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Weak polynomial identities of small degree for the Weyl algebra

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Abstract. In this paper we investigate weak polynomial identities for the Weyl algebra A_1 over an infinite field of arbitrary characteristic. Namely, we describe weak polynomial identities of the minimal degree, which is three, and of degrees 4 and 5. We also describe weak polynomial identities in two variables.

Contents

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1 Introduction

Assume that $\mathbb{F}$ is an infinite field of arbitrary characteristic $p = \mathrm{char}\,\mathbb{F} \geq 0$. All vector spaces and algebras are over $\mathbb{F}$ and all algebras are associative, unless stated otherwise. We write $\mathbb{F}\langle x_1, \ldots, x_n\rangle$ for the free unital $\mathbb{F}$-algebra with free generators $x_1, \ldots, x_n$. In case the free generators are $x_1, x_2, \ldots$ the corresponding free algebra is denoted by $\mathbb{F}\langle X\rangle$.

A polynomial identity for a unital $\mathbb{F}$-algebra $\mathcal{A}$ is an element $f(x_1, \ldots, x_m)$ of $\mathbb{F}\langle X\rangle$ such that $f(a_1, \ldots, a_m) = 0$ in $\mathcal{A}$ for all $a_1, \ldots, a_m \in \mathcal{A}$. The set $\mathrm{Id}_{\mathbb{F}}(\mathcal{A}) = \mathrm{Id}(\mathcal{A})$ of all polynomial identities for $\mathcal{A}$ is a T-ideal, i.e., $\mathrm{Id}(\mathcal{A})$ is an ideal of $\mathbb{F}\langle X\rangle$ such that $\phi(\mathrm{Id}(\mathcal{A})) \subset \mathrm{Id}(\mathcal{A})$ for every endomorphism ϕ of $\mathbb{F}\langle X\rangle$. Given an $\mathbb{F}$-subspace $\mathcal{V} \subset \mathcal{A}$, we write $\mathrm{Id}_{\mathbb{F}}(\mathcal{V}) = \mathrm{Id}(\mathcal{V})$ for the ideal of all polynomial identities for $\mathcal{V}$. Note that $\mathrm{Id}(\mathcal{V})$ is an *L-ideal* (or *weak T-ideal*), i.e., $\phi(\mathrm{Id}(\mathcal{V})) \subset \mathrm{Id}(\mathcal{V})$ for every linear endomorphism ϕ of $\mathbb{F}\langle X\rangle$, but $\mathrm{Id}(\mathcal{V})$ is not a T-ideal in general. We say that a space $\mathcal{V}$ generates the algebra $\mathcal{A}$, if any element of $\mathcal{A}$ can be written as a non-commutative polynomial without free term in some elements of $\mathcal{V}$. If a space $\mathcal{V}$ generates the algebra $\mathcal{A}$, then the polynomial identities for $\mathcal{V}$ are called *weak* polynomial identities for the pair $(\mathcal{A}, \mathcal{V})$ and we denote $\mathrm{Id}(\mathcal{V}) = \mathrm{Id}(\mathcal{A}, \mathcal{V})$.

Weak polynomial identities were introduced in 1973 by Razmyslov [24, 25] (see also book [26]), who applied them to study polynomial identities of matrices. Razmyslov [24], Drensky [4] and Koshlukov [17] described weak polynomial identities for the pair (M_2, sl_2) over a field of an arbitrary characteristic, where sl_2 is the space of all traceless matrices. Weak polynomial identities of small degrees for the pair (M_3, sl_3) were studies by Drensky, Rashkova [7] and by Blachar, Matzri, Rowen, Vishne [2].

For $p = 0$ weak polynomial identities for the pair (M_2, H_2) were described by Drensky [3], where H_n stands for the space of all symmetric $n \times n$ matrices. Minimal weak polynomial identities for the pair (M_n, H_n) for an arbitrary $n > 1$ were described by Ma and Racine [23] in case the characteristic of $\mathbb{F}$ satisfies certain restrictions.

Weak polynomial identities were also considered in [6, 14, 15, 16, 18], etc. More details on weak polynomial identities can be found in a recent survey by Drensky [5].

The Weyl algebra A_1 is generated by $\mathsf{V} = \mathbb{F}\text{-span}\{x,\ y\}$. In this paper we consider weak polynomial identities for the pair $(\mathsf{A}_1, \mathsf{V})$. In Lemma 4.1 we show that the following elements of $\mathbb{F}\langle X\rangle$ are weak polynomial identities for $(\mathsf{A}_1, \mathsf{V})$:

- $\Gamma_m(x_1, \ldots, x_m) = [[x_1, x_2], x_3 \cdots x_m]$ for $m \geq 3$,
- $\mathrm{St}_3(x_1, x_2, x_3) = x_1[x_2, x_3] - x_2[x_1, x_3] + x_3[x_1, x_2]$,
- $T_4(x_1, \ldots, x_4) = [x_1, x_2][x_3, x_4] - [x_1, x_3][x_2, x_4] + [x_2, x_3][x_1, x_4]$,

Denote by $\mathcal{I}$ the ideal of $\mathbb{F}\langle X\rangle$ generated by

$$\Gamma_3(x_i,x_j,x_k),\ \ \mathrm{St}_3(x_i,x_j,x_k),\ \ T_4(x_i,x_j,x_k,x_l)$$

for all $i,j,k,l>0$. In other words, $\mathcal{I}$ is the L-ideal generated by Γ_3, St_3, and T_4. Given $f_1,f_2\in\mathbb{F}\langle X\rangle$, we say that f_1 and f_2 are *equivalent* and write $f_1\equiv f_2$ in case $f_1-f_2\in\mathcal{I}$.

In Theorem 6.3 we describe weak polynomial identities for $(\mathsf{A}_1,\mathsf{V})$ of the minimal degree, which is three. In Theorem 6.1 we show that every weak polynomial identity for $(\mathsf{A}_1,\mathsf{V})$ in two variables lies in $\mathcal{I}$. Moreover, all weak polynomial identities for $(\mathsf{A}_1,\mathsf{V})$ of degrees 4 and 5 belong to $\mathcal{I}$ by Propositions 7.1 and 7.2. Therefore, we formulate the following conjecture:

Conjecture 1.1. *The ideal of all weak polynomial identities for the pair* $(\mathsf{A}_1,\mathsf{V})$ *is equal to* $\mathcal{I}$.

The key definitions are given in Section 2 and some properties are considered in Section 3. The proofs are based on the notion of a completely reduced form of elements of $\mathbb{F}\langle X\rangle$, which is introduced in Section 5.

2 Definitions and known results

2.1 Polynomial identities for the Weyl algebra A_1

The *Weyl algebra* A_1 is the unital associative algebra over $\mathbb{F}$ generated by letters x, y subject to the defining relation $yx=xy+1$ (equivalently, $[y,x]=1$, where $[y,x]=yx-xy$), i.e.,

$$\mathsf{A}_1=\mathbb{F}\langle x,y\rangle/\mathrm{id}\{yx-xy-1\}.$$

We say that algebras $\mathcal{A}$, $\mathcal{B}$ are called PI-equivalent and write $\mathcal{A}\sim_{\mathrm{PI}}\mathcal{B}$ if $\mathrm{Id}(\mathcal{A})=\mathrm{Id}(\mathcal{B})$. We say that an L-ideal $I\in\mathbb{F}\langle X\rangle$ is generated by $f_1,\ldots,f_k\in\mathbb{F}\langle X\rangle$ as an L-ideal, if I is an $\mathbb{F}$-span of $\{f^{(1)}f_i(g_1,\ldots,g_m)f^{(2)}\}$ for all $f^{(1)},f^{(2)}\in\mathbb{F}\langle X\rangle$, all linear combinations $g_1,\ldots,g_m$ of letters $\{x_1,x_2,\ldots\}$, and $1\le i\le k$. Obviously, in case f_i is multilinear (see Section 2.2 below) we can assume that $g_1,\ldots,g_m$ are letters.

Assume that $p=0$. It is well-known that the algebra A_1 does not have nontrivial polynomial identities. Nevertheless, some subspaces of A_1 satisfy certain polynomial identities. As an example, Dzhumadil'daev proved that the standard polynomial

$$\mathrm{St}_N(x_1,\ldots,x_N)=\sum_{\sigma\in\mathcal{S}_N}(-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(N)}$$

is a polynomial identity for $\mathsf{A}_1^{(-,s)}=\mathbb{F}\text{-span}\{ay^s\mid a\in\mathbb{F}[x]\}$ if and only if $N>2s$ (Theorem 1 of [9]). More results on polynomial identities for some subspaces of n^{th} Weyl algebra were obtained in [8, 10]. Considering $\mathsf{A}_1^{(-,1)}$ with respect to the Lie bracket we obtain a simple Lie algebra W_1, which is called Witt algebra. The well-known open conjecture claims that all polynomial identities for W_1 follow from the standard Lie identity of degree 5.

The $\mathbb{Z}$-graded identities for W_1 were described by Freitas, Koshlukov and Krasilnikov [13]. Moreover, $\mathbb{Z}$-graded identities for the related Lie algebra of the derivations of the algebra of Laurent polynomials were described in [11, 12].

The situation is drastically different in case $p > 0$. Namely, A_1 is PI-equivalent to the algebra M_p of all $p \times p$ matrices over $\mathbb{F}$. Moreover, the Weyl algebra A_1 over an arbitrary associative (but possible non-commutative) $\mathbb{F}$-algebra B is PI-equivalent to the algebra $M_p(\mathsf{B})$ of all $p \times p$ matrices over B (see Theorem 4.9 of [19] for more general result). Polynomial identities for $\mathsf{A}_1^{(-,s)}$ and other subspaces of A_1 were studied in [20, 21].

2.2 Notations

An algebra that satisfies a nontrivial polynomial identity is called a PI-algebra. A T-ideal I of $\mathbb{F}\langle X\rangle$ generated by $f_1, \ldots, f_k \in \mathbb{F}\langle X\rangle$ is the minimal T-ideal of $\mathbb{F}\langle X\rangle$ that contains $f_1, \ldots, f_k$. We denote by $\langle X\rangle_m$ and $\langle X\rangle$ the monoids (with unity) freely generated by the letters $x_1, \ldots, x_m$ and $x_1, x_2, \ldots$, respectively. Given $w \in \langle X\rangle_m$, we write $\deg_{x_i}(w)$ for the number of letters x_i in w and $\mathrm{mdeg}(w) \in \mathbb{N}_0^m$ for the multidegree $(\deg_{x_1}(w), \ldots, \deg_{x_m}(w))$ of w, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. An element $f \in \mathbb{F}\langle X\rangle$ is called (multi)homogeneous if it is a linear combination of monomials of the same (multi)degree. Given $f = f(x_1, \ldots, x_m)$ of $\mathbb{F}\langle X\rangle$, we write $f = \sum_{\underline{\delta} \in \mathbb{N}_0^m} f_{\underline{\delta}}$ for multihomogeneous components $f_{\underline{\delta}}$ of f with $\mathrm{mdeg}\, f_{\underline{\delta}} = \underline{\delta}$. If $f \in \mathbb{F}\langle X\rangle$ is multihomogeneous of multidegree $1^m = (1, \ldots, 1)$ (m times), then f is called *multilinear*. For $\underline{\delta} = (\delta_1, \ldots, \delta_m)$ we denote $|\underline{\delta}| = \delta_1 + \cdots + \delta_m$. Given $\underline{\delta} \in \mathbb{N}_0^m$, we write $\mathbb{F}\langle X\rangle_{\underline{\delta}}$ for all elements of $\mathbb{F}\langle X\rangle$ of multidegree $\underline{\delta}$ and we write $\mathrm{Id}(\mathcal{A}, \mathcal{V})_{\underline{\delta}}$ for all elements of $\mathrm{Id}(\mathcal{A}, \mathcal{V})$ of multidegree $\underline{\delta}$.

3 Properties

3.1 Properties of A_1

Given $a \in \mathbb{F}[x]$, we write $\partial(a)$ for the usual derivative of a polynomial a with respect to the variable x. Using the linearity of derivative and induction on the degree of $a \in \mathbb{F}[x]$ it is easy to see that

$$[y, a] = \partial(a) \text{ holds in } \mathsf{A}_1 \text{ for all } a \in \mathbb{F}[x]. \tag{1}$$

The following properties are well-known (for example, see [1]):

Proposition 3.1. *(a)* $\{x^i y^j \mid i, j \geq 0\}$ *and* $\{y^j x^i \mid i, j \geq 0\}$ *are* $\mathbb{F}$*-bases for* A_1.

(b) If $p = 0$*, then the center* $\mathsf{Z}(\mathsf{A}_1)$ *of* A_1 *is* $\mathbb{F}$*; if* $p > 0$*, then* $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$.

(c) If $p > 0$*, then* A_1 *is a free module over* $\mathsf{Z}(\mathsf{A}_1)$ *and the set* $\{x^i y^j \mid 0 \leq i, j < p\}$ *is a basis.*

(d) The algebra A_1 *is simple if and only if* $p = 0$.

3.2 Partial linearizations

Assume $f \in \mathbb{F}\langle X\rangle$ is multihomogeneous of multidegree $\underline{\delta} \in \mathbb{N}_0^m$. Given $1 \leq i \leq m$ and $\underline{\gamma} \in \mathbb{N}_0^k$ for some $k > 0$ with $|\underline{\gamma}| = \delta_i > 0$, the *partial linearization* $\mathrm{lin}_{x_i}^{\underline{\gamma}}(f)$ of f of multidegree $\underline{\gamma}$ with respect to x_i is the multihomogeneous component of

$$f(x_1, \ldots, x_{i-1}, x_i + \cdots + x_{i+k-1}, x_{i+k}, \ldots, x_{m+k-1})$$

of multidegree $(\delta_1, \ldots, \delta_{i-1}, \gamma_1, \ldots, \gamma_k, \delta_{i+1}, \ldots, \delta_m)$. As an example,

$$\mathrm{lin}_{\mathrm{x}_2}^{(2,1)}(\mathrm{x}_1^2\mathrm{x}_2^3\mathrm{x}_3^2) = \mathrm{x}_1^2(\mathrm{x}_2^2\mathrm{x}_3 + \mathrm{x}_2\mathrm{x}_3\mathrm{x}_2 + \mathrm{x}_3\mathrm{x}_2^2)\mathrm{x}_4^2.$$

The result of subsequent applications of partial linearizations to f is also called a partial linearization of f. The *complete linearization* $\mathrm{lin}(f)$ of f is the result of subsequent applications of $\mathrm{lin}_{x_1}^{1^{\delta_1}}, \ldots, \mathrm{lin}_{x_m}^{1^{\delta_m}}$ to f.

Since $\mathbb{F}$ is infinite, it is well-known that the following lemma holds (see also Lemma 2.3 of [21]).

Lemma 3.2. *Assume $\mathcal{A}$ is a unital $\mathbb{F}$-algebra and $\mathcal{V} \subset \mathcal{A}$ is an $\mathbb{F}$-subspace.*

1. *If f is a polynomial identity for $\mathcal{V}$, then all partial linearizations of f are also polynomial identities for $\mathcal{V}$.*

2. *Assume that all partial linearizations of a multihomogeneous element f of $\mathbb{F}\langle X\rangle$ are equal to zero over some basis of $\mathcal{V}$. Then f is a polynomial identity for $\mathcal{V}$.*

Note that part 1 of Lemma 3.2 does not hold in general for a finite field. As an example, see [22] for the case of $f(x_1) = x_1^n$ and

$$\mathcal{A} = \mathcal{V} = \frac{\mathbb{F}\langle X\rangle}{\mathrm{id}\{g^n \mid g \in \mathbb{F}\langle X\rangle \text{ without constant term}\}}.$$

4 Identities

Lemma 4.1. *The following elements of $\mathbb{F}\langle X\rangle$ are weak polynomial identities for $(\mathsf{A}_1, \mathsf{V})$:*

- $\Gamma_m(x_1, \ldots, x_m) = [[x_1, x_2], x_3 \cdots x_m]$ *for all $m \geq 3$,*
- $\mathrm{St}_3(x_1, x_2, x_3) = x_1[x_2, x_3] - x_2[x_1, x_3] + x_3[x_1, x_2]$,
- $T_4(x_1, \ldots, x_4) = [x_1, x_2][x_3, x_4] - [x_1, x_3][x_2, x_4] + [x_2, x_3][x_1, x_4]$,

Proof. **1.** Since $[x, x] = [y, y] = 0$ and $[x, y] = -[y, x] = -1$, we have

$$[u, v] \in Z(A_1) \text{ for all } u, v \in \mathsf{V}. \tag{2}$$

Thus $\Gamma_m \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$.

Since any g from the set $\{\mathrm{St}_3,\ T_4\}$ is multilinear, to show that $g \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$ it is enough to show that $g(u_1, \ldots, u_m) = 0$ in A_1 for all $u_1, \ldots, u_m \in \{x, y\}$. Obviously, $g(x, \ldots, x) = g(y, \ldots, y) = 0$.

2. Since $\mathrm{St}_3(x, x, y) = \mathrm{St}_3(x, y, y) = 0$, we obtain $\mathrm{St}_3 \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$.

3. If $(u_1, \ldots, u_4)$ is equal to (x, x, x, y) or (x, y, y, y), then $T_4(u_{\sigma(1)}, \ldots, u_{\sigma(4)}) = 0$ for all $\sigma \in S_4$. Similarly to $T_4(x, x, y, y) = 0 - 1 + 1 = 0$, we obtain that $T_4(u_{\sigma(1)}, \ldots, u_{\sigma(4)}) = 0$ for all $\sigma \in S_4$ and $(u_1, \ldots, u_4) = (x, x, y, y)$. Thus $T_4 \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$. □

Lemma 4.2. *Any weak identity for the pair $(\mathsf{A}_1, \mathsf{V})$ of degree ≤ 2 is zero.*

Proof. By Lemma 3.2 it is enough to show that $f = 0$ for every multihomogeneous weak identity $f \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$ of degree ≤ 2.

If $\mathrm{mdeg}(f) = (\delta)$ for $\delta \in \{1, 2\}$, then $f = \alpha x_1^{\delta}$ for $\alpha \in \mathbb{F}$ and equality $f(x) = 0$ implies $\alpha = 0$.

Assume that $\mathrm{mdeg}(f) = (1, 1)$ and $f = \alpha x_1 x_2 + \beta x_2 x_1$ for $\alpha, \beta \in \mathbb{F}$. Then $f(x, x) = 0$ implies that $\alpha + \beta = 0$, i.e., $f = \alpha [x_1, x_2]$. Hence $0 = f(x, y) = -\alpha$ implies $\alpha = 0$. □

Lemma 4.3. *The L-ideal generated by Γ_m, St_3, T_4, where $m \geq 3$, coincides with $\mathcal{I}$.*

Proof. For $m > 3$ consider

$$\begin{aligned} \Gamma_m(x_1, \ldots, x_m) &= [x_1, x_2] x_3 x_4 \cdots x_m - x_3 x_4 \cdots x_m [x_1, x_2] \\ &\equiv x_3 [x_1, x_2] x_4 \cdots x_m - x_3 x_4 \cdots x_m [x_1, x_2] \\ &\ \vdots \\ &\equiv x_3 x_4 \cdots x_m [x_1, x_2] - x_3 x_4 \cdots x_m [x_1, x_2] = 0. \end{aligned}$$

The claim is proven. □

5 Completely reduced bracket-monomials

Definition 5.1. A product

$$x_{t_1} \cdots x_{t_l} [x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}]$$

from $\mathbb{F}\langle X \rangle$, where $\underline{t} \in \mathbb{N}^l$, $\underline{r}, \underline{s} \in \mathbb{N}^k$ for some $l \geq 0$, $k > 0$ with $r_1 < s_1, \ldots, r_k < s_k$, is called a *bracket-monomial.*

Lemma 5.2. *If two bracket-monomials are equal in $\mathbb{F}\langle X \rangle$, then they are the same. In other words, if*

$$f = x_{t_1} \cdots x_{t_l} [x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}] \ \ \textit{and} \ \ f' = x_{t'_1} \ldots x_{t'_{l'}} [x_{r'_1}, x_{s'_1}] \cdots [x_{r'_{k'}}, x_{s'_{k'}}]$$

are bracket-monomials and $f = f'$ in $\mathbb{F}\langle X \rangle$, then $\underline{t} = \underline{t}'$, $\underline{r} = \underline{r}'$, $\underline{s} = \underline{s}'$.

Proof. Note that f can be written as a linear combinations of 2^k pairwise different monomials from $\langle X\rangle$ with coefficients ± 1.

We can assume that $l \geq l'$. Thus, $t_1 = t'_1, \ldots, t_{l'} = t'_{l'}$. Therefore, without loss of generality, we may assume that $l' = 0$. In case $l > 0$ we obtain that f is a linear combination of pairwise different monomials which start with x_{t_1}, but f' is a linear combination of pairwise different monomials which start with $x_{r'_1}$ and $x_{s'_1}$, where $r'_1 \neq s'_1$; a contradiction. Therefore, $l' = 0$.

We have that f is a linear combination of pairwise different monomials which start with x_{r_1} and x_{s_1}, but f' is a linear combination of pairwise different monomials which start with $x_{r'_1}$ and $x_{s'_1}$. Hence, $\{r_1, s_1\} = \{r'_1, s'_1\}$ and inequalities $r_1 < s_1$, $r'_1 < s'_1$ imply that $r_1 = r'_1$ and $s_1 = s'_1$. Therefore, without loss of generality we can assume that $f = [x_{r_2}, x_{s_2}]\cdots[x_{r_k}, x_{s_k}]$ and $f' = [x_{r'_2}, x_{s'_2}]\cdots[x_{r'_{k'}}, x_{s'_{k'}}]$. Repeating the above reasoning several times we conclude the proof. □

Definition 5.3. **(a)** A bracket-monomial

$$f = x_{t_1}\cdots x_{t_l}[x_{r_1}, x_{s_1}]\cdots[x_{r_k}, x_{s_k}] \in \mathbb{F}\langle X\rangle, \tag{3}$$

where $\underline{t} \in \mathbb{N}^l$, $\underline{r}, \underline{s} \in \mathbb{N}^k$ for some $l \geq 0$, $k > 0$ with $r_1 < s_1, \ldots, r_k < s_k$ is *semi-reduced* if

- $t_1 \leq \cdots \leq t_l$;
- $s_1 \leq \cdots \leq s_k$.

(b) A semi-reduced bracket-monomial $f \in \mathbb{F}\langle X\rangle$ defined by (3) is *reduced* if

- either $l = 0$ or $l \geq 1$, $t_l \leq s_1$.

(c) A reduced bracket-monomial $f \in \mathbb{F}\langle X\rangle$ defined by (3) is *completely reduced* if

- do not exist $1 \leq i \neq j \leq k$ with $r_j < r_i < s_i < s_j$.

Example 5.4. Consider the list of all completely reduced bracket-monomials of multidegree 1^m:

- $m = 2$: $[x_1, x_2]$;
- $m = 3$: $x_1[x_2, x_3]$, $x_2[x_1, x_3]$;
- $m = 4$: $x_1x_2[x_3, x_4]$, $x_1x_3[x_2, x_4]$, $x_2x_3[x_1, x_4]$, $[x_1, x_2][x_3, x_4]$, $[x_1, x_3][x_2, x_4]$;
- $m = 5$: $x_1x_2x_3[x_4, x_5]$, $x_1x_2x_4[x_3, x_5]$, $x_1x_3x_4[x_2, x_5]$, $x_2x_3x_4[x_1, x_5]$,$x_1[x_2, x_3][x_4, x_5]$, $x_1[x_2, x_4][x_3, x_5]$, $x_2[x_1, x_3][x_4, x_5]$, $x_2[x_1, x_4][x_3, x_5]$, $x_3[x_1, x_4][x_2, x_5]$

For $1 \leq i < j$ we consider $\mathbb{N}_0^i$ as a subset of $\mathbb{N}_0^j$ by

$$(r_1, \ldots, r_i) \to (r_1, \ldots, r_i, \underbrace{0, \ldots, 0}_{j-i}).$$

Assume $\underline{r} \in \mathbb{N}_0^i$ and $\underline{s} \in \mathbb{N}_0^j$ for some $i, j \geq 1$. Then we write $\underline{r} < \underline{s}$ for the lexicographical order on $\mathbb{N}_0^k$, where $k = \max\{i, j\}$ and we consider $\underline{r}$, $\underline{s}$ as elements of $\mathbb{N}_0^k$.

Definition 5.5. Consider a bracket-monomial

$$f = x_{t_1} \cdots x_{t_l} [x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}] \in \mathbb{F}\langle X \rangle, \tag{4}$$

where $\underline{t} \in \mathbb{N}^l$, $\underline{r}, \underline{s} \in \mathbb{N}^k$ for some $l \geq 0$, $k > 0$ and $r_1 < s_1, \ldots, r_k < s_k$. Then

(a) the *monomial weight* of f in case $l > 0$ is $\mathrm{mw}(f) = (t_{\sigma(1)}, \ldots, t_{\sigma(l)})$ for some permutation $\sigma \in \mathcal{S}_l$ such that
$$t_{\sigma(1)} \geq \cdots \geq t_{\sigma(l)}$$
and $\mathrm{mw}(f) = (0)$ in case $l = 0$.

(b) the *bracket weight* of f is $\mathrm{bw}(f) = (s_{\sigma(1)} - r_{\sigma(1)}, \ldots, s_{\sigma(k)} - r_{\sigma(k)})$ for some permutation $\sigma \in \mathcal{S}_k$ such that
$$s_{\sigma(1)} - r_{\sigma(1)} \geq \cdots \geq s_{\sigma(k)} - r_{\sigma(k)}.$$

Example 5.6. **(a)** We have $\mathrm{St}_3(x_1, x_2, x_3) = f_1 - f_2 + f_3$ for the semi-reduced bracket-monomials
$$f_1 = x_1[x_2, x_3], \quad f_2 = x_2[x_1, x_3], \quad f_3 = x_3[x_1, x_2].$$
Then $\mathrm{mw}(f_1) = (1)$, $\mathrm{mw}(f_2) = (2)$, $\mathrm{mw}(f_3) = (3)$ and $\mathrm{bw}(f_1) = (1)$, $\mathrm{bw}(f_2) = (2)$, $\mathrm{bw}(f_3) = (1)$. Note that f_1, f_2 are reduced, but f_3 is not reduced.
(b) We have $T_4(x_1, \ldots, x_4) = h_1 - h_2 + h_3$ for the reduced bracket-monomials

$$h_1 = [x_1, x_2][x_3, x_4], \quad h_2 = [x_1, x_3][x_2, x_4], \quad h_3 = [x_2, x_3][x_1, x_4].$$

Then $\mathrm{mw}(h_i) = (0)$ for $i = 1, 2, 3$ and $\mathrm{bw}(h_1) = (1, 1)$, $\mathrm{bw}(h_2) = (2, 2)$, $\mathrm{bw}(h_3) = (3, 1)$. Note that h_1, h_2 are completely reduced, but h_3 is not completely reduced.

Lemma 5.7. *Assume that $f \in \mathbb{F}\langle X \rangle$ is multihomogeneous of multidegree $\underline{\delta} \in \mathbb{N}_0^m$. Then there are semi-reduced bracket-monomials $f_i \in \mathbb{F}\langle X \rangle$ and $\alpha_i, \beta \in \mathbb{F}$ such that*

$$f \equiv \beta x_1^{\delta_1} \cdots x_m^{\delta_m} + \sum_i \alpha_i f_i,$$

where $\mathrm{mdeg}(f_i) = \underline{\delta}$ for all i. Moreover,

(a) if $f \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$, then $\beta = 0$;

(b) if f is a bracket-monomial, then $\beta = 0$ and $\mathrm{mw}(f_i) \leq \mathrm{mw}(f)$ for all i.

Proof. Assume that $f_1, f_2 \in \mathbb{F}\langle x_1, \ldots, x_m \rangle$ are multihomogeneous and $1 \leq i < j \leq m$. Since

$$f_1 x_j x_i f_2 = f_1 x_i x_j f_2 - f_1[x_i, x_j] f_2 = f_1 x_i x_j f_2 - f_1 f_2 [x_i, x_j] - f_1[[x_i, x_j], f_2],$$

by Lemma 4.3 we obtain that

$$f_1 x_j x_i f_2 \equiv f_1 x_i x_j f_2 - f_1 f_2 [x_i, x_j]. \tag{5}$$

Lemma 4.3 also implies that

$$f_1[x_i, x_j]f_0 f_2 \equiv f_1 f_0 [x_i, x_j] f_2 \tag{6}$$

for every $f_0 \in \mathbb{F}\langle X\rangle$. Since equivalences (5) and (6) preserve the multidegree, applying formulas (5) and (6) to f we obtain that $f \equiv \beta x_1^{\delta_1}\cdots x_m^{\delta_m} + \sum_i \alpha_i f_i$ for some semi-reduced bracket-monomials f_i and $\beta, \alpha_i \in \mathbb{F}$, where $\mathrm{mdeg}(f_i) = \underline{\delta}$.

Assume $f \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$. Since $[x, x] = 0$, we have $f_i(x, \ldots, x) = 0$ in A_1 for all i. Therefore, $0 = f(x, \ldots, x) = \beta\, x^{|\underline{\delta}|}$. Thus $\beta = 0$.

If f is a bracket-monomial, then it is easy to see that $\beta = 0$ and $\mathrm{mw}(f_i) \leq \mathrm{mw}(f)$ for all i. □

Lemma 5.8. *Consider a semi-reduced bracket-monomial*

$$f = x_{t_1}\cdots x_{t_l}[x_{r_1}, x_{s_1}]\cdots[x_{r_k}, x_{s_k}] \in \mathbb{F}\langle X\rangle.$$

Then $f \equiv \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{F}$, $f_i \in \mathbb{F}\langle X\rangle$ such that f_i is a reduced bracket-monomial, $\mathrm{mdeg}(f_i) = \mathrm{mdeg}(f)$, *and* $\mathrm{mw}(f_i) \leq \mathrm{mw}(f)$ *for all i.*

Proof. Since f is semi-reduced, we have $r_1 < s_1, \ldots, r_k < s_k$, $t_1 \leq \cdots \leq t_l$ and also $s_1 \leq \cdots \leq s_k$, where $l \geq 0$, $k > 0$.

We prove the lemma by induction on $\mathrm{mw}(f)$. If $\mathrm{mw}(f) = (0)$, then $l = 0$ and f is reduced.

Assume that $(0) < \mathrm{mw}(f)$ and for every semi-reduced bracket-monomial $f' \in \mathbb{F}\langle X\rangle$ with $\mathrm{mw}(f') < \mathrm{mw}(f)$ the statement of this lemma holds.

Assume that f is not reduced, i.e., $l \geq 1$ and $t_l > s_1$. Using St_3 from Lemma 4.1, we obtain

$$x_{t_l}[x_{r_1}, x_{s_1}] \equiv x_{s_1}[x_{r_1}, x_{t_l}] - x_{r_1}[x_{s_1}, x_{t_l}]. \tag{7}$$

Thus $f \equiv f_1 - f_2$ for

$$f_1 = x_{t_1}\cdots x_{t_{l-1}} x_{s_1}[x_{r_1}, x_{t_l}][x_{r_2}, x_{s_2}]\cdots[x_{r_k}, x_{s_k}],$$

$$f_2 = x_{t_1}\cdots x_{t_{l-1}} x_{r_1}[x_{s_1}, x_{t_l}][x_{r_2}, x_{s_2}]\cdots[x_{r_k}, x_{s_k}].$$

Note that $\mathrm{mdeg}(f_1) = \mathrm{mdeg}(f_2) = \mathrm{mdeg}(f)$. Using part (b) of Lemma 5.7, we obtain semi-reduced bracket-monomials $g_{1i}, g_{2j} \in \mathbb{F}\langle X\rangle$ and $\alpha_{1i}, \alpha_{2j} \in \mathbb{F}$ such that $f_1 \equiv \sum_i \alpha_{1i} g_{1i}$ and $f_2 \equiv \sum_j \alpha_{2j} g_{2j}$, where $\mathrm{mdeg}(g_{1i}) = \mathrm{mdeg}(g_{2j}) = \mathrm{mdeg}(f)$, $\mathrm{mw}(g_{1i}) \leq \mathrm{mw}(f_1) < \mathrm{mw}(f)$ and $\mathrm{mw}(g_{2j}) \leq \mathrm{mw}(f_2) < \mathrm{mw}(f)$. Applying the induction hypothesis to g_{1i}, g_{2j} we conclude the proof, since $f \equiv \sum_i \alpha_{1i} g_{1i} - \sum_i \alpha_{2i} g_{2i}$. □

Lemma 5.9. *Consider a reduced bracket-monomial*

$$f = x_{t_1}\cdots x_{t_l}[x_{r_1}, x_{s_1}]\cdots[x_{r_k}, x_{s_k}] \in \mathbb{F}\langle X\rangle.$$

Then $f \equiv \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{F}$, $f_i \in \mathbb{F}\langle X\rangle$ such that f_i is a completely reduced bracket-monomial, $\mathrm{mdeg}(f_i) = \mathrm{mdeg}(f)$, *and* $\mathrm{mw}(f_i) \leq \mathrm{mw}(f)$ *for all i.*

Proof. Since f is reduced, we have $r_1 < s_1, \ldots, r_k < s_k$, $t_1 \leq \cdots \leq t_l \leq s_1 \leq \cdots \leq s_k$, where $l \geq 0$, $k > 0$.

We prove the lemma by induction on $\mathrm{mw}(f)$.

1. Assume $\mathrm{mw}(f) = (0)$, i.e., $f = [x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}]$. To show that the statement of this lemma holds for f we use induction on $\mathrm{bw}(f)$.

Obviously, if $\mathrm{bw}(f) = (1)$, i.e., $f = [x_{r_1}, x_{s_1}]$ with $s_1 - r_1 = 1$, then f is completely reduced.

Assume $(1) < \mathrm{bw}(f)$ and for every reduced bracket-monomial $f' \in \mathbb{F}\langle X\rangle$ such that $\mathrm{mw}(f') = (0)$ and $\mathrm{bw}(f') < \mathrm{bw}(f)$ the statement of the lemma holds.

Assume that f is not completely reduced, i.e., there are $1 \leq i \neq j \leq k$ such that $r_j < r_i < s_i < s_j$. For short, denote $a_1 = r_j$, $a_2 = r_i$, $a_3 = s_i$, $a_4 = s_j$. Note that $a_1 < a_2 < a_3 < a_4$. Using equivalence (6) we obtain that

$$[x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}] \equiv [x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}]\, b$$

for some product $b = [x_{r'_1}, x_{s'_1}] \cdots [x_{r'_{k-2}}, x_{s'_{k-2}}]$ of brackets. Applying the equivalence $T_4(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}) \equiv 0$, we obtain

$$[x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}] \equiv -[x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}] + [x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}].$$

Thus $f \equiv -f_1 + f_2$ for

$$f_1 = [x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}]b,$$
$$f_2 = [x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}]b.$$

Note that $\mathrm{mdeg}(f_1) = \mathrm{mdeg}(f_2) = \mathrm{mdeg}(f)$. Since

$$\mathrm{bw}([x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}]) = (a_4 - a_1, a_3 - a_2)$$

is greater than both $\mathrm{bw}([x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}])$ and $\mathrm{bw}([x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}])$, we can see that $\mathrm{bw}(f_1) < \mathrm{bw}(f)$ and $\mathrm{bw}(f_2) < \mathrm{bw}(f)$.

We use equivalence (6) to obtain reduced bracket-monomials g_1 and g_2 such that $g_1 \equiv f_1$ and $g_2 \equiv f_2$, where $\mathrm{mdeg}(g_1) = \mathrm{mdeg}(g_2) = \mathrm{mdeg}(f)$, $\mathrm{mw}(g_1) = \mathrm{mw}(g_2) = (0)$,

$$\mathrm{bw}(g_1) = \mathrm{bw}(f_1) < \mathrm{bw}(f) \quad \text{and} \quad \mathrm{bw}(g_2) = \mathrm{bw}(f_2) < \mathrm{bw}(f).$$

Applying the induction hypothesis to g_1 and g_2, we can see that the statement of this lemma holds for f.

2. Assume that $(0) < \mathrm{mw}(f)$, that is, $l \geq 1$, and for every reduced bracket-monomial $f' \in \mathbb{F}\langle X\rangle$ with $\mathrm{mw}(f') < \mathrm{mw}(f)$ the claim of this lemma holds. To show that the statement of this lemma holds for f we use induction on $\mathrm{bw}(f)$.

Obviously, if $\mathrm{bw}(f) = (1)$, i.e., $f = x_{t_1} \cdots x_{t_l}[x_{r_1}, x_{s_1}]$ with $s_1 - r_1 = 1$, then f is completely reduced.

Assume $(1) < \mathrm{bw}(f)$ and for every reduced bracket-monomial $f' \in \mathbb{F}\langle X\rangle$ such that $\mathrm{mw}(f') = \mathrm{mw}(f)$ and $\mathrm{bw}(f') < \mathrm{bw}(f)$ the statement of the lemma holds.

Assume that f is not completely reduced, i.e., there are $1 \leq i \neq j \leq k$ such that $r_j < r_i < s_i < s_j$. For short, denote $a_1 = r_j$, $a_2 = r_i$, $a_3 = s_i$, $a_4 = s_j$. Note that $a_1 < a_2 < a_3 < a_4$ and $t_l \leq a_3$. Using equivalence (6) we obtain that

$$[x_{r_1}, x_{s_1}] \cdots [x_{r_k}, x_{s_k}] \equiv [x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}]\, b$$

for some product $b = [x_{r'_1}, x_{s'_1}] \cdots [x_{r'_{k-2}}, x_{s'_{k-2}}]$ of brackets, where $t_l \leq s'_1 \leq \cdots \leq s'_{k-2}$. Since $T_4(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}) \equiv 0$, we obtain

$$[x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}] \equiv -[x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}] + [x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}].$$

Thus $f \equiv -f_1 + f_2$ for

$$f_1 = x_{t_1} \cdots x_{t_l}[x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}]b,$$

$$f_2 = x_{t_1} \cdots x_{t_l}[x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}]b.$$

Note that $\mathrm{mdeg}(f_1) = \mathrm{mdeg}(f_2) = \mathrm{mdeg}(f)$. Since

$$\mathrm{bw}([x_{a_2}, x_{a_3}][x_{a_1}, x_{a_4}]) = (a_4 - a_1, a_3 - a_2)$$

is greater than $\mathrm{bw}([x_{a_1}, x_{a_2}][x_{a_3}, x_{a_4}])$ and $\mathrm{bw}([x_{a_1}, x_{a_3}][x_{a_2}, x_{a_4}])$, we can obtain the inequalities $\mathrm{bw}(f_1) < \mathrm{bw}(f)$ and $\mathrm{bw}(f_2) < \mathrm{bw}(f)$.

2.1. Assume that $l_t \leq a_2$. Then $l_t \leq a_2 < a_3 < a_4$ and $t_l \leq s'_1 \leq \cdots \leq s'_{k-2}$. We use equivalence (6) to obtain reduced bracket-monomials g_1 and g_2 such that $g_1 \equiv f_1$ and $g_2 \equiv f_2$, where $\mathrm{mdeg}(g_1) = \mathrm{mdeg}(g_2) = \mathrm{mdeg}(f)$, $\mathrm{mw}(g_1) = \mathrm{mw}(g_2) = \mathrm{mw}(f)$,

$$\mathrm{bw}(g_1) = \mathrm{bw}(f_1) < \mathrm{bw}(f) \quad \text{and} \quad \mathrm{bw}(g_2) = \mathrm{bw}(f_2) < \mathrm{bw}(f).$$

Applying induction on bracket weight to g_1 and g_2, we obtain that the statement of the lemma holds for f.

2.2. Assume $a_2 < t_l$. Using equivalence (7), we obtain $f_1 \equiv h_1 - h_2$ for

$$h_1 = x_{t_1} \cdots x_{t_{l-1}} x_{a_2}[x_{a_1}, x_{t_l}][x_{a_3}, x_{a_4}]b,$$

$$h_2 = x_{t_1} \cdots x_{t_{l-1}} x_{a_1}[x_{a_2}, x_{t_l}][x_{a_3}, x_{a_4}]b.$$

Since $t_l > a_1, a_2$, we have $\mathrm{mw}(h_1) < \mathrm{mw}(f)$ and $\mathrm{mw}(h_2) < \mathrm{mw}(f)$. Using part (b) of Lemma 5.7 and Lemma 5.8, we obtain reduced bracket-monomials $g_{1i'}, g_{2j'} \in \mathbb{F}\langle X\rangle$ and scalars $\alpha_{1i'}, \alpha_{2j'} \in \mathbb{F}$ such that $h_1 \equiv \sum_{i'} \alpha_{1i'} g_{1i'}$ and $h_2 \equiv \sum_{j'} \alpha_{2j'} g_{2j'}$, and where $\mathrm{mdeg}(g_{1i'}) = \mathrm{mdeg}(g_{2j'}) = \mathrm{mdeg}(f)$,

$$\mathrm{mw}(g_{1i'}) \leq \mathrm{mw}(h_1) < \mathrm{mw}(f) \quad \text{and} \quad \mathrm{mw}(g_{2j'}) \leq \mathrm{mw}(h_2) < \mathrm{mw}(f).$$

We apply induction on monomial weight to $g_{1i'}$ and $g_{2j'}$ to show that the statement of the lemma holds for f_1. We establish that the statement of the lemma holds for f_2 by repeating the proof from part 2.1. Therefore, the statement of the lemma holds for f. □

Theorem 5.10. *Assume that $f \in \mathbb{F}\langle X\rangle$ is multihomogeneous of multidegree $\underline{\delta} \in \mathbb{N}_0^m$. Then there are completely reduced bracket-monomials $f_i \in \mathbb{F}\langle X\rangle$ and $\alpha_i, \beta \in \mathbb{F}$ such that*

$$f \equiv \beta x_1^{\delta_1} \cdots x_m^{\delta_m} + \sum_i \alpha_i f_i,$$

where $\mathrm{mdeg}(f_i) = \underline{\delta}$ for all i. Moreover,

(a) if $f \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})$, then $\beta = 0$;

(b) if f is a bracket-monomial, then $\beta = 0$ and $\mathrm{mw}(f_i) \leq \mathrm{mw}(f)$ for all i.

Proof. Consequently applying Lemmas 5.7, 5.8, 5.9 we obtain the required. □

6 Minimal weak polynomial identities

Theorem 6.1. *Every weak polynomial identity for the pair $(\mathsf{A}_1, \mathsf{V})$ in two variables lies in the L-ideal $\mathcal{I}$ generated by St_3, Γ_3, T_4.*

Proof. Assume that $f \in \mathbb{F}\langle x_1, x_2\rangle$ is a weak polynomial identity in two variables for the pair $(\mathsf{A}_1, \mathsf{V})$. By Lemma 3.2, we can assume that f is multihomogeneous of multidegree (r, s) for some $r, s \geq 0$. Then Theorem 5.10 implies that f is equivalent to a linear combination of completely reduced bracket-monomials of multidegree (r, s).

Assume that $r \geq s$. If $s = 0$, then $f \equiv 0$ by the definition of completely reduced bracket-monomials. Assume $s > 0$. Then

$$f(x_1, x_2) \equiv \sum_{i=1}^{s} \alpha_i x_1^{r-i} x_2^{s-i} [x_1, x_2]^i$$

for some $\alpha_i \in \mathbb{F}$. Since $0 = f(x, y) = \sum_{i=1}^{s} (-1)^i \alpha_i x^{r-i} y^{s-i}$ in A_1, we obtain by part (a) of Proposition 3.1 that $\alpha_1 = \cdots = \alpha_s = 0$, i.e., $f \equiv 0$.

The case of $r < s$ can be considered similarly. The proof is completed. □

Lemma 6.2. *Every weak polynomial identity for the pair $(\mathsf{A}_1, \mathsf{V})$ of degree 3 lies in the L-ideal generated by Γ_3 and St_3.*

Proof. Assume that $f \in \mathbb{F}\langle X\rangle$ is a weak polynomial identity of degree 3 for the pair $(\mathsf{A}_1, \mathsf{V})$. By Lemma 3.2, we can assume that f is multihomogeneous of multidegree Δ with $|\Delta| = 3$. By Theorem 5.10, f is equivalent to a linear combination of completely reduced bracket-monomials of multidegree Δ.

Assume $\Delta = (1, 1, 1)$. Then

$$f(x_1, x_2, x_3) \equiv \alpha x_1 [x_2, x_3] + \beta x_2 [x_1, x_3],$$

where $\alpha, \beta \in \mathbb{F}$. Since $0 = f(x, y, y) = -\beta y$ and $0 = f(x, y, x) = \alpha x$ in A_1, we obtain $\alpha = \beta = 0$. The definition of the ideal $\mathcal{I}$ implies the required.

If $\Delta = (2, 1)$ or $\Delta = (3)$, then Theorem 6.1 concludes the proof. □

Theorem 6.3. **1.** *The minimal degree of a non-trivial weak polynomial identity for the pair* $(\mathsf{A}_1, \mathsf{V})$ *is three.*

2. *The vector space* $\mathrm{Id}(\mathsf{A}_1, \mathsf{V})_{\Delta}$ *for* $|\Delta| = 3$ *has the following basis:*

- $\Gamma_3(x_1, x_2, x_3)$, $\Gamma_3(x_1, x_3, x_2)$, *and* $\mathrm{St}_3(x_1, x_2, x_3)$, *in case* $\Delta = 1^3$;
- $\Gamma_3(x_1, x_2, x_1)$, *in case* $\Delta = (2, 1)$,
- $\emptyset$, *in case* $\Delta = (3)$.

Proof. **1.** It follows from Lemmas 4.1 and 4.2.

2. Lemma 6.2 implies that $\Gamma_3(x_1, x_2, x_1) \neq 0$ is a basis for $\mathrm{Id}(\mathsf{A}_1, \mathsf{V})_{(2,1)}$ and $\emptyset$ is a basis for $\mathrm{Id}(\mathsf{A}_1, \mathsf{V})_{(2,1)} = \{0\}$.

Since $\Gamma_3(x_1, x_2, x_3)$ and $\mathrm{St}_3(x_1, x_2, x_3)$ are multilinear, Lemma 6.2 implies that every element $f \in \mathrm{Id}(\mathsf{A}_1, \mathsf{V})_{1^3}$ lies in the $\mathbb{F}$-span of

$$\Gamma_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \ \text{ and } \ \mathrm{St}_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

for all $\sigma \in S_3$. Since $\Gamma_3(x_2, x_3, x_1) = -\Gamma_3(x_1, x_2, x_3) + \Gamma_3(x_1, x_3, x_2)$, we obtain that f lies in the $\mathbb{F}$-span of $\Gamma_3(x_1, x_2, x_3)$, $\Gamma_3(x_1, x_3, x_2)$, and $\mathrm{St}_3(x_1, x_2, x_3)$. The linear independence follows from straightforward calculations. ☐

7 Weak polynomial identities of degrees 4 and 5

Proposition 7.1. *Any weak polynomial identity for the pair* $(\mathsf{A}_1, \mathsf{V})$ *of degree* 4 *lies in the* L*-ideal* $\mathcal{I}$ *generated by* Γ_3, St_3, *and* T_4.

Proof. Assume that $f \in \mathbb{F}\langle X \rangle$ is a weak polynomial identity of degree 4 for the pair $(\mathsf{A}_1, \mathsf{V})$. By Lemma 3.2, we can assume that f is multihomogeneous of multidegree Δ with $|\Delta| = 4$. By Theorem 5.10, f is equivalent to a linear combination of completely reduced bracket-monomials of multidegree Δ.

Assume $\Delta = 1^4$. Using Example 5.4 we can see that

$$\begin{aligned} f(x_1, \ldots, x_4) \equiv\ & \alpha_1\, x_1x_2[x_3, x_4] + \alpha_2\, x_1x_3[x_2, x_4] + \alpha_3\, x_2x_3[x_1, x_4] \\ & + \beta_1[x_1, x_2][x_3, x_4] + \beta_2[x_1, x_3][x_2, x_4], \end{aligned}$$

where $\alpha_i, \beta_j \in \mathbb{F}$. Since we have $0 = f(x, x, y, x) = \alpha_1 x^2$, $0 = f(x, y, x, x) = -\alpha_2 x^2$, and $0 = f(y, x, x, x) = -\alpha_3 x^2$, we thus obtain $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Then equalities $0 = f(x, y, x, y) = \beta_1$ and $0 = f(x, x, y, y) = \beta_2$ imply that $f = 0$.

Assume $\Delta = (2, 1, 1)$. Then

$$f(x_1, x_2, x_3) \equiv \alpha_1\, x_1^2[x_2, x_3] + \alpha_2\, x_1x_2[x_1, x_3] + \alpha_3\, [x_1, x_2][x_1, x_3],$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$. We have $\alpha_1 = 0$, since $0 = f(x, y, x) = \alpha_1 x^2$. Thus, the equality $0 = f(x, x, y) = -\alpha_2 x^2$ implies $\alpha_2 = 0$. Finally, since $0 = f(y, x, x) = \alpha_3$, we obtain that $f = 0$.

If Δ belongs to the list $\{(3, 1),\ (2, 2),\ (4)\}$, then Theorem 6.1 concludes the proof. ☐

Proposition 7.2. *Any weak polynomial identity for the pair* $(\mathsf{A}_1,\mathsf{V})$ *of degree* 5 *lies in the L-ideal* $\mathcal{I}$ *generated by* Γ_3*,* St_3*, and* T_4*.*

Proof. Assume that $f\in\mathbb{F}\langle X\rangle$ is a weak polynomial identity of degree 5 for the pair $(\mathsf{A}_1,\mathsf{V})$. By Lemma 3.2, we can assume that f is multihomogeneous of multidegree Δ with $|\Delta|=5$. By Theorem 5.10, f is equivalent to a linear combination of completely reduced bracket-monomials of multidegree Δ.

Assume $\Delta=1^5$. Using Example 5.4 we can see that $f(x_1,\ldots,x_5)$ is equivalent to

$$\begin{aligned}\alpha_1\, x_1x_2x_3[x_4,x_5]+\alpha_2\, x_1x_2x_4[x_3,x_5]+\alpha_3\, x_1x_3x_4[x_2,x_5]+\alpha_4\, x_2x_3x_4[x_1,x_5]\\ +\beta_1\, x_1[x_2,x_3][x_4,x_5]+\beta_2\, x_1[x_2,x_4][x_3,x_5]+\beta_3\, x_2[x_1,x_3][x_4,x_5]\\ +\beta_4\, x_2[x_1,x_4][x_3,x_5]+\beta_5\, x_3[x_1,x_4][x_2,x_5]\end{aligned}$$

for some $\alpha_i,\beta_j\in\mathbb{F}$. Considering

$$f(x,x,x,y,x)=f(x,x,y,x,x)=f(x,y,x,x,x)=f(y,x,x,x,x)=0,$$

we obtain $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$. Equalities

$$f(y,y,x,x,x)=f(y,x,y,x,x)=f(x,y,y,x,x)=f(y,x,x,y,x)=0$$

imply that $\beta_5=\beta_4=\beta_2=\beta_3=0$. Finally, $0=f(x,y,x,y,x)=\beta_1 x$ implies $\beta_1=0$, i.e., $f=0$.

Assume $\Delta=(3,1,1)$. Then

$$f(x_1,x_2,x_3)\equiv\alpha_1x_1^3[x_2,x_3]+\alpha_2x_1^2x_2[x_1,x_3]+\alpha_3x_1[x_1,x_2][x_1,x_3]$$

for some $\alpha_i\in\mathbb{F}$. We have $\alpha_1=0$, since $0=f(x,y,x)=\alpha_1x^3$. Thus, the equality $0=f(x,x,y)=-\alpha_2x^3$ implies $\alpha_2=0$. Finally, since $0=f(y,x,x)=\alpha_3y$, we obtain that $f=0$.

Assume $\Delta=(2,2,1)$. Then $f(x_1,x_2,x_3)$ is equivalent to

$$\alpha_1x_1^2x_2[x_2,x_3]+\alpha_2x_1x_2^2[x_1,x_3]+\alpha_3x_1[x_1,x_2][x_2,x_3]+\alpha_4x_2[x_1,x_2][x_1,x_3]$$

for some $\alpha_i\in\mathbb{F}$. We have $\alpha_1=\alpha_3=0$, since $0=f(x,y,x)=\alpha_1x^2y-\alpha_3x$. Thus, the equality $0=f(x,x,y)=-\alpha_2x^3$ implies $\alpha_2=0$. Finally, since $0=f(y,x,x)=\alpha_4x$, we obtain that $f=0$.

Assume $\Delta=(2,1,1,1)$. Then $f(x_1,x_2,x_3,x_4)$ is equivalent to

$$\begin{aligned}\alpha_1x_1^2x_2[x_3,x_4]+\alpha_2x_1^2x_3[x_2,x_4]+\alpha_3x_1x_2x_3[x_1,x_4]\\ +\beta_1x_1[x_1,x_2][x_3,x_4]+\beta_2x_1[x_1,x_3][x_2,x_4]+\beta_3x_2[x_1,x_3][x_1,x_4]\end{aligned}$$

for some $\alpha_i,\beta_i\in\mathbb{F}$. Since $0=f(x,x,y,x)=\alpha_1x^3$ and $0=f(x,y,x,x)=\alpha_2x^3$, we have $\alpha_1=\alpha_2=0$,. Thus, the equality $0=f(x,x,x,y)=-\alpha_3x^3$ implies $\alpha_3=0$. Considering $0=f(y,x,x,x)=\beta_3x$, we obtain $\beta_3=0$. Finally, equalities $0=f(y,y,x,x)=\beta_2y$ and $0=f(y,x,y,x)=\beta_1y$ imply that $\beta_1=\beta_2=0$, i.e., $f=0$.

If Δ belongs to the list $\{(4,1),\ (3,2),\ (5)\}$, then Theorem 6.1 concludes the proof. □

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