An element from $\mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$ is called *reduced* if it is a linear combination of reduced monomials. Note that Γ, Λ, Δ are reduced.

Lemma 4.1. *For every homogeneous $f \in \mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$ there exist multilinear $f_1, f_2 \in \mathbb{F}\langle X\rangle$ of degree 4 such that $f = f_1 + f_2$,*

- *f_1 is reduced;*
- *f_2 is a linear combination of polynomials of the form $x_i\mathrm{St}_3(x_j, x_k, x_l)$, $\mathrm{St}_3(x_i, x_j, x_k)x_l$ where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

Proof. Consider the usual lexicographical order on the set of all monomials from $\mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$. Denote by L the subspace of $\mathbb{F}\langle X\rangle$ generated by $x_i\mathrm{St}_3(x_j, x_k, x_l)$, $\mathrm{St}_3(x_i, x_j, x_k)x_l$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Given a monomial $w \in \mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$, we write $w \equiv 0$, if $w - h \in L$ for some $h \in \mathbb{F}\langle X\rangle$ such that all monomials of h are less than w. Since $x_1\mathrm{St}_3(x_2, x_3, x_4) \in L$, we obtain that $x_1x_4x_3x_2 \equiv 0$. Similarly, considering $x_2\mathrm{St}_3(x_1, x_3, x_4) \in L$, $x_3\mathrm{St}_3(x_1, x_2, x_4) \in L$, $x_4\mathrm{St}_3(x_1, x_2, x_3) \in L$, respectively, we obtain that

$$x_2x_4x_3x_1 \equiv 0,\quad x_3x_4x_2x_1 \equiv 0,\quad x_4x_3x_2x_1 \equiv 0,$$

respectively. Moreover, considering $\mathrm{St}_3(x_1, x_3, x_4)x_2 \in L$, $\mathrm{St}_3(x_1, x_2, x_3)x_3 \in L$ and also $\mathrm{St}_3(x_1, x_2, x_3)x_4 \in L$, respectively, we can see that

$$x_4x_3x_1x_2 \equiv 0,\quad x_4x_2x_1x_3 \equiv 0,\quad x_3x_2x_1x_4 \equiv 0,$$

respectively. Consequently, applying the obtained equivalences to f, it is easy to see that the claim holds. ∎

Lemma 4.2. *The elements Γ, Λ are polynomial identities for $\mathsf{A}_1^{(-,1)}$. If $p = 2$, then Δ is also a polynomial identity for $\mathsf{A}_1^{(-,1)}$.*

Proof. Using Lemma 3.1 and straightforward calculations (by means of a computer program) we can see that Γ, Λ are equal to zero over the set $\{c_i \,|\, i \geq 0\}$. Since the set $\{c_i \,|\, i \geq 0\}$ is a basis of $\mathsf{A}_1^{(-,1)}$, Lemma 2.3 concludes the proof. Similarly, we prove Lemma 4.2 for Δ in case $p = 2$. ☐

The following remark can be verified by straightforward calculations.

Remark 4.1. The following equalities hold in $\mathbb{F}\langle X\rangle$:

1.

$$4\,\Gamma - 2\,\Lambda = \Phi_{\rm lin} + x_1\mathrm{St}_3(x_2,x_3,x_4) + x_2\mathrm{St}_3(x_1,x_3,x_4) + 2\,x_3\mathrm{St}_3(x_1,x_2,x_4) + \mathrm{St}_3(x_2,x_3,x_4)x_1 + \mathrm{St}_3(x_1,x_3,x_4)x_2 + 2\,\mathrm{St}_3(x_1,x_2,x_4)x_3.$$

2.

$$x_1\mathrm{St}_3(x_2,x_3,x_4) - x_2\mathrm{St}_3(x_1,x_3,x_4) + x_3\mathrm{St}_3(x_1,x_2,x_4) - x_4\mathrm{St}_3(x_1,x_2,x_3) + \mathrm{St}_3(x_2,x_3,x_4)x_1 - \mathrm{St}_3(x_1,x_3,x_4)x_2 + \mathrm{St}_3(x_1,x_2,x_4)x_3 - \mathrm{St}_3(x_1,x_2,x_3)x_4 = 0.$$

Proposition 4.3. *The following set is an $\mathbb{F}$-basis of the space of all polynomial identities for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$ in case $p \neq 2$:*

$$\Gamma,\ \Phi_{\rm lin},\ x_1\mathrm{St}_3(x_2,x_3,x_4),\ x_2\mathrm{St}_3(x_1,x_3,x_4),\ x_3\mathrm{St}_3(x_1,x_2,x_4),\ x_4\mathrm{St}_3(x_1,x_2,x_3),$$

$$\mathrm{St}_3(x_2,x_3,x_4)x_1,\ \mathrm{St}_3(x_1,x_3,x_4)x_2,\ \mathrm{St}_3(x_1,x_2,x_4)x_3.$$

Proof. By Proposition 2.2 and Lemmas 3.3, 4.2 all elements from the formulation of the proposition are identities for $\mathsf{A}_1^{(-,1)}$. By part 1 of Remark 4.1 we can consider Λ instead of $\Phi_{\rm lin}$ in the formulation of the proposition.

Assume that $f \in \mathbb{F}\langle X\rangle$ is a polynomial identity for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$. By Lemma 4.1 and part 2 of Remark 4.1 we can assume that f is reduced, i.e.,

$$\begin{aligned} f = \;& \alpha_1\, x_1x_2x_3x_4 + \alpha_2\, x_1x_2x_4x_3 + \alpha_3\, x_1x_3x_2x_4 + \alpha_4\, x_1x_3x_4x_2 + \alpha_5\, x_1x_4x_2x_3 \\ & + \alpha_6\, x_2x_1x_3x_4 + \alpha_7\, x_2x_1x_4x_3 + \alpha_8\, x_2x_3x_1x_4 + \alpha_9\, x_2x_3x_4x_1 + \alpha_{10}\, x_2x_4x_1x_3 \\ & + \alpha_{11}\, x_3x_1x_2x_4 + \alpha_{12}\, x_3x_1x_4x_2 + \alpha_{13}\, x_3x_2x_4x_1 + \alpha_{14}\, x_3x_4x_1x_2 \\ & + \alpha_{15}\, x_4x_1x_2x_3 + \alpha_{16}\, x_4x_1x_3x_2 + \alpha_{17}\, x_4x_2x_3x_1, \end{aligned} \tag{17}$$

where $\alpha_1,\ldots,\alpha_{17} \in \mathbb{F}$. Note that

$$\Gamma = h_1 - x_4x_2x_3x_1 \;\text{ and }\; \Lambda = h_2 + x_4x_1x_3x_2 - x_4x_2x_3x_1,$$

where h_1, h_2 are linear combinations of reduced monomials different from $x_4x_1x_3x_2$ and $x_4x_2x_3x_1$. Then $h = f + (\alpha_{16} + \alpha_{17})\Gamma - \alpha_{16}\Lambda$ does not contain monomials $x_4x_1x_3x_2$, $x_4x_2x_3x_1$. Hence, considering polynomial identity h instead of f, we can assume that $\alpha_{16} = \alpha_{17} = 0$.

To obtain equations on $\alpha_1,\ldots,\alpha_{15}$ we consider $f(c_i,c_j,c_k,c_l)=0$ and take the coefficient of $x^r y^s$ for certain i,j,k,l,r,s. The resulting linear equation $\gamma_1\alpha_1+\cdots+\gamma_{15}\alpha_{15}=0$ for some $\gamma_1,\ldots,\gamma_{15}\in\mathbb{F}$ we write down as the line $(\gamma_1,\ldots,\gamma_{15})$ in the matrix A below. Here is the list of parameters i,j,k,l,r,s which we consider:

- $f(c_1,c_0,c_0,c_0)$, xy^4;
- $f(c_1,c_0,c_0,c_0)$, y^3;
- $f(c_0,c_1,c_0,c_0)$, y^3;
- $f(c_0,c_0,c_1,c_0)$, y^3;
- $f(c_2,c_0,c_0,c_0)$, y^2;
- $f(c_0,c_2,c_0,c_0)$, y^2;
- $f(c_0,c_0,c_2,c_0)$, y^2;
- $f(c_1,c_1,c_0,c_0)$, y^2;
- $f(c_1,c_0,c_1,c_0)$, y^2;
- $f(c_1,c_1,c_1,c_0)$, y;
- $f(c_1,c_1,c_0,c_1)$, y;
- $f(c_1,c_0,c_1,c_1)$, y;
- $f(c_2,c_1,c_0,c_0)$, y;
- $f(c_2,c_0,c_1,c_0)$, y;
- $f(c_0,c_2,c_1,c_0)$, y.

The resulting matrix is

$$A=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&2&3&2&1&1&3&2&1\\
1&1&2&3&2&0&0&0&0&0&2&3&1&3&2\\
2&3&1&1&3&2&3&1&1&3&0&0&0&0&3\\
0&0&0&0&0&0&0&2&6&2&0&0&6&2&0\\
0&0&2&6&2&0&0&0&0&0&2&6&0&6&2\\
2&6&0&0&6&2&6&0&0&6&0&0&0&0&6\\
0&0&0&0&0&0&0&0&0&0&1&2&2&4&1\\
0&0&0&0&0&1&2&1&2&4&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&2&0\\
0&0&0&0&0&0&0&2&2&0&0&0&0&0&0\\
0&0&0&2&2&0&0&0&0&0&0&0&0&0&2
\end{pmatrix}.$$

Since $\det(A)=-64$ is non-zero, we obtain that $\alpha_1=\cdots=\alpha_{15}=0$, i.e, $f=0$. Thus any polynomial identity $f\in\mathbb{F}\langle X\rangle$ for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$ is an $\mathbb{F}$-linear combination of polynomial identities from the formulation of the proposition.

Note that we have proven that any element $f\in\mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$ can be written as a linear combination of 15 monomials, modulo the subspace generated by elements from the formulation of the proposition. Comparing the dimensions, we obtain that elements from the formulation of the proposition are linearly independent. □

Proposition 4.4. *The following set is an $\mathbb{F}$-basis of the space of all polynomial identities for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$ in case $p=2$:*

$$\Gamma,\ \Psi,\ \Delta,\ \Lambda,\ x_1\mathrm{St}_3(x_2,x_3,x_4),\ x_2\mathrm{St}_3(x_1,x_3,x_4),\ x_3\mathrm{St}_3(x_1,x_2,x_4),\ x_4\mathrm{St}_3(x_1,x_2,x_3),$$

$$\mathrm{St}_3(x_2,x_3,x_4)x_1,\ \mathrm{St}_3(x_1,x_3,x_4)x_2,\ \mathrm{St}_3(x_1,x_2,x_4)x_3,$$

$$[[x_1,x_3],[x_2,x_4]].$$

Proof. By Proposition 2.2 and Lemmas 3.3, 4.2 all elements from the formulation of the proposition are identities for $\mathsf{A}_1^{(-,1)}$.

Assume that $f \in \mathbb{F}\langle X\rangle$ is a polynomial identity for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$. By Lemma 4.1 and part 2 of Remark 4.1 we can assume that f is reduced, i.e., can be written as in formula (17). Note that

$$\begin{aligned} g &= [[x_1,x_3],[x_2,x_4]] + x_2\mathrm{St}_3(x_1,x_3,x_4) + \mathrm{St}_3(x_1,x_2,x_4)x_3 \\ &= x_1x_2x_4x_3 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_1x_4x_2x_3 \\ &+ x_2x_1x_3x_4 + x_2x_3x_1x_4 + x_2x_3x_4x_1 + x_2x_4x_1x_3 \\ &+ x_3x_1x_2x_4 + x_3x_1x_4x_2 + x_4x_1x_2x_3 + x_4x_2x_3x_1. \end{aligned}$$

is reduced. Thus

$$\begin{array}{rcl} \Gamma &=& h_1 + x_3x_4x_1x_2 + x_4x_2x_3x_1, \\ \Psi &=& h_2 + x_3x_4x_1x_2 \\ \Delta &=& h_3 + x_3x_4x_1x_2 + x_4x_1x_3x_2, \\ \Lambda &=& h_4 + x_1x_2x_4x_3 + x_3x_2x_4x_1 + x_4x_1x_3x_2 + x_4x_2x_3x_1, \\ g &=& h_5 + x_1x_2x_4x_3 + x_4x_2x_3x_1, \end{array} \tag{18}$$

where $h_1,\ldots,h_5$ are linear combinations of reduced monomials which do not lie in the set S:

$$S = \{x_1x_2x_4x_3,\ x_3x_2x_4x_1,\ x_3x_4x_1x_2,\ x_4x_1x_3x_2, x_4x_2x_3x_1\}.$$

Consider (18) as a system of liner equations on elements of S. Since the corresponding matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is invertible, the elements from the set S are linear combinations of Γ, Ψ, Δ, Λ, g and reduced monomials which do not lie in the set S. Hence, without loss of generality we can assume that $\alpha_2 = \alpha_{13} = \alpha_{14} = \alpha_{16} = \alpha_{17} = 0$.

To obtain equations on $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{12}, \alpha_{15}$ we consider $f(c_i, c_j, c_k, c_l) = 0$ and take the coefficient of $x^r y^s$ for certain i, j, k, l, r, s. The resulting linear equation

$$\gamma_1\alpha_1 + \gamma_3\alpha_3 + \cdots + \gamma_{12}\alpha_{12} + \gamma_{15}\alpha_{15} = 0$$

for some $\gamma_i \in \mathbb{F}$ we write down as the line $(\gamma_1, \gamma_3, \gamma_4, \gamma_5, \ldots, \gamma_{12}, \gamma_{15})$ in the matrix A below. Here is the list of parameters i, j, k, l, r, s which we consider:

- $f(c_1, c_0, c_0, c_0)$, xy^4;
- $f(c_1, c_0, c_0, c_0)$, y^3;
- $f(c_0, c_1, c_0, c_0)$, y^3;
- $f(c_0, c_0, c_1, c_0)$, y^3;
- $f(c_1, c_1, c_0, c_0)$, y^2;
- $f(c_1, c_0, c_1, c_0)$, y^2;
- $f(c_1, c_1, c_1, c_0)$, y;
- $f(c_1, c_1, c_1, c_0)$, xy^2;
- $f(c_1, c_1, c_0, c_1)$, y;
- $f(c_1, c_1, c_0, c_1)$, xy^2;
- $f(c_1, c_0, c_1, c_1)$, y;
- $f(c_1, c_0, c_1, c_1)$, xy^2.

The resulting matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\det(A) = 1$, we obtain that $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \cdots = \alpha_{12} = \alpha_{15} = 0$, i.e, $f = 0$. Thus any polynomial identity $f \in \mathbb{F}\langle X\rangle$ for $\mathsf{A}_1^{(-,1)}$ of multidegree $(1,1,1,1)$ is an $\mathbb{F}$-linear combination of polynomial identities from the formulation of the proposition.

Note that we have proven that any element $f \in \mathbb{F}\langle X\rangle$ of multidegree $(1,1,1,1)$ can be written as a linear combination of 12 monomials, modulo the subspace generated by elements from the from formulation of the proposition. Comparing the dimensions, we obtain that elements from the formulation of the proposition are linearly independent. $\square$

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