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Cohomology and deformations of left-symmetric Rinehart algebras

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Abstract. We introduce a notion of left-symmetric Rinehart algebras, which is a generalization of the notion of left-symmetric algebras. The left multiplication gives rise to a representation of the corresponding sub-adjacent Lie-Rinehart algebra. We construct left-symmetric Rinehart algebras from $\mathcal{O}$-operators on Lie-Rinehart algebras. We extensively investigate representations of left-symmetric Rinehart algebras. Moreover, we construct a graded Lie algebra on the space of multi-derivations whose Maurer–Cartan elements characterize left-symmetric Rinehart algebras and study deformations of left-symmetric Rinehart algebras, which are controlled by the second cohomology class in the deformation cohomology. We also give the relationships between $\mathcal{O}$-operators and Nijenhuis operators on left-symmetric Rinehart algebras.

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1 Introduction

Left-symmetric algebras are algebras for which the associator

$$(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$$

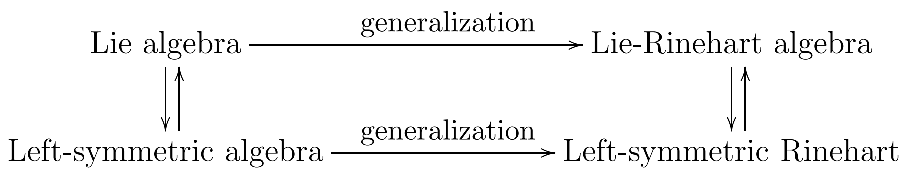
satisfies the identity $(x, y, z) = (y, x, z)$. These algebras appeared as early as 1896 in the work of Cayley [8] as rooted tree algebras. In the 1960s, they also arose from the study of several topics in geometry and algebra, such as convex homogenous cones [30], affine manifolds and affine structures on Lie groups [19,27] and deformations of associative algebras [14]. In 2006, Burde [6] wrote an interesting survey showing the importance of left-symmetric algebras in many areas, such as vector fields, rooted tree algebras, vertex algebras, operad theory, deformation complexes of algebras, convex homogeneous cones, affine manifolds and left-invariant affine structures on Lie groups [6].

Left symmetric algebras are the underlying algebraic structures of non-abelian phase spaces of Lie algebras [1, 21], leading to a bialgebra theory of left-symmetric algebras [3]. They can also be seen as the algebraic structures behind the classical Yang-Baxter equations. Precisely, they provide a construction of solutions of the classical Yang-Baxter equations in certain semidirect product Lie algebra structures (that is, over the double spaces) induced by left-symmetric algebras [2,22].

The notion of Lie-Rinehart algebras was introduced by J. Herz in [15] and further developed in [28, 29]. A a notion of (Poincaré) duality for this class of algebras was introduced in [16,17]. Lie-Rinehart structures have been the subject of extensive studies, in relations to symplectic geometry, Poisson structures, Lie groupoids and algebroids and other kinds of quantizations (see [18,20,23,24,25,26]). For further details and a history of the notion of Lie-Rinehart algebra, we refer the reader to [18]. Lie-Rinehart algebras have been investigated furthermore in [4,7,11,12].

A left-symmetric algebroid is a geometric generalization of a left-symmetric algebra. See [23,24,25] for more details and applications. The notion of a Nijenhuis operator on a left-symmetric algebroid was introduced in [24], which could generate a trivial deformation. More details on deformations of left-symmetric algebras can be found in [31].

In this paper, we introduce a notion of left-symmetric Rinehart algebras, which is a generalization of a left-symmetric algebra and an algebraic version of left symmetric algebroids. The following diagram shows how left-symmetric Rinehart algebras fit in relation to Lie algebras, left-symmetric algebras and Lie-Rinehart algebras.



The paper is organized as follows. In Section 2, we recall some definitions concerning left-symmetric algebras and Lie-Rinehart algebra. In Section 3, we introduce the notion of left-symmetric Rinehart algebra and give some of its properties. As in the case of a left-symmetric algebras, one can obtain the sub-adjacent Lie-Rinehart algebra from a left-symmetric Rinehart algebra by using the commutator. The left multiplication gives rise to a representation of the sub-adjacent Lie-Rinehart algebra. We construct left-symmetric Rinehart algebras using $\mathcal{O}$-operators. Section 4 is devoted to the study of representations and cohomology of left-symmetric Rinehart algebra. In Section 5, we construct a graded Lie algebra whose Maurer-Cartan elements are left-symmetric Rinehart algebras which give rise to a coboundary operator. Section 6 is devoted to introduce the deformation cohomology associated to a left-symmetric Rinehart algebra, which controls the deformations. In Section 7, we introduce the notion of a Nijenhuis operator, which could generate a trivial deformation. In addition, we investigate some connection between $\mathcal{O}$-operators and Nijenhuis operators.

Throughout this paper all vector spaces are over a field $\mathbb{K}$ of characteristic zero.

2 Preliminaries

In this section, we briefly recall some basics of left-symmetric algebras and Lie-Rinehart algebras [6].

Definition 2.1. A left-symmetric algebra is a vector space L endowed with a linear map $\cdot : L \otimes L \longrightarrow L$ such that for any $x, y, z \in L$,

$$(x, y, z) = (y, x, z), \quad \text{or equivalently,} \quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z),$$

where the associator $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$.

Let ad^L (resp. ad^R) be the left multiplication operator (resp. right multiplication operator) on L that is, i.e. $\mathrm{ad}^L(x)y = x \cdot y$ (resp. $\mathrm{ad}^R(x)y = y \cdot x$), for any $x, y \in L$. The following lemma is given in [6].

Lemma 2.2. *Let $(L,\cdot)$ be a left-symmetric algebra. The commutator $[x,y]=x\cdot y-y\cdot x$ defines a Lie algebra L, which is called the sub-adjacent Lie algebra of L. The algebra L is also called a compatible left-symmetric algebra on the Lie algebra L. Furthermore, the map $ad^L : L \to \mathfrak{gl}(L)$ with $x \mapsto L_x$ gives a representation of the Lie algebra $(L,[\cdot,\cdot])$.*

Definition 2.3. Let $(L,\cdot)$ be a left-symmetric algebra and M a vector space. A representation of L on M consists of a pair (ρ,μ), where $\rho : L \longrightarrow \mathfrak{gl}(M)$ is a representation of the sub-adjacent Lie algebra L on M and $\mu : L \longrightarrow \mathfrak{gl}(M)$ is a linear map satisfying:

$$\rho(x)\circ\mu(y)-\mu(y)\circ\rho(x)=\mu(x\cdot y)-\mu(y)\circ\mu(x), \quad \forall\ x,y\in L. \tag{1}$$

The map ρ is called a left representation and μ is a right representation. Usually, we denote a representation by $(M;\rho,\mu)$. Then $(L;\mathrm{ad}^L,\mathrm{ad}^R)$ is a representation of $(L,\cdot)$ which is called adjoint representation.

The cohomology complex for a left-symmetric algebra $(L,\cdot)$ with a representation $(M;\rho,\mu)$ is given as follows. The set of $(n+1)$-cochains is given by

$$C^{n+1}(L,M)=\mathrm{Hom}(\wedge^n L\otimes L,M), \quad \forall n\geq 0. \tag{2}$$

For all $\omega\in C^n(L,M)$, the coboundary operator $\delta : C^n(L,M)\longrightarrow C^{n+1}(L,M)$ is given by

$$\begin{aligned}\delta\omega(x_1,x_2,\dots,x_{n+1}) &= \sum_{i=1}^{n}(-1)^{i+1}\rho(x_i)\omega(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\ &\quad+\sum_{i=1}^{n}(-1)^{i+1}\mu(x_{n+1})\omega(x_1,\dots,\widehat{x_i},\dots,x_n,x_i)\\ &\quad-\sum_{i=1}^{n}(-1)^{i+1}\omega(x_1,\dots,\widehat{x_i},\dots,x_n,x_i\cdot x_{n+1})\\ &\quad+\sum_{1\leq i<j\leq n}(-1)^{i+j}\omega([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{n+1}).\end{aligned}$$

We then have the following lemma whose proof comes from a direct computation using identity (1).

Lemma 2.4 (See [5])**.** *The map δ satisfies $\delta^2=0$.*

Definition 2.5. A Lie-Rinehart algebra L over an associative commutative algebra A is a Lie algebra over $\mathbb{K}$ with an A-module structure and a linear map $\rho : L\to Der(A)$, such that the following conditions hold:

1. For all $a\in A$ and $x,y\in L$

$$\rho([x,y])=\rho(x)\rho(y)-\rho(x)\rho(y) \quad \text{and} \quad \rho(ax)=a\rho(x).$$

2. The compatibility condition:

$$[x, ay] = \rho(x)ay + a[x, y], \quad \forall a \in A, x, y \in L. \tag{3}$$

Let $(L, A, [\cdot,\cdot]_L, \rho)$ and $(L', A', [\cdot,\cdot]_{L'}, \rho')$ be two Lie-Rinehart algebras, then a Lie-Rinehart algebra homomorphism is defined as a pair of maps (g, f), where the maps $f : L \to L'$ and $g : A \to A'$ are two algebra homomorphisms such that:

(1) $f(ax) = g(a)f(x)$ for all $x \in L$ and $a \in A$,

(2) $g(\rho(x)a) = \rho'(f(x))g(a)$ for all $x \in L$ and $a \in A$.

Now, we recall the definition of module over a Lie-Rinehart algebra (for more details see [10]).

Definition 2.6. Let M be an A-module. Then M is a module over a Lie-Rinehart algebra $(L, A, [\cdot,\cdot], \rho)$ if there exists a map $\theta : L \otimes M \to M$ such that:

1. θ is a representation of the Lie algebra $(L, [\cdot,\cdot])$ on M.
2. $\theta(ax, m) = a\theta(x, m)$ for all $a \in A, x \in L, m \in M$.
3. $\theta(x, am) = a\theta(x, m) + \rho(x)am$ for all $x \in L, a \in A, m \in M$.

We have the following lemma giving a characterization of of the θ which are representations.

Lemma 2.7. *The map θ is representation if and only if $L \oplus M$ is Lie-Rinehart algebra over A, where $[\cdot,\cdot]_{L\oplus M}$ and $\theta_{L\oplus M}$ are given by*

$$\begin{aligned}
[x_1 + m_1, x_2 + m_2]_{L\oplus M} &= [x_1, x_2] + \rho(x_1)m_2 - \rho(x_2)m_1, \\
\theta_{L\oplus M}(x_1 + m_1) &= \theta(x_1)
\end{aligned}$$

for all $x_1, x_2 \in L$ and $m_1, m_2 \in M$.

3 Some basic properties of a left-symmetric Rinehart algebras

In this section, we introduce a notion of left-symmetric Rinehart algebras illustrated by some examples. As in the case of a left-symmetric algebra, we obtain the sub-adjacent Lie-Rinehart algebra from a left-symmetric Rinehart algebra using the commutator. In addition, we construct left-symmetric Rinehart algebras using $\mathcal{O}$-operators.

Definition 3.1. A left-symmetric Rinehart algebra is a quadruple $(L, A, \cdot, \ell)$ where $(L, \cdot)$ is a left-symmetric algebra, A is an associative commutative algebra and $\ell : L \to Der(A)$ a linear map such that the following conditions hold:

1. L is an A-module.

2. For all $a \in A$ and $x, y \in L$

$$\ell(x \cdot y - y \cdot x) = \ell(x)\ell(y) - \ell(x)\ell(y) \quad \text{and} \quad \ell(ax) = a\ell(x).$$

3. The compatibility conditions: for all $a \in A$ and $x, y \in L$

$$x \cdot (ay) = \ell(x)ay + a(x \cdot y), \tag{4}$$

$$(ax) \cdot y = a(x \cdot y). \tag{5}$$

Example 3.2. It is clear that any left-symmetric algebra is a left-symmetric Rinehart algebra.

Example 3.3. A Novikov Poisson algebra is a left-symmetric Rinehart algebra (see [32]).

Example 3.4. Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and let $L \oplus A$be the direct sum of L and A. Then $(L \oplus A, A, \cdot_{L\oplus A}, \ell_{L\oplus A})$ is a left-symmetric Rinehart algebra, where the $\cdot_{L\oplus A}$ is defined by the following expression, for all $x_1, x_2 \in L$, $a_1, a_2 \in A$;

$$(x_1 + a_1) \cdot_{L\oplus A} (x_2 + a_2) = x_1 \cdot x_2 + \ell(x_1)(a_2);$$

and $\ell_{L\oplus A} : L \oplus A \to Der(A)$ is defined by $\ell_{L\oplus A}(a_1 + x_1) = \ell(x_1)$. Indeed, it obvious that $(L \oplus A, \cdot_{L\oplus A})$ is a left-symmetric algebra, $\ell_{L\oplus A}$ is a representation of left-symmetric algebra $L \oplus A$ and $\ell_{L\oplus A} \in Der(A)$.
By direct calculation, we have $\ell_{L\oplus A}(b(x_1 + a_1)) = b\ell_{L\oplus A}(x_1 + a_1)$ for all $b, a_1 \in A$ and $x_1 \in L$. On the other hand, letting $x_1, x_2 \in L$ and $b, a_1, a_2 \in A$, we have

$$\begin{aligned}(x_1 + a_1) \cdot_{L\oplus A} b(x_2 + a_2) =&(a_1 + x_1) \cdot_{L\oplus A} (bx_2 + ba_2)\\ =&x_1 \cdot (bx_2) + \ell(x_1)(ba_2)\\ =&b(x_1 \cdot x_2) + \ell(x_1)b(x_2) + \ell(x_1)(b)a_2 + b\ell(x_1)(a_2)\\ =&b\big(x_1 \cdot x_2 + \ell(x_1)(a_2)\big) + \ell(x_1)b(x_2) + \ell(x_1)(b)a_2\\ =&b\big((x_1 + a_1) \cdot_{L\oplus A} (x_2 + a_2)\big) + \ell_{L\oplus A}(x_1 + a_1)b(x_2 + a_2).\end{aligned}$$

Moreover,

$$\begin{aligned}b(x_1 + a_1) \cdot_{L\oplus A} (x_2 + a_2) =&(bx_1 + ba_1) \cdot_{L\oplus A} (x_2 + a_2)\\ =&(bx_1) \cdot x_2 + \ell(bx_1)(a_2)\\ =&b(x_1 \cdot x_2) + b\ell(x_1)(a_2)\\ =&b\big(x_1 \cdot x_2 + \ell(x_1)a_2\big)\\ =&b\big((x_1 + a_1) \cdot_{L\oplus A} (x_2 + a_2)\big).\end{aligned}$$

Now we have the following theorem.

Theorem 3.5. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra. Then, $(L, A, [\cdot,\cdot], \ell)$ is a Lie-Rinehart algebra, denoted by L^C, called the sub-adjacent Lie-Rinehart algebra of $(L, A, \cdot, \ell)$.*

Proof. Since $(L, \cdot)$ is a left-symmetric algebra, we have that $(L, [\cdot,\cdot])$ is a Lie algebra. For any $a \in A$, by direct computations, we have

$$\begin{aligned}[x, ay] &= x\cdot(ay) - (ay)\cdot x = a(x\cdot y) + \ell(x)ay - a(y\cdot x)\\ &= a[x,y] + \ell(x)(a)y,\end{aligned}$$

which implies that $(L, A, [\cdot,\cdot], \ell)$ is a Lie-Rinehart algebra.

To see that the linear map $\ell : L \longrightarrow Der(A)$ is a representation, we only need to show that $\ell_{[x,y]} = [\ell_x, \ell_y]_{Der(A)}$, which follows directly from the fact that $(L, \cdot)$ is a left-symmetric algebra. This ends the proof. □

Definition 3.6. Let $(L_1, A_1, \cdot_1, \ell_1)$ and $(L_2, A_2, \cdot_2, \ell_2)$ be two left-symmetric Rinehart algebras. A homomorphism of left-symmetric Rinehart algebras is a pair of two algebra homomorphisms (f, g) where $f : L_1 \longrightarrow L_2$ and $g : A_1 \longrightarrow A_2$ such that:

$$f(ax) = g(a)f(x), \quad g(\ell_1(x)a) = \ell_2(f(x))g(a), \quad \forall x, y \in L_1, a \in A_1.$$

The following proposition is immediate.

Proposition 3.7. *Let (f, g) be a homomorphism of left-symmetric Rinehart algebras from $(L_1, A_1, \cdot_1, \ell_1)$ to $(L_2, A_2, \cdot_2, \ell_2)$. Then (f, g) is also a Lie-Rinehart algebra homomorphism of the corresponding sub-adjacent Lie-Rinehart algebras.*

Now we give the definition of an $\mathcal{O}$-operator.

Definition 3.8. Let $(L, A, [\cdot,\cdot], \rho)$ be a Lie-Rinehart algebra and $\theta : L \longrightarrow End(M)$ be a representation over M. A linear map $T : M \longrightarrow L$ is called an $\mathcal{O}$-operator if for all $u, v \in M$ and $a \in A$ we have

$$T(au) = aT(u), \tag{6}$$

$$[T(u), T(v)] = T\big(\theta(T(u))(v) - \theta(T(v))(u)\big). \tag{7}$$

Remark 3.9. Consider the semidirect product Lie-Rinehart algebra

$$(L \ltimes_\theta M, A, [\cdot,\cdot]_{L\ltimes_\theta M}, \rho_{L\ltimes_\theta M}),$$

where $\rho_{L\ltimes_\theta M}(x+u) := \rho(x)(u)$ and the bracket $[\cdot,\cdot]_{L\ltimes_\theta M}$ is given by

$$[x+u, y+v]_{L\ltimes_\theta M} = [x,y] + \theta(x)(v) - \theta(y)(u).$$

Any $\mathcal{O}$-operator $T : M \longrightarrow L$ gives a Nijenhuis operator $\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ on the Lie-Rinehart algebra $L \ltimes_\theta M$. More precisely, we have

$$[\tilde{T}(x+u), \tilde{T}(y+v)]_{L\ltimes_\theta M} = \tilde{T}\Big([\tilde{T}(x+u), y+v]_{L\ltimes_\theta M} + [x+u, \tilde{T}(y+v)]_{L\ltimes_\theta M} - \tilde{T}[x+u, y+v]_{L\ltimes_\theta M}\Big).$$

Fore more details on Nijenhuis operators and their applications the reader should consult [13].

Let $T : M \longrightarrow L$ be an $\mathcal{O}$-operator. Define the multiplication $\cdot_T$ on M by

$$u \cdot_T v = \theta(T(u))(v), \ \forall u, v \in M.$$

We then have the following proposition.

Proposition 3.10. *With the above notations,* $(M, A, \cdot_T, \ell_T = \ell \circ T)$ *is a left-symmetric Rinehart algebra, and the map* T *is Lie-Rinehart algebra homomorphism from* $(M, [\cdot, \cdot])$ *to* $(L, [\cdot, \cdot])$.

Proof. It is easy to see that $(M, \cdot_T)$ is a left-symmetric algebra. For any $a \in A$, using Definition 3.1 and equation (6) we have

$$\ell_M(au) = \ell(T(au)) = a\ell(T(u)) = a\ell_M(u),$$

Similarly, using Definition 2.6 we obtain

$$\begin{array}{rcl} (au) \cdot_T v &=& \theta(T(au))(v) = \theta aT(u))(v) = a\theta(T(u))(v), \\ u \cdot_T (av) &=& \theta(T(u))(av) = a\theta(T(u))(v) + \ell \circ T(u)(a)v. \end{array}$$

Thus, $(M, A, \cdot_T, \ell_M)$ is a left-symmetric Rinehart algebra. Let $[\cdot, \cdot]$ be the sub-adjacent Lie bracket on M. Then we have

$$T[u, v] = T(u \cdot_T v - v \cdot_T u) = T(\theta(T(u))(v) - \theta(T(v))(u)) = [T(u), T(v)].$$

So T is a homomorphism of Lie algebras. □

4 Representations of left-symmetric Rinehart algebras

In this section, we develop the notion of representations of a left-symmetric Rinehart algebra and give a cohomology theory with coefficients in a representation.

Definition 4.1. Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and M be an A-module. A representation of A on M consists of a pair (ρ, μ), where ρ is a representation of the sub-adjacent Lie-Rinehart algebra $(L, A, [\cdot, \cdot]^C, \ell)$ and $\mu : L \to End(M)$ is a linear map, such that for all $x, y \in L$ and $m \in M$, we have

$$\begin{array}{l} \mu(ax)m = a\mu(x)m = \mu(x)(am) \\ \rho(x)\mu(y) - \mu(y)\rho(x) = \mu(x \cdot y) - \mu(y)\mu(x). \end{array} \quad (8)$$

We will denote this representation by $(M; \rho, \mu)$.

For a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ and a representation $(M; \rho, \mu)$, the following proposition gives a construction of a left-symmetric Rinehart algebra called semidirect product and denoted by $L \ltimes_{\rho,\mu} M$.

Proposition 4.2. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $(M; \rho, \mu)$ a representation. Then, $(L \oplus M, A, \cdot_{L\oplus M}, \ell_{L\oplus M})$ is a left-symmetric Rinehart algebra, where $\cdot_{L\oplus M}$ and $\ell_{L\oplus M}$ are given by*

$$\begin{aligned} (x_1 + m_1) \cdot_{L\oplus M} (x_2 + m_2) &= x_1 \cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1, &(9)\\ \ell_{L\oplus M}(x_1 + m_1) &= \ell(x_1), &(10) \end{aligned}$$

for all $x_1, x_2 \in L$ and $m_1, m_2 \in M$.

Proof. Let $(M; \rho, \mu)$ be a representation. It is straightforward to see that $(L \oplus M, A, \cdot_{L\oplus M})$ is a left-symmetric algebra. For any $x_1, x_2 \in L$ and $m_1, m_2 \in M$, we have

$$\begin{aligned} (x_1 + m_1) \cdot_{L\oplus M} (a(x_2 + m_2)) &= x_1 \cdot (ax_2) + \rho(x_1)am_2 + \mu(ax_2)m_1\\ &= a(x_1 \cdot x_2) + \ell(x_1)(ax_2) + a\rho(x_1)m_2\\ &\qquad +\ell(x_1)(am_2) + a\mu(x_2)m_1\\ &= a((x_1 + m_1) \cdot_{L\oplus M} (x_2 + m_2)) + \ell_{L\oplus M}(x_1)(a)(x_2 + m_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (a(x_1 + m_1)) \cdot_{L\oplus M} (x_2 + m_2) &= (ax_1) \cdot x_2 + \rho(ax_1)m_2 + \mu(x_2)(am_1)\\ &= a((x_1 + m_1) \cdot_{L\oplus M} (x_2 + m_2)). \end{aligned}$$

Therefore, $(L \oplus M, A, \cdot_{L\oplus M}, \ell_{L\oplus M})$ is a left-symmetric Rinehart algebra. □

Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $(M; \rho, \mu)$ be a representation. Let $\rho^* : L \otimes M^* \to M^*$ and $\mu^* : M^* \otimes L \longrightarrow M^*$ be defined by

$$\langle \rho^*(x)\xi, m\rangle = \ell(x)\langle \xi, m\rangle - \langle \rho(x)m, \xi\rangle \quad \text{and} \quad \langle \mu^*(x)\xi, m\rangle = -\langle \xi, \mu(x)m\rangle,$$

where $M^* = Hom_A(M, A)$. Then, we have the following proposition.

Proposition 4.3. *With the above notations, we obtain that*

(i) *$(M, \rho - \mu)$ is a representation of the sub-adjacent Lie-Rinehart algebra $(L, A, [\cdot, \cdot], \ell)$.*

(ii) *$(M^*, \rho^* - \mu^*, -\mu^*)$ is a representation be a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$.*

Proof. Since $(M; \rho, \mu)$ is a representation of the left-symmetric algebra $(L, A, \cdot, \ell)$, using Proposition 4.2 we have that $(L \oplus M, A, \cdot_{L\oplus M}, \ell_{L\oplus M})$ is a left-symmetric Rinehart algebra. Consider its sub-adjacent Lie-Rinehart algebra $(L \oplus M, A, \cdot_{L\oplus M}, [\cdot, \cdot]_{L\oplus M}, \ell_{L\oplus M})$. We have

$$\begin{aligned} [(x_1 + m_1), (x_2 + m_2)]_{L\oplus M} &= (x_1 + m_1) \cdot_{L\oplus M} (x_2 + m_2) - (x_2 + m_2) \cdot_{L\oplus M} (x_1 + m_1)\\ &= x_1 \cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1\\ &\qquad -x_2 \cdot x_1 - \rho(x_2)m_1 - \mu(x_1)m_2\\ &= [x_1, x_2]^C + (\rho - \mu)(x_1)(m_2) + (\rho - \mu)(x_2)(m_1). \end{aligned}$$

From Lemma 2.7 we deduce that $(M, \rho - \mu)$ is a representation of Lie-Rinehart algebra L on M. This finishes the proof of (i).

For item (ii), it is clear that $\rho^* - \mu^*$ is just the dual representation of the representation $\theta = \rho - \mu$ of the sub-adjacent Lie-Rinehart algebra of L. We can directly check that $-\mu^*(ax)\xi = -a\mu^*(x)\xi = -\mu^*(x)(a\xi)$. For any $x, y \in L$, $\xi \in M^*$ and $m \in M$ we have

$$\begin{aligned}
&-\langle(\rho^* - \mu^*)(x)\mu^*(y)\xi, m\rangle + \langle\mu^*(y)(\rho^* - \mu^*)(x)\xi, m\rangle \\
&\quad= -\langle\rho^*(x)\mu^*(y)\xi, m\rangle + \langle\mu^*(x)\mu^*(y)\xi, m\rangle + \langle\mu^*(y)\rho^*(x)\xi, m\rangle - \langle\mu^*(y)\mu^*(x)\xi, m\rangle \\
&\quad= \ell(x)\langle\xi, \mu(y)m\rangle - \langle\xi, \mu(y)\rho(x)m\rangle + \langle\xi, \mu(y)\mu(x)m\rangle \\
&\qquad -\ell(x)\langle\xi, \mu(y)m\rangle + \langle\xi, \rho(x)\mu(y)m\rangle - \langle\xi, \mu(x)\mu(y)m\rangle \\
&\quad= \langle\xi, \mu(x \cdot y)m\rangle - \langle\xi, \mu(y)\mu(x)m\rangle + \langle\xi, \mu(y)\mu(x)m\rangle - \langle\xi, \mu(x)\mu(y)m\rangle \\
&\quad= \langle(-\mu^*(x \cdot y) - \mu^*(y)\mu^*(x))\xi, m\rangle.
\end{aligned}$$

Therefore $(M^*, \rho^* - \mu^*, -\mu^*)$ is a representation of L. □

Corollary 4.4. *With the above notations, we have*

(i) *The left-symmetric Rinehart algebras $L \ltimes_{\rho,\mu} M$ and $L \ltimes_{\rho-\mu,0} M$ have the same sub-adjacent Lie-Rinehart algebra $L \ltimes_{\rho-\mu} M$.*

(ii) *The left-symmetric Rinehart algebras $L \ltimes_{\rho^*,0} M^*$ and $L \ltimes_{\rho^*-\mu^*,-\mu^*} M^*$ have the same sub-adjacent Lie-Rinehart algebra $L \ltimes_{\rho^*} M^*$.*

Let $(M; \rho, \mu)$ be a representation of a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$. In general, (M^*, ρ^*, μ^*) is not a representation. But we have the following proposition.

Proposition 4.5. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $(M; \rho, \mu)$ be a representation. Then the following conditions are equivalent:*

(1) *$(M; \rho - \mu, -\mu)$ is a representation of $(L, A, \cdot, \ell)$.*

(2) *(M^*, ρ^*, μ^*) is a representation of $(L, A, \cdot, \ell)$.*

(3) *$\mu(x)\mu(y) = \mu(y)\mu(x)$ for all $x, y \in L$.*

5 The Matsushima-Nijenhuis bracket for left-symmetric Rinehart algebras

In this section we construct a graded Lie algebra whose Maurer-Cartan elements are left-symmetric Rinehart algebras which give rise to a coboundary operator.

Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebras. A multiderivation of degree n is a multilinear map $P \in \mathrm{Hom}(\Lambda^n L \otimes L, L)$ such that, for every $a \in A$, $x_i \in L$ and $i \in \{1, 2, \ldots, n+1\}$, we have

$$P(x_1, \ldots, ax_i, \ldots, x_n, x_{n+1}) = aP(x_1, \ldots, x_i, \ldots, x_n, x_{n+1}), \tag{11}$$

$$P(x_1, \cdots, x_n, ax_{n+1}) = aP(x_1, \cdots, x_n, x_{n+1}) + \Xi_P(x_1, \cdots, x_n)(a)x_{n+1}, \tag{12}$$

where $\Xi_P : L^{\otimes n} \to Der(A)$ is called the symbol map. The space of all multiderivations of degree n will be denoted by $\mathfrak{D}^n(L)$. Set $\mathfrak{D}^*(L) = \oplus_{n\geq -1}\mathfrak{D}^n(L)$ with $\mathfrak{D}^{-1}(L) = L$, the space of multiderivations on L.

A permutation $\sigma \in \mathbb{S}_n$ is called an $(i, n-i)$-unshuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. If $i = 0$ and $i = n$, we assume $\sigma = Id$. The set of all $(i, n-i)$-unshuffles will be denoted by $\mathbb{S}_{(i,n-i)}$. The notion of an $(i_1, \cdots, i_k)$-unshuffle and the set $\mathbb{S}_{(i_1,\cdots,i_k)}$ are defined similarly.

Let $P \in \mathfrak{D}^m(L)$ and $Q \in \mathfrak{D}^n(L)$. We define the Matsushima–Nijenhuis bracket $[\cdot,\cdot]_{MN} : \mathfrak{D}^m(L) \times \mathfrak{D}^n(L) \to \mathfrak{D}^{m+n}(L)$ by

$$[P, Q]_{MN} = P \diamond Q - (-1)^{mn} Q \diamond P,$$

where

$$\begin{aligned}
&P \diamond Q(x_1, x_2, \cdots, x_{m+n+1})\\
&= \sum_{\sigma\in\mathbb{S}_{(m,1,n-1)}} (-1)^\sigma P(Q(x_{\sigma(1)}, \cdots, x_{\sigma(m+1)}), x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\
&\quad +(-1)^{mn} \sum_{\sigma\in\mathbb{S}_{(n,m)}} (-1)^\sigma P(x_{\sigma(1)}, \cdots, x_{\sigma(n)}, Q(x_{\sigma(n+1)}, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})).
\end{aligned}$$

Theorem 5.1. *With the above notations, we have*

(i) The pair $(\mathfrak{D}^*(L), [\cdot,\cdot]_{MN})$ is a graded Lie algebra.

(ii) There is a one-to-one correspondence between the set of Maurer-Cartan elements of the graded Lie algebra $(\mathfrak{D}^*(L), [\cdot,\cdot]_{MN})$ and left-symmetric Rinehart algebra structures on L.

Proof. (i) We begin by check that the Matsushima-Nijenhuis bracket is well defined. For $P \in \mathfrak{D}^m(L)$ and $Q \in \mathfrak{D}^n(L)$, by a direct calculation, we have

$$\begin{aligned}
&[P, Q]_{MN}(ax_1, x_2, \cdots, x_{m+n+1})\\
&= aP \diamond Q(x_1, x_2, \cdots, x_{m+n+1}) - (-1)^{mn} aQ \diamond P(x_1, x_2, \cdots, x_{m+n+1})\\
&\quad + \sum_{\sigma\in\mathbb{S}_{(m-1,1,n-1)}} (-1)^\sigma \Xi_Q(x_{\sigma(2)}, \cdots, x_{\sigma(m+1)})(a)P(x_1, x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\
&\quad +(-1)^{mn} \sum_{\sigma\in\mathbb{S}_{(n-1,1,m-1)}} (-1)^\sigma \Xi_P(x_{\sigma(2)}, \cdots, x_{\sigma(n+1)})(a)Q(x_1, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\
&\quad -(-1)^{mn} \sum_{\sigma\in\mathbb{S}_{(n-1,1,m-1)}} (-1)^\sigma \Xi_P(x_{\sigma(2)}, \cdots, x_{\sigma(n+1)})(a)Q(x_1, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\
&\quad - \sum_{\sigma\in\mathbb{S}_{(m-1,1,n-1)}} (-1)^\sigma \Xi_Q(x_{\sigma(2)}, \cdots, x_{\sigma(m+1)})(a)P(x_1, x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\
&= a[P, Q]_{MN}(x_1, x_2, \cdots, x_{m+n+1}),
\end{aligned}$$

which implies that

$$[P,Q]_{MN}(ax_1,x_2,\cdots,x_{m+n+1}) = a[P,Q]_{MN}(x_1,x_2,\cdots,x_{m+n+1}).$$

It is straightforward to check that $[P,Q]_{MN}$ is skew-symmetric with respect to its first $m+n$ arguments. Thus $[P,Q]_{MN}$ is A-linear with respect to its first $m+n$ arguments.

On the other hand, following a straightforward calculation, we have

$$\begin{aligned}[P,Q]_{MN}(x_1,x_2,\cdots,ax_{m+n+1}) &= a[P,Q]_{MN}(x_1,x_2,\cdots,x_{m+n+1})\\ &\quad +\Xi_{[P,Q]_{MN}}(x_1,x_2,\cdots,x_{m+n})(a)x_{m+n+1},\end{aligned}$$

where the symbol map $\Xi_{[P,Q]_{MN}}$ is given by

$$\begin{aligned}&\Xi_{[P,Q]_{MN}}(x_1,x_2,\cdots,x_{m+n})(a)\\ &= \sum_{\sigma\in\mathbb{S}_{(m,1,n-1)}} (-1)^{\sigma}\Xi_P(Q(x_{\sigma(1)},\cdots,x_{\sigma(m+1)}),x_{\sigma(m+2)},\cdots,x_{\sigma(m+n)}))(a)\\ &\quad + \sum_{\sigma\in\mathbb{S}_{(n,1,m-1)}} (-1)^{\sigma}\Xi_Q(P(x_{\sigma(1)},\cdots,x_{\sigma(n+1)}),x_{\sigma(n+2)},\cdots,x_{\sigma(m+n)})(a)\\ &\quad + (-1)^{mn}\sum_{\sigma\in\mathbb{S}_{(m,n)}} (-1)^{\sigma}\Xi_P(x_{\sigma(1)},\cdots,x_{\sigma(n)})(\Xi_Q(x_{\sigma(n+1)},\cdots,x_{\sigma(n+m)}))(a)\\ &\quad + \sum_{\sigma\in\mathbb{S}_{(m,n)}} (-1)^{\sigma}\Xi_Q(x_{\sigma(1)},\cdots,x_{\sigma(m)})(\Xi_P(x_{\sigma(m+1)},\cdots,x_{\sigma(m+n)}))(a).\end{aligned}$$

Thus $[P,Q]_{MN}\in\mathfrak{D}^{m+n}(L)$.

It was shown in [9] that the Matsushima-Nijenhuis bracket provides a graded Lie algebra structure on the graded vector space $\oplus_{n\geq 1}Hom(\Lambda^{n-1}L\otimes L,L)$. Therefore, $(\mathfrak{D}^*(L),[\cdot,\cdot]_{MN})$ is a graded Lie algebra.

(ii) Let $\pi\in\mathfrak{D}^1(L)$, we have

$$\pi(ax_1,x_2)=a\pi(x_1,x_2),\quad \pi(x_1,ax_2)=a\pi(x_1,x_2)+\Xi_\pi(x_1)(a)x_2,\quad \forall\ x_1,x_2\in L.$$

In addition, we can easily check that

$$\begin{aligned}[\pi,\pi]_{MN}(x_1,x_2,x_3) &= 2(\pi(\pi(x_1,x_2),x_3)-\pi(\pi(x_2,x_1),x_3)\\ &\quad -\pi(x_1,\pi(x_2,x_3))+\pi(x_2,(x_1,x_3))).\end{aligned}$$

Thus (L,A,π,Ξ_π) is a left-symmetric Rinehart algebra if and only if $[\pi,\pi]_{MN}=0$. □

Remark 5.2. The cohomology of left-symmetric algebras first appeared in the unpublished paper of Y. Matsushima. Then A. Nijenhuis constructed a graded Lie bracket, which produces the cohomology theory for left-symmetric algebras. Thus the aforementioned graded Lie bracket is usually called the Matsushima–Nijenhuis bracket.

Let (L, A, π, ℓ) be a left-symmetric Rinehart algebra. According to Theorem 5.1, we have $[\pi, \pi]_{MN} = 0$. Using the graded Jacobi identity, we get a coboundary operator $\delta : \mathfrak{D}^{n-1}(L) \to \mathfrak{D}^n(L)$, by putting

$$\delta(P) = (-1)^{n-1}[\pi, P]_{MN}, \quad \forall P \in \mathfrak{D}^{n-1}(L). \tag{13}$$

By straightforward computation, we obtain

Proposition 5.3. *For any $P \in \mathfrak{D}^{n-1}(L)$, we have*

$$\begin{aligned} \delta P(x_1, x_2, \cdots, x_{n+1}) &= \sum_{i=1}^{n}(-1)^{i+1}\pi(x_i, P(x_1, x_2, \cdots, \hat{x_i}, \cdots, x_{n+1})) \\ &+ \sum_{i=1}^{n}(-1)^{i+1}\pi(P(x_1, x_2, \cdots, \hat{x_i}, \cdots, x_n, x_i), x_{n+1}) \\ &- \sum_{i=1}^{n}(-1)^{i+1}P(x_1, x_2, \cdots, \hat{x_i}, \cdots, x_n, \pi(x_i, x_{n+1})) \\ &+ \sum_{1\le i<j\le n}(-1)^{i+j}P(\pi(x_i, x_j) - \pi(x_j, x_i), x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_{n+1}) \end{aligned} \tag{14}$$

for all $x_i \in L, i = 1, 2\cdots, n+1$ and $\Xi_{\delta P}$ is given by

$$\begin{aligned} \Xi_P(x_1, x_2, \cdots, x_n) &= \sum_{i=1}^{n}(-1)^{i+1}[\Xi_\pi(x_i), \Xi_P(x_1, x_2, \cdots, \hat{x_i}, \cdots, x_n)] \\ &+ \sum_{1\le i<j\le n}(-1)^{i+j}\Xi_P(\pi(x_i, x_j) - \pi(x_j, x_i), x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_n) \\ &+ \sum_{i=1}^{n}(-1)^{i+1}\Xi_\pi(P(x_1, x_2, \cdots, \hat{x_i}, \cdots, x_n, x_i)). \end{aligned} \tag{15}$$

Definition 5.4. The cochain complex $(\mathfrak{D}^*(L) = \oplus_{n\ge 0}\mathfrak{D}^n(L), \delta)$ is called the deformation complex of the left-symmetric Rinehart algebra L. The corresponding k-th cohomology group, which we denote by $H^k(L)$, is called the k-th deformation cohomology group.

6 Deformation of left-symmetric Rinehart algebra

We investigate in this section a deformation theory of left-symmetric Rinehart algebras. But first let us introduce some notation. For a left-symmetric Rinehart algebra $(L, A, \mathfrak{m}, \ell)$ we will denote the left-symmetric multiplication " $\cdot$ " by $\mathfrak{m}$ in the sequel of the paper. Let $\mathbb{K}[[t]]$ be the formal power series ring in one variable t and coefficients in $\mathbb{K}$. Let $L[[t]]$ be the set of formal power series whose coefficients are elements of L (note that $L[[t]]$ is obtained by extending the coefficients domain of $\mathbb{K}[[t]]$ from $\mathbb{K}$ to L). Thus, $L[[t]]$ is a $\mathbb{K}[[t]]$-module.

6.1 Formal deformations

Definition 6.1. A deformation of a left-symmetric Rinehart algebra $(L, A, \mathfrak{m}, \ell)$ is a $\mathbb{K}[[t]]$-bilinear map

$$\mathfrak{m}_t : L[[t]] \otimes L[[t]] \to L[[t]]$$

which is given by $\mathfrak{m}_t(x, y) = \sum_{i\geq 0} t^i \mathfrak{m}_i(x, y)$, where $\mathfrak{m}_0 = \mathfrak{m}$ and the $\mathfrak{m}_i \in \mathfrak{D}^1(L)$ satisfy the condition $[\mathfrak{m}_t, \mathfrak{m}_t]_{MN} = 0$.

Note that $\mathfrak{m}_t$ is a 1-degree multiderivation of A with symbol $\Xi_{\mathfrak{m}_t} : L \to Der(A)$ given by

$$\Xi_{\mathfrak{m}_t} = \sum_{i\geq 0} t^i \Xi_{\mathfrak{m}_i}.$$

Moreover, since $[\mathfrak{m}_t, \mathfrak{m}_t]_{MN} = 0$, it corresponds to a left symmetric Lie Rinehart algebra structure. In particular, it yields a t-parameterized family of products $\mathfrak{m}_t : L\otimes L \to L$ and a family of maps $\ell_t : L \to Der(A)$, which satisfy the following identities for all $x, y \in L$:

$$\begin{aligned}\mathfrak{m}_t(x, y) =& x \cdot y + \sum_{i\geq 1} t^i \mathfrak{m}_i(x, y),\\ \ell_t(x) =& \ell(x) + \sum_{i\geq 1} t^i \Xi_{\mathfrak{m}_i}(x).\end{aligned}$$

The t-parametrized family $(L, A, \mathfrak{m}_t, \ell_t)$ is called a 1-parameter formal deformation of $(L, A, \mathfrak{m}, \ell)$ generated by $\mathfrak{m}_1, \cdots, \mathfrak{m}_m \in \mathfrak{D}^1(L)$.

Let $(L, A, \mathfrak{m}_t, \ell_t)$ be a deformation of $\mathfrak{m}$. Then, for all $a \in A, x, y, z \in L$

$$\mathfrak{m}_t(\mathfrak{m}_t(x, y), z) - \mathfrak{m}_t(x, \mathfrak{m}_t(y, z)) = \mathfrak{m}_t(\mathfrak{m}_t(y, x), z) - \mathfrak{m}_t(y, \mathfrak{m}_t(x, z)). \tag{16}$$

$$\mathfrak{m}_t(ax, y) = a\mathfrak{m}_t(x, y), \tag{17}$$

$$\mathfrak{m}_t(x, ay) = a\mathfrak{m}_t(x, y) + \ell_t(x)ay \tag{18}$$

The identities (17)–(18), mean that $\mathfrak{m}_i \in \mathfrak{D}^1(L)$. Comparing the coefficients of t^n for $n \geq 0$ in equation (16), we get the following:

$$\sum_{i+j=n} \mathfrak{m}_i(\mathfrak{m}_j(x, y), z) - \mathfrak{m}_i(x, \mathfrak{m}_j(y, z)) - \mathfrak{m}_i(\mathfrak{m}_j(y, x), z) + \mathfrak{m}_i(y, \mathfrak{m}_j(x, z)) = 0. \tag{19}$$

For $n = 1$, equation (19) implies

$$\begin{aligned}&\mathfrak{m}_1(\mathfrak{m}(x, y), z) + \mathfrak{m}(\mathfrak{m}_1(x, y), z) - \mathfrak{m}_1(x, \mathfrak{m}(y, z)) - \mathfrak{m}(x, \mathfrak{m}_1(y, z))\\ &-\mathfrak{m}_1(\mathfrak{m}(y, x), z) - \mathfrak{m}(\mathfrak{m}_1(y, x), z) + \mathfrak{m}_1(y, \mathfrak{m}(x, z)) + \mathfrak{m}(y, \mathfrak{m}_1(x, z)) = 0.\end{aligned}$$

Or equivalently $\delta(\mathfrak{m}_1) = [\mathfrak{m}, \mathfrak{m}_1]_{MN} = 0$.

The 1-degree multiderivation $\mathfrak{m}_1$ is called the infinitesimal of the deformation $\mathfrak{m}_t$. More generally, if $\mathfrak{m}_i = 0$ for $1 \leq i \leq n-1$ and $\mathfrak{m}_n$ is non zero 1-degree multiderivation then $\mathfrak{m}_n$ is called the n-infinitesimal of the deformation $\mathfrak{m}_t$. By the above discussion, the following proposition follows immediately.

Proposition 6.2. *The infinitesimal of the deformation $\mathfrak{m}_t$ is a 2-cocycle in $\mathfrak{D}^1(L)$. More generally, the n-infinitesimal is a 2-cocycle.*

Now we give a notion of equivalence of two deformations. Let us denote a deformation $(L, A, \mathfrak{m}_t, \ell_t)$ of $(L, A, \mathfrak{m}, \ell)$ simply by L_t. Let us consider two deformations L_t and L'_t of $(L, A, \mathfrak{m}, \ell)$, generated by $\mathfrak{m}_i$ and $\mathfrak{m}'_i$, respectively, for $i \geq 0$.

Definition 6.3. Two deformations L_t and L'_t are said to be equivalent if there exists a formal automorphism

$$\Phi_t : L[[t]] \to L[[t]] \text{ defined as } \Phi_t = id_L + \sum_{i\geq 1} t^i \phi_i$$

where for each $i \geq 1$, $\phi_i : L \to L$ is a $\mathbb{K}$-linear map such that

$$\mathfrak{m}'_t(x, y) = \Phi_t^{-1}\mathfrak{m}_t(\Phi_t(x), \Phi_t(y)) \qquad \text{and} \qquad \ell'_t(\Phi_t(x)) = \ell_t(x).$$

Definition 6.4. Any deformation that is equivalent to the deformation $\mathfrak{m}_0 = \mathfrak{m}$ is said to be a trivial deformation.

Theorem 6.5. *The cohomology class of the infinitesimal of a deformation $\mathfrak{m}_t$ is determined by the equivalence class of $\mathfrak{m}_t$.*

Proof. Let Φ_t be an equivalence of deformation between $\mathfrak{m}_t$ and $\tilde{\mathfrak{m}}_t$. Then we get,

$$\tilde{\mathfrak{m}}_t(x, y) = \Phi_t^{-1}\mathfrak{m}_t(\Phi_t x, \Phi_t y).$$

Comparing the coefficients of t from both sides of the above equation we have

$$\tilde{\mathfrak{m}}_1(x, y) + \Phi_1(\mathfrak{m}_0(x, y)) = \mathfrak{m}_1(x, y) + \mathfrak{m}_0(\Phi_1(x), y) + \mathfrak{m}_0(x, \Phi_1(y)),$$

or equivalently,

$$\mathfrak{m}_1 - \tilde{\mathfrak{m}}_1 = \delta(\phi_1).$$

This establishes the result. ∎

Definition 6.6. A left-symmetric Rinehart algebra is said to be rigid if and only if every deformation of it is trivial.

Theorem 6.7. *A non-trivial deformation of a left-symmetric Rinehart algebra is equivalent to a deformation whose n-infinitesimal is not a coboundary for some $n \geq 1$.*

Proof. Let $\mathfrak{m}_t$ be a deformation of left-symmetric Rinehart algebra with n-infinitesimal $\mathfrak{m}_n$ for some $n \geq 1$. Assume that there exists a 2-cochain $\phi \in C^1(L, L)$ with $\delta(\phi) = \mathfrak{m}_n$. Then set

$$\Phi_t = id_L + \phi t^n \ \text{ and define } \ \bar{\mathfrak{m}}_t = \Phi_t \circ \mathfrak{m}_t \circ \Phi_t^{-1}.$$

Then by computing the expression and comparing coefficients of t^n, we get

$$\bar{\mathfrak{m}}_n - \mathfrak{m}_n = -\delta(\phi).$$

So, $\bar{\mathfrak{m}}_n = 0$. We can repeat the argument to kill off any infinitesimal, which is a coboundary. □

Corollary 6.8. *If $H^2(L, L) = 0$, then all deformations of L are equivalent to a trivial deformation.*

6.2 Obstructions to the extension theory of deformations

Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra. Now we consider the problem of extending a deformation of $\mathfrak{m}$ of order n to a deformation of $\mathfrak{m}$ of order $(n+1)$. Let $\mathfrak{m}_t$ and ℓ_t be a deformation of order n of $\mathfrak{m}$ and ℓ respectively. That is

$$\mathfrak{m}_t = \sum_{i=0}^{n} \mathfrak{m}_i t^i = \mathfrak{m} + \sum_{i=1}^{n} \mathfrak{m}_i t^i \quad \text{and} \quad \ell_t = \sum_{i=0}^{n} \ell_i t^i = \ell + \sum_{i=1}^{n} \ell_i t^i,$$

where $\mathfrak{m}_i \in \mathfrak{D}^1(L)$ and $\ell_i : L \to Der(A)$ a linear map for each $1 \leq i \leq n$ such that

$$\mathfrak{m}_i(\mathfrak{m}_j(x, y), z) - \mathfrak{m}_i(x, \mathfrak{m}_j(y, z)) = \mathfrak{m}_i(\mathfrak{m}_j(y, x), z) - \mathfrak{m}_i(y, \mathfrak{m}_j(x, z)), \tag{20}$$
$$\mathfrak{m}_i(ax, y) = a\mathfrak{m}_i(x, y), \tag{21}$$
$$\mathfrak{m}_i(x, ay) = a\mathfrak{m}_i(x, y) + \ell_i(x)ay \tag{22}$$

for all $1 \leq i, j \leq n$. If there exists a 2-cochain $\mathfrak{m}_{n+1} \in \mathfrak{D}^1(L)$ and $\ell_{n+1} : L \to Der(A)$ such that $(L, A, \tilde{\mathfrak{m}}_t, \tilde{\ell}_t)$ is a deformation of $(L, A, \mathfrak{m}, \ell)$ of order $n+1$, where

$$\tilde{\mathfrak{m}}_t = \mathfrak{m}_t + \mathfrak{m}_{n+1} t^{n+1} \quad \text{and} \quad \tilde{\ell}_t = \ell_t + \ell_{n+1} t^{n+1}.$$

Then we say that $\mathfrak{m}_t$ extends to a deformation of order $(n+1)$. In this case $\mathfrak{m}_t$ is called extendable.

Definition 6.9. Let $\mathfrak{m}_t$ be a deformation of $\mathfrak{m}$ of order n. Consider the cochain in $C^3(L, L)$ defined as

$$\begin{aligned} Obs_L(x, y, z) = \sum_{\substack{i+j=n+1;\\ i,j>0}} \Big(& \mathfrak{m}_i(\mathfrak{m}_j(x, y), z) - \mathfrak{m}_i(x, \mathfrak{m}_j(y, z)) \\ & - \mathfrak{m}_i(\mathfrak{m}_j(y, x), z) + \mathfrak{m}_i(y, \mathfrak{m}_j(x, z))) \Big), \end{aligned} \tag{23}$$

for $x, y, z \in L$. The 3-cochain Obs_L is called an obstruction cochain for extending the deformation of $\mathfrak{m}$ of order n to a deformation of order $n+1$.

A straightforward computation gives the following

Proposition 6.10. *The obstructions are left-symmetric Rinehart algebra 3-cocycles.*

Theorem 6.11. *Let $\mathfrak{m}_t$ be a deformation of $\mathfrak{m}$ of order n. Then $\mathfrak{m}_t$ extends to a deformation of order $n+1$ if and only if the cohomology class of Obs_L vanishes.*

Proof. Suppose that a deformation $\mathfrak{m}_t$ of order n extends to a deformation of order $n+1$. Then

$$\sum_{\substack{i+j=n+1;\\ i,j\geq 0}} \Big(\mathfrak{m}_i(\mathfrak{m}_j(x,y),z)-\mathfrak{m}_i(x,\mathfrak{m}_j(y,z))-\mathfrak{m}_i(\mathfrak{m}_j(y,x),z)+\mathfrak{m}_i(y,\mathfrak{m}_j(x,z))\Big)=0.$$

As a result, we get $Obs_L=\delta(m_{n+1})$. So, the cohomology class of Obs_L vanishes.

Conversely, let Obs_L be a coboundary. Suppose that

$$Obs_L=\delta(\mathfrak{m}_{n+1})$$

for some 2-cochain $\mathfrak{m}_{n+1}$. Define a map $\tilde{\mathfrak{m}_t}: L[[t]]\times L[[t]]\to L[[t]]$ as follows

$$\tilde{\mathfrak{m}_t}=\mathfrak{m}_t+\mathfrak{m}_{n+1}t^{n+1}.$$

Then for any $x,y,z\in L$, the map $\tilde{\mathfrak{m}_t}$ satisfies the following identity

$$\sum_{\substack{i+j=n+1;\\ i,j\geq 0}} \Big(\mathfrak{m}_i(\mathfrak{m}_j(x,y),z)-\mathfrak{m}_i(x,\mathfrak{m}_j(y,z))-\mathfrak{m}_i(\mathfrak{m}_j(y,x),z)+\mathfrak{m}_i(y,\mathfrak{m}_j(x,z))\Big)=0.$$

This, in turn, implies that $\tilde{\mathfrak{m}_t}$ is a deformation of $\mathfrak{m}$ extending $\mathfrak{m}_t$. □

Corollary 6.12. *If $H^3(L,L)=0$, then every 2-cocycle in $C^2(L,L)$ is the infinitesimal of some deformation of $\mathfrak{m}$.*

6.3 Trivial deformation

We study deformations of left-symmetric Rinehart algebras using the deformation cohomology. Let $(L,A,\cdot,\ell)$ be a left-symmetric Rinehart algebra, and $\mathfrak{m}\in C^2(L,L)$. Consider a t-parameterized family of multiplications $\mathfrak{m}_t: L[[t]]\otimes L[[t]]\to L[[t]]$ and linear maps $\ell_t: L\to Der(A)$ given by

$$\mathfrak{m}_t(x,y)=x\cdot y+t\mathfrak{m}(x,y), \tag{24}$$
$$\ell_t=\ell+t\Xi_{\mathfrak{m}}. \tag{25}$$

If $L_t=(L,A,\mathfrak{m}_t,\ell_t)$ is a left-symmetric Rinehart algebra for all t, we say that $\mathfrak{m}$ generates a 1-parameter infinitesimal deformation of $(L,A,\cdot,\ell)$

Since $\mathfrak{m}$ is a 2-cochain, we have

$$\mathfrak{m}(ax,y)=a\mathfrak{m}(x,y),\ \text{and}\ \mathfrak{m}(x,ay)=a\mathfrak{m}_t(x,y)+\Xi_{\mathfrak{m}}(x)(a)y,$$

which implies that conditions (4) and (5) in Definition 3.1 are satisfied for $\mathfrak{m}_t$. Then we can deduce that $(L,A,\mathfrak{m}_t,\ell_t)$ is a deformation of $(L,A,\cdot,\ell)$ if and only if

$$\begin{aligned} &x\cdot\mathfrak{m}(y,z)-y\cdot\mathfrak{m}(x,z)+\mathfrak{m}(y,x)\cdot z-\mathfrak{m}(x,y)\cdot z\\ &=\mathfrak{m}(y,x\cdot z)-\mathfrak{m}(x,y\cdot z)-\mathfrak{m}([x,y],z), \end{aligned} \tag{26}$$

and

$$\mathfrak{m}(\mathfrak{m}(x,y),z)-\mathfrak{m}(x,\mathfrak{m}(y,z)) \;=\; \mathfrak{m}(\mathfrak{m}(y,x),z)-\mathfrak{m}(y,\mathfrak{m}(x,z)). \tag{27}$$

Equation (26) means that $\mathfrak{m}$ is a 2-cocycle, and equation (27) means that $(L, A, \mathfrak{m}, \Xi_{\mathfrak{m}})$ is a left-symmetric Rinehart algebra.

Recall that a deformation is said to be trivial if there exists a family of left-symmetric Rinehart algebra isomorphisms $\mathrm{Id} + tN : L_t \longrightarrow L$.

By direct computations, L_t is trivial if and only if

$$\mathfrak{m}(x,y) \;=\; x\cdot N(y)+N(x)\cdot y-N(x\cdot y), \tag{28}$$
$$N\mathfrak{m}(x,y) \;=\; N(x)\cdot N(y), \tag{29}$$
$$\ell\circ N \;=\; \Xi_{\mathfrak{m}}. \tag{30}$$

Again, equation (30) can be obtained from equation (28). It follows from (28) and (29) that N must satisfy the following integrability condition

$$N(x)\cdot N(y)-x\cdot N(y)-N(x)\cdot y+N^2(x\cdot y)=0. \tag{31}$$

Now we give the following definition.

Definition 6.13. An A-linear map $N : L \longrightarrow L$ is called a Nijenhuis operator on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ if the Nijenhuis condition (31) holds.

Obviously, any Nijenhuis operator on a left-symmetric Rinehart algebra is also a Nijenhuis operator on the corresponding sub-adjacent Lie-Rinehart algebra.

We have seen that a trivial deformation of a left-symmetric Rinehart algebra gives rise to a Nijenhuis operator. In fact, the converse is also true as can be seen from the following theorem.

Theorem 6.14. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and N be a Nijenhuis operator. Then a deformation of $(L, A, \cdot, \ell)$ can be obtained by putting*

$$\mathfrak{m}(x,y)=\delta N(x,y).$$

Furthermore, this deformation is trivial.

Proof. Since $\mathfrak{m}$ is a coboundary, then it is a cocycle, i.e. equation (26) holds. To see that $\mathfrak{m}$ generates a deformation, we only need to show that (27) holds, which follows from the Nijenhuis condition (31). At the end, we can easily check that

$$(\mathrm{Id}+tN)(x\cdot_t y)=(\mathrm{Id}+tN)(x)\cdot(\mathrm{Id}+tN)(y), \quad \ell\circ(\mathrm{Id}+tN)=\ell_t,$$

which implies that the deformation is trivial. □

Theorem 6.15. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and N be a Nijenhuis operator. Then $(L, A, \cdot_N, \ell_N = \ell\circ N)$ is a left-symmetric Rinehart algebra, where*

$$x\cdot_N y = x\cdot N(y)+N(x)\cdot y-N(x\cdot y), \forall x,y\in L.$$

Proof. It is obvious to show that $(L, \cdot_N)$ is a left-symmetric algebra and ℓ_N is a representation of L on $Der(A)$. Evidently, we have

$$\ell_N(ax) = a\ell_N(x), \forall x \in L, a \in A.$$

Furthermore, for any $x, y \in L$ and $a \in A$ we have

$$\begin{aligned} x \cdot_N (ay) =& x \cdot N(ay) + N(x) \cdot (ay) - N(x \cdot (ay)) \\ =& a(x \cdot N(y)) + \ell(x)aN(y) + a(N(x) \cdot y) + \ell(N(x))ay - aN(x \cdot y) - N(\ell(x)ay) \\ =& a(x \cdot N(y) + N(x) \cdot y - N(x \cdot y)) + \ell_N(x)ay + N(\ell(x)ay) - N(\ell(x)ay). \\ =& a(x \cdot_N y) + \ell_N(x)ay. \end{aligned}$$

Moreover,

$$\begin{aligned} (ax) \cdot_N y =& (ax) \cdot N(y) + N(ax) \cdot y - N((ax) \cdot y) \\ =& a(x \cdot N(y)) + a(N(x) \cdot y) - aN(x \cdot y) \\ =& a(x \cdot N(y) + N(x) \cdot y - N(x \cdot y)) \\ =& a(x \cdot_N y). \end{aligned}$$

Then, $(L, A, \cdot_N, \ell_N = \ell \circ N)$ is a a left-symmetric Rinehart algebra. □

Immediately, we have the following result.

Lemma 6.16. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and N be a Nijenhuis operator. Then for arbitrary positive $j, k \in \mathbb{N}$, the following equation holds*

$$N^j(x) \cdot N^k(y) - N^k(N^j(x) \cdot y) - N^j(x \cdot N^k(y)) + N^{j+k}(x \cdot y) = 0, \quad \forall\ x, y \in L. \tag{32}$$

If N is invertible, this formula becomes valid for arbitrary $j, k \in \mathbb{Z}$.

By direct calculations, we have the following corollary.

Corollary 6.17. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and N a Nijenhuis operator.*

(i) *For all $k \in \mathbb{N}$, $(L, A, \cdot_{N^k}, \ell_{N^k} = \ell \circ N^k)$ is a left-symmetric Rinehart algebra.*

(ii) *For all $l \in \mathbb{N}$, N^l is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L, A, \cdot_{N^k}, \ell_{N^k})$.*

(iii) *The left-symmetric Rinehart algebras $(L, A, (\cdot_{N^k})_{N^l}, \ell_{N^{k+l}})$ and $(L, A, \cdot_{N^{k+l}}, \ell_{N^{k+l}})$ are the same.*

(iv) *N^l is a left-symmetric Rinehart algebra homomorphism from $(L, A, \cdot_{N^{k+l}}, \ell_{N^{k+l}})$ to $(L, A, \cdot_{N^k}, \ell_{N^k})$.*

Theorem 6.18. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and N be a Nijenhuis operator. Then the operator $P(N) = \sum_{i=0}^{n} c_i N^i$ is a Nijenhuis operator. If N is invertible, then $Q(N) = \sum_{i=-m}^{n} c_i N^i$ is also a Nijenhuis operator.*

Proof. According to Lemma 6.16, we obtain, $\forall x, y \in L$,

$$\begin{aligned} & P(N)(x) \cdot P(N)(y) - P(N)(P(N)(x) \cdot y) - P(N)(x \cdot P(N)(y)) + P^2(N)(x \cdot y) \\ = & \sum_{i,j=0}^{n} c_j c_k \Big(N^j(x) \cdot N^k(y) - N^k(N^j(x) \cdot y) - N^j(x \cdot N^k(y)) + N^{j+k}(x \cdot y) \Big) = 0, \end{aligned}$$

which implies that $P(N)$ is a Nijenhuis operator. Similarly we can easy check the second statement. □

7 $\mathcal{O}$-operators and Nijenhuis operators

In this section, we highlight the relationships between $\mathcal{O}$-operators and Nijenhuis operators on left-symmetric Rinehart algebras. Moreover, we illustrate some connections between Nijenhuis operators and compatible $\mathcal{O}$-operators on left-symmetric Rinehart algebras.

7.1 Relationships between $\mathcal{O}$-operators and Nijenhuis operators

We first give the definitions of an $\mathcal{O}$-operator and of Rota-Baxter operator.

Definition 7.1. An $\mathcal{O}$-operator on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$ is a linear map $T : M \longrightarrow L$ satisfying

$$T(au) = aT(u), \tag{33}$$

$$T(u) \cdot T(v) = T\Big(\rho(T(u))(v) + \mu(T(v))(u)\Big), \quad \forall u, v \in M, a \in A. \tag{34}$$

Definition 7.2. Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $\mathcal{R} : ker(\ell) \longrightarrow L$ a linear operator. If $\mathcal{R}$ satisfies

$$\mathcal{R}(ax) = a\mathcal{R}(x), \tag{35}$$

$$\mathcal{R}(x) \cdot \mathcal{R}(y) = \mathcal{R}(\mathcal{R}(x) \cdot y + x \cdot \mathcal{R}(y)), \quad \forall x, y \in L, a \in A, \tag{36}$$

then $\mathcal{R}$ is called a Rota-Baxter operator of weight 0 on L.

Notice that a Rota-Baxter operator of weight zero on a left-symmetric Rinehart algebra L is exactly an $\mathcal{O}$-operator associated to the adjoint representation $(L; ad^L, ad^R)$.

The following proposition gives connections between Nijenhuis operators and Rota-Baxter operators.

Proposition 7.3. *Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $N : L \longrightarrow L$ a linear operator.*

(i) *If* $N^2 = \mathrm{Id}$*, then* N *is a Nijenhuis operator if and only if* $N \pm \mathrm{Id}$ *is a Rota-Baxter operator of weight* ∓ 2 *on* $(L, A, \cdot, \ell)$*.*

(ii) *If* $N^2 = 0$*, then* N *is a Nijenhuis operator if and only if* N *is a Rota-Baxter operator of weight zero on* $(L, A, \cdot, \ell)$*.*

(iii) *If* $N^2 = N$*, then* N *is a Nijenhuis operator if and only if* N *is a Rota-Baxter operator of weight* -1 *on* $(L, A, \cdot, \ell)$*.*

Proof. For Item (i), for all $x, y \in L$ then we have

$$\begin{aligned}
&(N-Id)(x)\cdot(N-Id)(y)\\
&\qquad\qquad -(N-Id)\big((N-Id)(x)\cdot y + x\cdot(N-Id)(y)\big) + 2(N-Id)(x\cdot y)\\
&\quad = N(x)\cdot N(y) - N\big(N(x)\cdot y + x\cdot N(y)\big) + x\cdot y\\
&\quad = N(x)\cdot N(y) - N\big(N(x)\cdot y + x\cdot N(y)\big) + N^2(x\cdot y).
\end{aligned}$$

So N is a Nijenhuis operator if and only if $N - Id$ is a Rota-Baxter operator of weight 2 on L. Similarly, we obtain that N is a Nijenhuis operator if and only if $N + Id$ is a Rota-Baxter operator of weight -2 on L.

Items (ii) and (iii) are obvious from the definitions of Nijenhuis operators and Rota-Baxter operator. □

Proposition 7.4. *Let* $(L, A, \cdot, \ell)$ *be a left-symmetric Rinehart algebra and* $(M; \rho, \mu)$ *be a representation on L. Let* $T : M \to L$ *be a linear map. For any* λ*,* T *is an* $\mathcal{O}$*-operator on* L *associated to* $(M; \rho, \mu)$ *if and only if the linear map* $\mathcal{R}_{T,\lambda} := \begin{pmatrix} 0 & T \\ 0 & -\lambda\mathrm{Id} \end{pmatrix}$ *is a Rota-Baxter operator of weight* λ *on the semidirect product left-symmetric Rinehart algebra* $(L \oplus M, \cdot_{L\oplus M})$*, where the multiplication* $\cdot_{L\oplus M}$ *is given by* (9)*.*

Proof. It is easy to check the equation (35). Let $x_1, x_2 \in L$ and $m_1, m_2 \in M$,

$$\begin{aligned}
\mathcal{R}_{T,\lambda}(x_1+m_1)\cdot_{L\oplus M}\mathcal{R}_{T,\lambda}(x_2+m_2) &= (T(m_1)-\lambda m_1)\cdot_{L\oplus M}(T(m_2)-\lambda m_2)\\
&= T(m_1)\cdot T(m_2) - \lambda\rho(T(m_1)m_2 - \lambda\mu(T(m_2))m_1.
\end{aligned} \tag{37}$$

On the other hand,

$$\begin{aligned}
&\mathcal{R}_{T,\lambda}\big(\mathcal{R}_{T,\lambda}(x_1+m_1)\cdot_{L\oplus M}(x_2+m_2) + (x_1+m_1)\cdot_{L\oplus M}\mathcal{R}_{T,\lambda}(x_2+m_2)\big)+\\
&\quad \lambda\mathcal{R}_{T,\lambda}((x_1+m_1)\cdot_{L\oplus M}(x_2+m_2))\\
=&\mathcal{R}_{T,\lambda}\big((T(m_1)-\lambda m_1)\cdot_{L\oplus M}(x_2+m_2) + (x_1+m_1)\cdot_{L\oplus M}(T(m_2)-\lambda m_2)\big)+\\
&\quad \lambda\mathcal{R}_{T,\lambda}(x_1\cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1)\\
=&\mathcal{R}_{T,\lambda}\big(T(m_1)\cdot x_2 + \rho(T(m_1))m_2 - \lambda\mu(x_2)m_1\\
&\qquad + x_1\cdot T(m_2) - \lambda\rho(x_1)m_2 + \mu(T(m_2))m_1\big)
\end{aligned} \tag{38}$$

$$
\begin{aligned}
&+\lambda\big(T(\rho(x_1)m_2)+T(\mu(x_2)m_1)-\lambda(\rho(x_1)m_2+\mu(x_2)m_1)\big)\\
=&T\big(\rho(T(m_1))m_2+\mu(T(m_2))m_1\big)-\lambda T\big(\mu(x_2)m_1+\rho(x_1)m_2\big) \qquad (39)\\
&-\lambda\Big(\rho(T(m_1))m_2+\mu(T(m_2))m_1\Big)+\lambda^2\Big(\mu(x_2)m_1+\rho(x_1)m_2\Big)\\
&+\lambda\big(T(\rho(x_1)m_2)+T(\mu(x_2)m_1)\big)-\lambda^2\big(\rho(x_1)m_2+\mu(x_2)m_1)\big)\\
=&T\big(\rho(T(m_1))m_2+\mu(T(m_2))m_1\big)-\lambda\rho(T(m_1))m_2-\lambda\mu(T(m_2))m_1. \qquad (40)
\end{aligned}
$$

According to equations (37) and (40), $\mathcal{R}_{T,\lambda}$ is a Rota-Baxter operator of weight λ on the semidirect product left-symmetric Rinehart algebra $(L\oplus M,\cdot_{L\oplus M})$ if and only if T is an $\mathcal{O}$-operator on $(L,A,\cdot,\ell)$ associated to $(M;\rho,\mu)$. □

Proposition 7.5. *Let $(L,A,\cdot,\ell)$ be a left-symmetric Rinehart algebra and let $(M;\rho,\mu)$ be a representation on L. Let $T:M\to L$ be a linear map. Then the following statements are equivalent.*

(i) *T is an $\mathcal{O}$-operator on the left-symmetric Rinehart algebra $(L,A,\cdot,\ell)$.*

(ii) *$\mathcal{N}_T := \begin{pmatrix} 0 & T \\ 0 & \mathrm{Id}\end{pmatrix}$ is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L\oplus M,\cdot_{L\oplus M})$.*

(iii) *$N_T := \begin{pmatrix} 0 & T \\ 0 & 0\end{pmatrix}$ is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L\oplus M,\cdot_{L\oplus M})$.*

Proof. Note that $\mathcal{N}_T=\mathcal{R}_{T,-1}$ and $(\mathcal{N}_T)^2=\mathcal{N}_T$, thus $\mathcal{N}_T$ is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L\oplus M,\cdot_{L\oplus M})$, using (iii) in Proposition 7.3.

Similarly $N_T=\mathcal{R}_{T,0}$ and $(N_T)^2=0$, then N_T is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L\oplus M,\cdot_{L\oplus M})$, according to (ii) in Proposition 7.3. □

7.2 Compatible $\mathcal{O}$-operators and Nijenhuis operators

In this subsection we study compatibility of $\mathcal{O}$-operators and Nijenhuis operators. First we start with the following definition.

Definition 7.6. Let $(L,A,\cdot,\ell)$ be a left-symmetric Rinehart algebra and let $(M;\rho,\mu)$ be a representation. Let $T_1,T_2:M\longrightarrow L$ be two $\mathcal{O}$-operators associated to $(M;\rho,\mu)$. Then T_1 and T_2 are called compatible if T_1+T_2 is an $\mathcal{O}$-operator associated to $(M;\rho,\mu)$.

Let $T_1,T_2:M\longrightarrow L$ be two $\mathcal{O}$-operators on a left-symmetric Rinehart algebra $(L,A,\cdot,\ell)$ associated to a representation $(M;\rho,\mu)$ such that

$$
\begin{aligned}
T_1(u)\cdot T_2(v)+T_2(u)\cdot T_1(v)=&\,T_1\Big(\rho(T_2(u))(v)+\mu(T_2(v))(u)\Big)\\
&+T_2\Big(\rho(T_1(u))(v)+\mu(T_1(v))(u)\Big), \qquad (41)
\end{aligned}
$$

for all $u,v\in M$.

Lemma 7.7. *Two operators T_1 and T_2 are compatible if and only if the equation (41) holds.*

Proof. For all $u, v \in M$, $a \in A$, we have

$$\begin{aligned}(T_1 + T_2)(au) &= T_1(au) + T_2(au)\\ &= aT_1(u) + aT_2(u)\\ &= a(T_1 + T_2)(u).\end{aligned}$$

Furthermore,

$$\begin{aligned}&(T_1 + T_2)(u) \cdot (T_1 + T_2)(v) - (T_1 + T_2)\Big(\rho((T_1 + T_2)(u))(v) + \mu((T_1 + T_2)(v))(u)\Big)\\ &= T_1(u) \cdot T_1(v) + T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v) + T_2(u) \cdot T_2(v)\\ &\quad - (T_1 + T_2)\Big(\rho(T_1(u))(v) + \rho(T_2(u))(v) + \mu(T_1(v))(u) + \mu(T_2(v))(u)\Big)\\ &= T_1(u) \cdot T_1(v) + T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v) + T_2(u) \cdot T_2(v)\\ &\quad - T_1\big(\rho(T_1(u))(v) + \mu(T_1(v))(u)\big) - T_1\big(\rho(T_2(u))(v) + \mu(T_2(v))(u)\big)\\ &\quad - T_2\big(\rho(T_1(u))(v) + \mu(T_1(v))(u)\big) - T_2\big(\rho(T_2(u))(v) + \mu(T_2(v))(u)\big)\\ &= T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v)\\ &\quad - T_1\Big(\rho(T_2(u))(v) + \mu(T_2(v))(u)\Big) - T_2\Big(\rho(T_1(u))(v) + \mu(T_1(v))(u)\Big)\end{aligned}$$

Then $T_1 + T_2$ is an $\mathcal{O}$-operator associated to $(M; \rho, \mu)$ if and only if equation (41) holds. □

Remark 7.8. Equation (41) implies that for any k_1, k_2 the linear combination $k_1T_1 + k_2T_2$ is an $\mathcal{O}$-operator.

There is a close relationship between a Nijenhuis operator and a pair of compatible $\mathcal{O}$-operators as can be seen from the following proposition.

Proposition 7.9. *Let $T_1, T_2 : M \longrightarrow L$ be two $\mathcal{O}$-operators on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$. Suppose that T_2 is invertible. If T_1 and T_2 are compatible, then $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$.*

Proof. For all $x, y \in L$, since T_2 is invertible, there exist $u, v \in M$ such that $T_2(u) = x$, $T_2(v) = y$. Hence $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator if and only if the following equation holds:

$$NT_2(u) \cdot NT_2(v) = N(NT_2(u) \cdot T_2(v) + T_2(u) \cdot NT_2(v)) - N^2(T_2(u) \cdot T_2(v)). \tag{42}$$

Since $T_1 = N \circ T_2$ is an $\mathcal{O}$-operator, the left-hand side of the above equation is

$$NT_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)).$$

Using the fact that T_2 and $T_1 = N \circ T_2$ are two compatible $\mathcal{O}$-operators, we get

$$\begin{aligned} & NT_2(u) \cdot T_2(v) + T_2(u) \cdot NT_2(v) \\ = \ & T_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)) + NT_2(\rho(T_2(u))(v) + \mu(T_2(v))(u)) \\ = \ & T_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)) + N(T_2(u) \cdot T_2(v)). \end{aligned}$$

Hence, equation (42) holds by acting N on both sides of the last equality. □

Using an $\mathcal{O}$-operator and a Nijenhuis operator, we can construct a pair of compatible $\mathcal{O}$-operators.

Proposition 7.10. *Let $T : M \longrightarrow L$ be an $\mathcal{O}$-operator on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$ and let N be a Nijenhuis operator on $(L, A, \cdot, \ell)$. Then $N \circ T$ is an $\mathcal{O}$-operator on the left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to $(M; \rho, \mu)$ if and only if for all $u, v \in M$, the following equation holds:*

$$\begin{aligned} & N\Big(NT(u) \cdot T(v) + T(u) \cdot NT(v)\Big) \\ =& N\Big(T\big(\rho(NT(u))(v) + \mu(NT(v))(u)\big) + NT\big(\rho(T(u))(v) + \mu(T(v))(u)\big)\Big). \end{aligned} \tag{43}$$

If in addition N is invertible, then T and NT are compatible. More explicitly, for any $\mathcal{O}$-operator T, if there exists an invertible Nijenhuis operator N such that NT is also an $\mathcal{O}$-operator, then T and NT are compatible.

Proof. Let $u, v \in M$ and $a \in A$, we have

$$NT(au) = N(T(au)) = N(aT(u)) = aNT(u).$$

In addition, since N is a Nijenhuis operator and T is an $\mathcal{O}$-operator we have

$$\begin{aligned} NT(u) \cdot NT(v) &= N\Big(NT(u) \cdot T(v) + T(u) \cdot NT(v)\Big) - N^2(T(u) \cdot T(v)) \\ &= NT\Big(\rho(NT(u))(v) + \mu(NT(v))(u)\Big) \end{aligned}$$

if and only if (43) holds.

If NT is an $\mathcal{O}$-operator and N is invertible, then we have

$$\begin{aligned} & T(u) \cdot T(v) + T(u) \cdot NT(v) \\ & = T\big(\rho(NT(u))(v) + \mu(NT(v))(u)\big) + NT\big(\rho(T(u))(v) + \mu(T(v))(u)\big), \end{aligned}$$

which is exactly the condition that NT and T are compatible. □

The following result is an immediate consequence of the last two propositions.

Corollary 7.11. *Let $T_1, T_2 : M \longrightarrow L$ be two $\mathcal{O}$-operators on a left-symmetric Rinheart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$. Suppose that T_1 and T_2 are invertible. Then T_1 and T_2 are compatible if and only if $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator.*

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References

[1] C. Bai. A further study on non-abelian phase spaces: left-symmetric algebraic approach and related geometry. *Rev. Math. Phys*, 18:545–564, 2006.

[2] C. Bai. A unified algebraic approach to the classical yang-baxter equation. *J. Phys. A: Math. Theor*, 40:11073–11082, 2007.

[3] C. Bai. Left-symmetric bialgebras and an analogue of the classical yang-baxter equation. *Commun. Contemp. Math*, 10:221–260, 2008.

[4] R. Bkouche. Structures (k, a)-linéaires. *C. R. Acad. Sci. Paris Sér*, A-B 262:A373–A376, 1966.

[5] D. Burde. Simple left-symmetric algebras with solvable Lie algebra. *Manuscripta Math*, 95(no. 3):397–411, 1998.

[6] D. Burde. Left-symmetric algebras, or pre-Lie algebras in geometry and physics. *Cent. Eur. J. Math*, 4:323–357, 2006.

[7] J. M. Casas. Obstructions to Lie-Rinehart algebra extensions. *Algebra Colloq*, 18(1):83–104, 2011.

[8] A. Cayley. On the theory of analytic forms called trees. *Cayley, A., ed. Collected Mathematical Papers of Arthur Cayley*, 3:242–246, 1890.

[9] F. Chapoton. M. livernet, pre-Lie algebras and the rooted trees operad. *Int. Math. Res. Not*, pages 395–408, 2001.

[10] S. Chemla. Operations for modules on Lie-Rinehart superalgebras. *Manuscripta Math*, 87(2):199–224, 1995.

[11] L. Chen, M. Liu, and J. Liu. Cohomologies and crossed modules for pre-Lie Rinehart algebras. *J. of Geom. and Phy*, 176:104501, 2022.

[12] Z. Chen, Z. Liu, and D. Zhong. Lie-Rinehart bialgebras for crossed products. *J. Pure Appl. Algebra*, 215(6):1270–1283, 2011.

[13] I. Dorfman. Dirac structures and integrability of nonlinear evolution equation. *Wiley,*, 1993.

[14] M. Gerstenhaber. The cohomology structure of an associative ring. *Ann. Math*, 78:267–288, 1963.

[15] J. Herz. Pseudo-algèbres de Lie. *C. R. Acad. Sci. Paris*, 236:1935–1937, 1953.

[16] J. Huebschmann. Poisson cohomology and quantization. *J. Reine Angew. Math*, 408:57–113, 1990.

[17] J. Huebschmann. Duality for Lie-Rinehart algebras and the modular class. *J. Reine Angew. Math*, 510:103–159, 1999.

[18] J. Huebschmann. Lie-Rinehart algebras, descent, and quantization, in: Galois theory, hopf algebras, and semiabelian categories. *Fields Inst. Commun*, 43:295–316, 2004.

[19] J. L. Koszul. Domaines bornés homogènes et orbites de groupes de transformations affines. *Bull. Soc. Math. France*, 89:515—533, 1961.

[20] U. Krahmer and A. Rovi. A Lie-Rinehart algebra with no antipode. *Comm. Algebra*, 43(10):4049–4053, 2015.

[21] B. A. Kupershmidt. Non-abelian phase spaces. *J. Phys. A*, 27:2801–2809, 1994.

[22] B. A. Kupershmidt. What a classical r-matrix really is. *J. Nonlinear Math. Phys*, 6(4):448–488, 1999.

[23] J. Liu, Y. Sheng, and C. Bai. Left-symmetric bialgebroids and their corresponding manin triples. *Diff. Geom. Appl*, 59:91–111, 2018.

[24] J. Liu, Y. Sheng, and C. Bai. Pre-symplectic algebroids and their applications. *Lett. Math. Phys*, 108(3):779–804, 2018.

[25] J. Liu, Y. Sheng, C. Bai, and Z. Chen. Left-symmetric algebroids. *Math. Nach*, 289(14–15):1893–1908, 2016.

[26] K. Mackenzie. Lie groupoids and Lie algebroids in differential geometry. *London Mathematical Society Lecture Note Series*, 124, 1987.

[27] Y. Matsushima. Affine structures on complex mainfolds. *Osaka J. Math*, 5:215–222, 1968.

[28] R. Palais. The cohomology of Lie rings. *Amer. Math. Soc., Providence, R. I., Proc. Symp. Pure Math*, pages 130–137, 1961.

[29] G. Rinehart. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc*, 108:195–222, 1963.

[30] E. B. Vinberg. Convex homogeneous cones. *Transl. Moscow Math. Soc*, 12:340–403, 1963.

[31] Q. Wang, C. Bai, J. Liu, and Y. Sheng. Nijenhuis operators on pre-Lie algebras. *Commun. Contemp. Math*, 21(7):1850050, 2019.

[32] X. Xu. On simple novikov algebras and their irreducible modules. *J. Algebra*, 185(3):905–934, 1996.

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