Cohomology and deformations of left-symmetric Rinehart algebras

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Abstract. We introduce a notion of left-symmetric Rinehart algebras, which is a generalization of the notion of left-symmetric algebras. The left multiplication gives rise to a representation of the corresponding sub-adjacent Lie-Rinehart algebra. We construct left-symmetric Rinehart algebras from $\mathcal{O}$-operators on Lie-Rinehart algebras. We extensively investigate representations of left-symmetric Rinehart algebras. Moreover, we construct a graded Lie algebra on the space of multi-derivations whose Maurer–Cartan elements characterize left-symmetric Rinehart algebras and study deformations of left-symmetric Rinehart algebras, which are controlled by the second cohomology class in the deformation cohomology. We also give the relationships between $\mathcal{O}$-operators and Nijenhuis operators on left-symmetric Rinehart algebras.

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1 Introduction

Left-symmetric algebras are algebras for which the associator

\[(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)\]

satisfies the identity \((x, y, z) = (y, x, z)\). These algebras appeared as early as 1896 in the work of Cayley [8] as rooted tree algebras. In the 1960s, they also arose from the study of several topics in geometry and algebra, such as convex homogenous cones [30], affine manifolds and affine structures on Lie groups [19,27] and deformations of associative algebras [14]. In 2006, Burde [6] wrote an interesting survey showing the importance of left-symmetric algebras in many areas, such as vector fields, rooted tree algebras, vertex algebras, operad theory, deformation complexes of algebras, convex homogeneous cones, affine manifolds and left-invariant affine structures on Lie groups [6].

Left symmetric algebras are the underlying algebraic structures of non-abelian phase spaces of Lie algebras [1,21], leading to a bialgebra theory of left-symmetric algebras [3]. They can also be seen as the algebraic structures behind the classical Yang-Baxter equations. Precisely, they provide a construction of solutions of the classical Yang-Baxter equations in certain semidirect product Lie algebra structures (that is, over the double spaces) induced by left-symmetric algebras [2,22].

The notion of Lie-Rinehart algebras was introduced by J. Herz in [15] and further developed in [28,29]. A notion of (Poincaré) duality for this class of algebras was introduced in [16,17]. Lie-Rinehart structures have been the subject of extensive studies, in relations to symplectic geometry, Poisson structures, Lie groupoids and algebroids and other kinds of quantizations (see [18,20,23,24,25,26]). For further details and a history of the notion of Lie-Rinehart algebra, we refer the reader to [18]. Lie-Rinehart algebras have been investigated furthermore in [4,7,11,12].

A left-symmetric algebroid is a geometric generalization of a left-symmetric algebra. See [23,24,25] for more details and applications. The notion of a Nijenhuis operator on a
left-symmetric algebroid was introduced in [24], which could generate a trivial deformation. More details on deformations of left-symmetric algebras can be found in [31].

In this paper, we introduce a notion of left-symmetric Rinehart algebras, which is a generalization of a left-symmetric algebra and an algebraic version of left symmetric algebroids. The following diagram shows how left-symmetric Rinehart algebras fit in relation to Lie algebras, left-symmetric algebras and Lie-Rinehart algebras.

\[
\begin{array}{ccc}
\text{Lie algebra} & \xrightarrow{\text{generalization}} & \text{Lie-Rinehart algebra} \\
\downarrow & & \downarrow \\
\text{Left-symmetric algebra} & \xrightarrow{\text{generalization}} & \text{Left-symmetric Rinehart algebra}
\end{array}
\]

The paper is organized as follows. In Section 2, we recall some definitions concerning left-symmetric algebras and Lie-Rinehart algebra. In Section 3, we introduce the notion of left-symmetric Rinehart algebra and give some of its properties. As in the case of a left-symmetric algebras, one can obtain the sub-adjacent Lie-Rinehart algebra from a left-symmetric Rinehart algebra by using the commutator. The left multiplication gives rise to a representation of the sub-adjacent Lie-Rinehart algebra. We construct left-symmetric Rinehart algebras using $O$-operators. Section 4 is devoted to the study of representations and cohomology of left-symmetric Rinehart algebra. In Section 5, we construct a graded Lie algebra whose Maurer-Cartan elements are left-symmetric Rinehart algebras which give rise to a coboundary operator. Section 6 is devoted to introduce the deformation cohomology associated to a left-symmetric Rinehart algebra, which controls the deformations. In Section 7, we introduce the notion of a Nijenhuis operator, which could generate a trivial deformation. In addition, we investigate some connection between $O$-operators and Nijenhuis operators.

Throughout this paper all vector spaces are over a field $\mathbb{K}$ of characteristic zero.

## 2 Preliminaries

In this section, we briefly recall some basics of left-symmetric algebras and Lie-Rinehart algebras [6].

**Definition 2.1.** A left-symmetric algebra is a vector space $L$ endowed with a linear map $\cdot : L \otimes L \rightarrow L$ such that for any $x, y, z \in L$,

\[(x, y, z) = (y, x, z), \quad \text{or equivalently,} \quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z),\]

where the associator $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$.

Let $\text{ad}^L$ (resp. $\text{ad}^R$) be the left multiplication operator (resp. right multiplication operator) on $L$ that is, i.e. $\text{ad}^L(x)y = x \cdot y$ (resp. $\text{ad}^R(x)y = y \cdot x$), for any $x, y \in L$. The following lemma is given in [6].
Lemma 2.2. Let \((L, \cdot)\) be a left-symmetric algebra. The commutator \([x, y] = x \cdot y - y \cdot x\) defines a Lie algebra \(L\), which is called the sub-adjacent Lie algebra of \(L\). The algebra \(L\) is also called a compatible left-symmetric algebra on the Lie algebra \(\bar{L}\). Furthermore, the map \(ad^L : L \to \mathfrak{gl}(L)\) with \(x \mapsto L_x\) gives a representation of the Lie algebra \((\bar{L}, [\cdot, \cdot])\).

Definition 2.3. Let \((L, \cdot)\) be a left-symmetric algebra and \(M\) a vector space. A representation of \(L\) on \(M\) consists of a pair \((\rho, \mu)\), where \(\rho : L \to \mathfrak{gl}(M)\) is a representation of the sub-adjacent Lie algebra \(L\) on \(M\) and \(\mu : L \to \mathfrak{gl}(M)\) is a linear map satisfying:

\[
\rho(x) \circ \mu(y) - \mu(y) \circ \rho(x) = \mu(x \cdot y) - \mu(y) \circ \mu(x), \quad \forall x, y \in L.
\] (1)

The map \(\rho\) is called a left representation and \(\mu\) is a right representation. Usually, we denote a representation by \((M; \rho, \mu)\). Then \((L; ad^L, ad^R)\) is a representation of \((L, \cdot)\) which is called adjoint representation.

The cohomology complex for a left-symmetric algebra \((L, \cdot)\) with a representation \((M; \rho, \mu)\) is given as follows. The set of \((n+1)\)-cochains is given by

\[
C^{n+1}(L, M) = \text{Hom}(\wedge^n L \otimes L, M), \quad \forall n \geq 0.
\] (2)

For all \(\omega \in C^n(L, M)\), the coboundary operator \(\delta : C^n(L, M) \to C^{n+1}(L, M)\) is given by

\[
\delta \omega(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} \rho(x_i) \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})
+ \sum_{i=1}^{n} (-1)^{i+1} \mu(x_{n+1}) \omega(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_i)
- \sum_{i=1}^{n} (-1)^{i+1} \omega(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_i \cdot x_{n+1})
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).
\]

We then have the following lemma whose proof comes from a direct computation using identity (1).

Lemma 2.4 (See \([5]\)). The map \(\delta\) satisfies \(\delta^2 = 0\).

Definition 2.5. A Lie-Rinehart algebra \(L\) over an associative commutative algebra \(A\) is a Lie algebra over \(\mathbb{K}\) with an \(A\)-module structure and a linear map \(\rho : L \to \text{Der}(A)\), such that the following conditions hold:

1. For all \(a \in A\) and \(x, y \in L\)

\[
\rho([x, y]) = \rho(x) \rho(y) - \rho(x) \rho(y), \quad \text{and} \quad \rho(ax) = a \rho(x).
\]
2. The compatibility condition:

\[ [x, ay] = \rho(x)ay + a[x, y], \quad \forall a \in A, x, y \in L. \tag{3} \]

Let \((L, A, [\cdot, \cdot], \rho)\) and \((L', A', [\cdot, \cdot]_{L'}, \rho')\) be two Lie-Rinehart algebras, then a Lie-Rinehart algebra homomorphism is defined as a pair of maps \((g, f)\), where the maps \(f : L \rightarrow L'\) and \(g : A \rightarrow A'\) are two algebra homomorphisms such that:

1. \(f(ax) = g(a)f(x)\) for all \(x \in L\) and \(a \in A\),
2. \(g(\rho(x)a) = \rho'(f(x))g(a)\) for all \(x \in L\) and \(a \in A\).

Now, we recall the definition of module over a Lie-Rinehart algebra (for more details see [10]).

**Definition 2.6.** Let \(M\) be an \(A\)-module. Then \(M\) is a module over a Lie-Rinehart algebra \((L, A, [\cdot, \cdot], \rho)\) if there exists a map \(\theta : L \otimes M \rightarrow M\) such that:

1. \(\theta\) is a representation of the Lie algebra \((L, [\cdot, \cdot])\) on \(M\).
2. \(\theta(ax, m) = a\theta(x, m)\) for all \(a \in A, x \in L, m \in M\).
3. \(\theta(x, am) = a\theta(x, m) + \rho(x)am\) for all \(x \in L, a \in A, m \in M\).

We have the following lemma giving a characterization of of the \(\theta\) which are representations.

**Lemma 2.7.** The map \(\theta\) is representation if and only if \(L \oplus M\) is Lie-Rinehart algebra over \(A\), where \([\cdot, \cdot]_{L \oplus M}\) and \(\theta_{L \oplus M}\) are given by:

\[
[x_1 + m_1, x_2 + m_2]_{L \oplus M} = [x_1, x_2] + \rho(x_1)m_2 - \rho(x_2)m_1,
\]

\[
\theta_{L \oplus M}(x_1 + m_1) = \theta(x_1)
\]

for all \(x_1, x_2 \in L\) and \(m_1, m_2 \in M\).

### 3 Some basic properties of a left-symmetric Rinehart algebras

In this section, we introduce a notion of left-symmetric Rinehart algebras illustrated by some examples. As in the case of a left-symmetric algebra, we obtain the sub-adjacent Lie-Rinehart algebra from a left-symmetric Rinehart algebra using the commutator. In addition, we construct left-symmetric Rinehart algebras using \(O\)-operators.

**Definition 3.1.** A left-symmetric Rinehart algebra is a quadruple \((L, A, \cdot, \ell)\) where \((L, \cdot)\) is a left-symmetric algebra, \(A\) is an associative commutative algebra and \(\ell : L \rightarrow \text{Der}(A)\) a linear map such that the following conditions hold:
By direct calculation, we have \( \ell(x \cdot y - y \cdot x) = \ell(x)\ell(y) - \ell(x)\ell(y) \) and \( \ell(ax) = a\ell(x) \).

3. The compatibility conditions: for all \( a \in A \) and \( x, y \in L \)
   \[
   x \cdot (ay) = \ell(x)ay + a(x \cdot y), \tag{4}
   \]
   \[
   (ax) \cdot y = a(x \cdot y). \tag{5}
   \]

**Example 3.2.** It is clear that any left-symmetric algebra is a left-symmetric Rinehart algebra.

**Example 3.3.** A Novikov Poisson algebra is a left-symmetric Rinehart algebra (see [32]).

**Example 3.4.** Let \( (L, A, \cdot, \ell) \) be a left-symmetric Rinehart algebra and let \( L \oplus A \) be the direct sum of \( L \) and \( A \). Then \( (L \oplus A, A, \cdot_{L \oplus A}, \ell_{L \oplus A}) \) is a left-symmetric Rinehart algebra, where the \( \cdot_{L \oplus A} \) is defined by the following expression, for all \( x, y \in L, a, b \in A \);

\[
(x_1 + a_1) \cdot_{L \oplus A} (x_2 + a_2) = x_1 \cdot x_2 + \ell(x_1)(a_2);
\]

and \( \ell_{L \oplus A} : L \oplus A \to \text{Der}(A) \) is defined by \( \ell_{L \oplus A}(a_1 + x_1) = \ell(x_1) \). Indeed, it obvious that \( (L \oplus A, \cdot_{L \oplus A}) \) is a left-symmetric algebra, \( \ell_{L \oplus A} \) is a representation of left-symmetric algebra \( L \oplus A \) and \( \ell_{L \oplus A} \in \text{Der}(A) \).

By direct calculation, we have \( \ell_{L \oplus A}(b(x_1 + a_1)) = b\ell_{L \oplus A}(x_1 + a_1) \) for all \( b, a_1 \in A \) and \( x_1 \in L \). On the other hand, letting \( x_1, x_2 \in L \) and \( b, a_1, a_2 \in A \), we have

\[
(x_1 + a_1) \cdot_{L \oplus A} b(x_2 + a_2) = (a_1 + x_1) \cdot_{L \oplus A} (bx_2 + ba_2) = x_1 \cdot (bx_2) + \ell(x_1)(ba_2) = b(x_1 \cdot x_2) + \ell(x_1)(bx_2) + \ell(x_1)(b_2) + b\ell(x_1)(a_2) = b(x_1 \cdot x_2 + \ell(x_1)(a_2)) + \ell(x_1)(bx_2) + \ell(x_1)(ba_2) = b((x_1 + a_1) \cdot_{L \oplus A} (x_2 + a_2)) + \ell_{L \oplus A}(x_1 + a_1)b(x_2 + a_2).
\]

Moreover,

\[
b(x_1 + a_1) \cdot_{L \oplus A} (x_2 + a_2) = (bx_1 + ba_1) \cdot_{L \oplus A} (x_2 + a_2) = (bx_1) \cdot x_2 + \ell(bx_1)(a_2) = b(x_1 \cdot x_2) + b\ell(x_1)(a_2) = b(x_1 \cdot x_2 + \ell(x_1)a_2) = b((x_1 + a_1) \cdot_{L \oplus A} (x_2 + a_2)).
\]

Now we have the following theorem.
Theorem 3.5. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra. Then, \((L, A, [\cdot, \cdot], \ell)\) is a Lie-Rinehart algebra, denoted by \(L^\ell\), called the sub-adjacent Lie-Rinehart algebra of \((L, A, \cdot, \ell)\).

Proof. Since \((L, \cdot)\) is a left-symmetric algebra, we have that \((L, [\cdot, \cdot])\) is a Lie algebra. For any \(a \in A\), by direct computations, we have

\[
[x, ay] = x \cdot (ay) - (ay) \cdot x = a(x \cdot y) + \ell(x)ay - a(y \cdot x)
\]

which implies that \((L, A, [\cdot, \cdot], \ell)\) is a Lie-Rinehart algebra.

To see that the linear map \(\ell : L \rightarrow Der(A)\) is a representation, we only need to show that \(\ell_{[x,y]} = [\ell_x, \ell_y]_{Der(A)}\), which follows directly from the fact that \((L, \cdot)\) is a left-symmetric algebra. This ends the proof. \(\square\)

Definition 3.6. Let \((L_1, A_1, \cdot_1, \ell_1)\) and \((L_2, A_2, \cdot_2, \ell_2)\) be two left-symmetric Rinehart algebras. A homomorphism of left-symmetric Rinehart algebras is a pair of two algebra homomorphisms \((f, g)\) where \(f : L_1 \rightarrow L_2\) and \(g : A_1 \rightarrow A_2\) such that:

\[
f(ax) = g(a)f(x), \quad g(\ell_1(x)a) = \ell_2(f(x))g(a), \quad \forall x, y \in L_1, a \in A_1.
\]

The following proposition is immediate.

Proposition 3.7. Let \((f, g)\) be a homomorphism of left-symmetric Rinehart algebras from \((L_1, A_1, \cdot_1, \ell_1)\) to \((L_2, A_2, \cdot_2, \ell_2)\). Then \((f, g)\) is also a Lie-Rinehart algebra homomorphism of the corresponding sub-adjacent Lie-Rinehart algebras.

Now we give the definition of an \(O\)-operator.

Definition 3.8. Let \((L, A, [\cdot, \cdot], \rho)\) be a Lie-Rinehart algebra and \(\theta : L \rightarrow End(M)\) be a representation over \(M\). A linear map \(T : M \rightarrow L\) is called an \(O\)-operator if for all \(u, v \in M\) and \(a \in A\) we have

\[
T(au) = aT(u),
\]

\[
[T(u), T(v)] = T(\theta(T(u))(v) - \theta(T(v))(u)).
\]

Remark 3.9. Consider the semidirect product Lie-Rinehart algebra

\[
(L \ltimes_\theta M, [\cdot, \cdot]_{L \ltimes_\theta M}, \rho_{L \ltimes_\theta M}),
\]

where \(\rho_{L \ltimes_\theta M}(x + u) := \rho(x)(u)\) and the bracket \([\cdot, \cdot]_{L \ltimes_\theta M}\) is given by

\[
[x + u, y + v]_{L \ltimes_\theta M} = [x, y] + \theta(x)(v) - \theta(y)(u).
\]

Any \(O\)-operator \(T : M \rightarrow L\) gives a Nijenhuis operator \(\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\) on the Lie-Rinehart algebra \(L \ltimes_\theta M\). More precisely, we have

\[
[\tilde{T}(x+u), \tilde{T}(y+v)]_{L \ltimes_\theta M} = \tilde{T}\left([\tilde{T}(x+u), y+v]_{L \ltimes_\theta M} + [x+u, \tilde{T}(y+v)]_{L \ltimes_\theta M} - \tilde{T}(x+u, y+v)_{L \ltimes_\theta M}\right).
\]
Fore more details on Nijenhuis operators and their applications the reader should consult [13].

Let $T : M \rightarrow L$ be an $\mathcal{O}$-operator. Define the multiplication $\cdot_T$ on $M$ by

$$u \cdot_T v = \theta(T(u))(v), \ \forall u, v \in M.$$  

We then have the following proposition.

**Proposition 3.10.** With the above notations, $(M, A, \cdot_T, \ell_T = \ell \circ T)$ is a left-symmetric Rinehart algebra, and the map $T$ is Lie-Rinehart algebra homomorphism from $(M, [\cdot, \cdot])$ to $(L, [\cdot, \cdot])$.

**Proof.** It is easy to see that $(M, \cdot_T)$ is a left-symmetric algebra. For any $a \in A$, using Definition 3.1 and equation (6) we have

$$\ell_M (au) = \ell(T(au)) = a\ell(T(u)) = a\ell_M(u),$$

Similarly, using Definition 2.6 we obtain

$$ (au) \cdot_T v = \theta(T(au))(v) = \theta a T(u)(v) = a\theta(T(u))(v),$$

$$ u \cdot_T (av) = \theta(T(u))(av) = a\theta(T(u))(v) + \ell \circ T(u)(a)v. $$

Thus, $(M, A, \cdot_T, \ell_M)$ is a left-symmetric Rinehart algebra. Let $[\cdot, \cdot]$ be the sub-adjacent Lie bracket on $M$. Then we have

$$T[u, v] = T(u \cdot_T v - v \cdot_T u) = T(\theta(T(u))(v) - \theta(T(v))(u)) = [T(u), T(v)].$$

So $T$ is a homomorphism of Lie algebras. \qed

### 4 Representations of left-symmetric Rinehart algebras

In this section, we develop the notion of representations of a left-symmetric Rinehart algebra and give a cohomology theory with coefficients in a representation.

**Definition 4.1.** Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $M$ be an $A$-module. A representation of $A$ on $M$ consists of a pair $(\rho, \mu)$, where $\rho$ is a representation of the sub-adjacent Lie-Rinehart algebra $(L, A, [\cdot, \cdot]^C, \ell)$ and $\mu : L \rightarrow End(M)$ is a linear map, such that for all $x, y \in L$ and $m \in M$, we have

$$\mu(ax)m = a\mu(x)m = \mu(x)(am)$$

$$\rho(x)\mu(y) - \mu(x)\rho(y) = \mu(x \cdot y) - \mu(y)\mu(x).$$

(8)

We will denote this representation by $(M; \rho, \mu)$. 
For a left-symmetric Rinehart algebra \((L, A, \cdot, \ell)\) and a representation \((M; \rho, \mu)\), the following proposition gives a construction of a left-symmetric Rinehart algebra called semidirect product and denoted by \(L \ltimes_{\rho, \mu} M\).

**Proposition 4.2.** Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \((M; \rho, \mu)\) a representation. Then, \((L \oplus M, A, \cdot_{L \oplus M}, \ell_{L \oplus M})\) is a left-symmetric Rinehart algebra, where \(\cdot_{L \oplus M}\) and \(\ell_{L \oplus M}\) are given by

\[
(x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2) = x_1 \cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1, \\
\ell_{L \oplus M}(x_1 + m_1) = \ell(x_1),
\]

for all \(x_1, x_2 \in L\) and \(m_1, m_2 \in M\).

**Proof.** Let \((M; \rho, \mu)\) be a representation. It is straightforward to see that \((L \oplus M, A, \cdot_{L \oplus M})\) is a left-symmetric algebra. For any \(x_1, x_2 \in L\) and \(m_1, m_2 \in M\), we have

\[
(x_1 + m_1) \cdot_{L \oplus M} (a(x_2 + m_2)) = x_1 \cdot (ax_2) + \rho(x_1)am_2 + \mu(ax_2)m_1 \\
= a(x_1 \cdot x_2) + \ell(x_1)(ax_2) + \rho(x_1)m_2 + \mu(x_2)m_1 \\
= a((x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2)) + \ell_{L \oplus M}(x_1)(a)(x_2 + m_2).
\]

On the other hand, we have

\[
(a(x_1 + m_1)) \cdot_{L \oplus M} (x_2 + m_2) = (ax_1) \cdot x_2 + \rho(ax_1)m_2 + \mu(x_2)(am_1) \\
= a((x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2)).
\]

Therefore, \((L \oplus M, A, \cdot_{L \oplus M}, \ell_{L \oplus M})\) is a left-symmetric Rinehart algebra. \(\Box\)

Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \((M; \rho, \mu)\) a representation. Let \(\rho^* : L \otimes M^* \rightarrow M^*\) and \(\mu^* : M^* \otimes L \rightarrow M^*\) be defined by

\[
\langle \rho^*(x)\xi, m \rangle = \ell(x)(\xi, m) - \langle \rho(x)m, \xi \rangle \quad \text{and} \quad \langle \mu^*(x)\xi, m \rangle = -\langle \xi, \mu(x)m \rangle,
\]

where \(M^* = Hom_A(M, A)\). Then, we have the following proposition.

**Proposition 4.3.** With the above notations, we obtain that

(i) \((M, \rho - \mu)\) is a representation of the sub-adjacent Lie-Rinehart algebra \((L, A, [\cdot, \cdot], \ell)\).

(ii) \((M^*, \rho^* - \mu^*, -\mu^*)\) is a representation be a left-symmetric Rinehart algebra \((L, A, \cdot, \ell)\).

**Proof.** Since \((M; \rho, \mu)\) is a representation of the left-symmetric algebra \((L, A, \cdot, \ell)\), using Proposition 4.2 we have that \((L \oplus M, A, \cdot_{L \oplus M}, \ell_{L \oplus M})\) is a left-symmetric Rinehart algebra. Consider its sub-adjacent Lie-Rinehart algebra \((L \oplus M, A, \cdot_{L \oplus M}, [\cdot, \cdot]_{L \oplus M}, \ell_{L \oplus M})\). We have

\[
[(x_1 + m_1), (x_2 + m_2)]_{L \oplus M} = (x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2) - (x_2 + m_2) \cdot_{L \oplus M} (x_1 + m_1) \\
= x_1 \cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1 \\
- x_2 \cdot x_1 - \rho(x_2)m_1 - \mu(x_1)m_2 \\
= [x_1, x_2]^C + (\rho - \mu)(x_2)(m_2) + (\rho - \mu)(x_1)(m_1).
\]
From Lemma 2.7 we deduce that \((M, \rho - \mu)\) is a representation of Lie-Rinehart algebra \(L\) on \(M\). This finishes the proof of (i).

For item (ii), it is clear that \(\rho^* - \mu^*\) is just the dual representation of the representation \(\theta = \rho - \mu\) of the sub-adjacent Lie-Rinehart algebra of \(L\). We can directly check that 
\[-\mu^*(ax)\xi = -a\mu^*(x)\xi = -\mu^*(x)(a\xi).\]
For any \(x, y \in L, \xi \in M^*\) and \(m \in M\) we have
\[-((\rho^* - \mu^*)(x)\mu^*(y)\xi, m) + \langle \mu^*(y)(\rho^* - \mu^*)(x)\xi, m \rangle
\]
\[= -\langle \rho^*(x)\mu^*(y)\xi, m \rangle + \langle \mu^*(y)\rho^*(x)\xi, m \rangle + \langle \mu^*(y)\mu^*(x)\xi, m \rangle - \langle \mu^*(y)\mu^*(x)\xi, m \rangle
\]
\[= \ell(x)\langle \xi, \mu(y)m \rangle - \langle \xi, \mu(x)\rho(y)m \rangle + \langle \xi, \rho(y)\mu(x)m \rangle - \langle \xi, \mu(y)\mu(x)m \rangle
\]
\[= \langle \xi, \mu(x\cdot y)m \rangle - \langle \xi, \mu(y)\mu(x)m \rangle + \langle \xi, \mu(y)\mu(x)m \rangle - \langle \xi, \mu(x)\mu(y)m \rangle
\]
\[= \langle (-\mu^*(x\cdot y) - \mu^*(y)\mu^*(x))\xi, m \rangle.\]

Therefore \((M^*, \rho^* - \mu^*, -\mu^*)\) is a representation of \(L\).

**Corollary 4.4.** With the above notations, we have

(i) The left-symmetric Rinehart algebras \(L \ltimes_{\rho, \mu} M\) and \(L \ltimes_{\rho - \mu, 0} M\) have the same sub-adjacent Lie-Rinehart algebra \(L \ltimes_{\rho - \mu} M\).

(ii) The left-symmetric Rinehart algebras \(L \ltimes_{\rho^*, 0} M^*\) and \(L \ltimes_{\rho^* - \mu^*, -\mu^*} M^*\) have the same sub-adjacent Lie-Rinehart algebra \(L \ltimes_{\rho^*} M^*\).

Let \((M; \rho, \mu)\) be a representation of a left-symmetric Rinehart algebra \((L, A, \cdot, \ell)\). In general, \((M^*, \rho^*, \mu^*)\) is not a representation. But we have the following proposition.

**Proposition 4.5.** Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \((M; \rho, \mu)\) be a representation. Then the following conditions are equivalent:

1. \((M; \rho - \mu, -\mu)\) is a representation of \((L, A, \cdot, \ell)\).
2. \((M^*, \rho^*, \mu^*)\) is a representation of \((L, A, \cdot, \ell)\).
3. \(\mu(x)\mu(y) = \mu(y)\mu(x)\) for all \(x, y \in L\).

5 The Matsushima-Nijenhuis bracket for left-symmetric Rinehart algebras

In this section we construct a graded Lie algebra whose Maurer-Cartan elements are left-symmetric Rinehart algebras which give rise to a coboundary operator.

Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebras. A multiderivation of degree \(n\) is a multilinear map \(P \in \text{Hom}(\Lambda^n L \otimes L, L)\) such that, for every \(a \in A, x_i \in L\) and \(i \in \{1, 2, \ldots, n + 1\}\), we have

\[P(x_1, \ldots, ax_i, \ldots, x_n, x_{n+1}) = aP(x_1, \ldots, x_i, \ldots, x_n, x_{n+1}),\]
\[P(x_1, \ldots, x_n, ax_{n+1}) = aP(x_1, \ldots, x_n, x_{n+1}) + \Xi_P(x_1, \ldots, x_n)(a)x_{n+1},\]
where $\Xi : L^{*} \to \text{Der}(A)$ is called the symbol map. The space of all multiderivations of degree $n$ will be denoted by $\mathfrak{D}^{n}(L)$. Set $\mathfrak{D}^{n}(L) = \bigoplus_{n \geq -1} \mathfrak{D}^{n}(L)$ with $\mathfrak{D}^{-1}(L) = L$, the space of multiderivations on $L$.

A permutation $\sigma \in S_{n}$ is called an $(i, n - i)$-unshuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i + 1) < \cdots < \sigma(n)$. If $i = 0$ and $i = n$, we assume $\sigma = \text{Id}$. The set of all $(i, n - i)$-unshuffles will be denoted by $S_{(i, n - i)}$. The notion of an $(i_{1}, \cdots, i_{k})$-unshuffle and the set $S_{(i_{1}, \cdots, i_{k})}$ are defined similarly.

Let $P \in \mathfrak{D}^{m}(L)$ and $Q \in \mathfrak{D}^{n}(L)$. We define the Matsushima–Nijenhuis bracket $[\cdot, \cdot]_{\text{MN}} : \mathfrak{D}^{m}(L) \times \mathfrak{D}^{n}(L) \to \mathfrak{D}^{m+n}(L)$ by

$$[P, Q]_{\text{MN}} = P \circ Q - (-1)^{mn}Q \circ P,$$

where

$$P \circ Q(x_{1}, x_{2}, \cdots, x_{m+n+1})$$

$$= \sum_{\sigma \in S_{(m, 1, n-1)}} (-1)^{\sigma} P(Q(x_{\sigma(1)}, \cdots, x_{\sigma(m+1)}), x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})$$

$$+ (-1)^{mn} \sum_{\sigma \in S_{(n, m)}} (-1)^{\sigma} P(x_{\sigma(1)}, \cdots, x_{\sigma(n)}, Q(x_{\sigma(n+1)}, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})).$$

**Theorem 5.1.** With the above notations, we have

(i) The pair $(\mathfrak{D}^{*}(L), [\cdot, \cdot]_{\text{MN}})$ is a graded Lie algebra.

(ii) There is a one-to-one correspondence between the set of Maurer-Cartan elements of the graded Lie algebra $(\mathfrak{D}^{*}(L), [\cdot, \cdot]_{\text{MN}})$ and left-symmetric Rinehart algebra structures on $L$.

**Proof.** (i) We begin by check that the Matsushima-Nijenhuis bracket is well defined. For $P \in \mathfrak{D}^{m}(L)$ and $Q \in \mathfrak{D}^{n}(L)$, by a direct calculation, we have

$$[P, Q]_{\text{MN}}(ax_{1}, x_{2}, \cdots, x_{m+n+1})$$

$$= aP \circ Q(x_{1}, x_{2}, \cdots, x_{m+n+1}) - (-1)^{mn}aQ \circ P(x_{1}, x_{2}, \cdots, x_{m+n+1})$$

$$+ \sum_{\sigma \in S_{(m-1, 1, n-1)}} (-1)^{\sigma}\Xi P(x_{\sigma(2)}, \cdots, x_{\sigma(m+1)})(a)P(x_{1}, x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})$$

$$+ (-1)^{mn} \sum_{\sigma \in S_{(n-1, 1, m-1)}} (-1)^{\sigma}\Xi P(x_{\sigma(2)}, \cdots, x_{\sigma(n+1)})(a)Q(x_{1}, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})$$

$$- (-1)^{mn} \sum_{\sigma \in S_{(n-1, 1, m-1)}} (-1)^{\sigma}\Xi P(x_{\sigma(2)}, \cdots, x_{\sigma(n+1)})(a)Q(x_{1}, x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})$$

$$- \sum_{\sigma \in S_{(m-1, 1, n-1)}} (-1)^{\sigma}\Xi Q(x_{\sigma(2)}, \cdots, x_{\sigma(m+1)})(a)P(x_{1}, x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})$$

$$= a[P, Q]_{\text{MN}}(x_{1}, x_{2}, \cdots, x_{m+n+1}),$$
which implies that
\[
[P, Q]_{MN}(ax_1, x_2, \cdots, x_{m+n+1}) = a[P, Q]_{MN}(x_1, x_2, \cdots, x_{m+n+1}).
\]
It is straightforward to check that \([P, Q]_{MN}\) is skew-symmetric with respect to its first
\(m + n\) arguments. Thus \([P, Q]_{MN}\) is \(A\)-linear with respect to its first \(m + n\) arguments.

On the other hand, following a straightforward calculation, we have
\[
[P, Q]_{MN}(x_1, x_2, \cdots, ax_{m+n+1}) = a[P, Q]_{MN}(x_1, x_2, \cdots, x_{m+n+1}) + \Xi_{[P, Q]_{MN}}(x_1, x_2, \cdots, x_{m+n})(a)x_{m+n+1},
\]
where the symbol map \(\Xi_{[P, Q]_{MN}}\) is given by
\[
\Xi_{[P, Q]_{MN}}(x_1, x_2, \cdots, x_{m+n})(a) = \sum_{\sigma \in S_{(m,1,n-1)}} (-1)^{\sigma} \Xi_P(Q(x_{\sigma(1)}, \cdots, x_{\sigma(m+1)}), x_{\sigma(m+2)}, \cdots, x_{\sigma(m+n)}))(a)
\]
\[+ \sum_{\sigma \in S_{(n,1,m-1)}} (-1)^{\sigma} \Xi_Q(P(x_{\sigma(1)}, \cdots, x_{\sigma(n+1)}), x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}))(a)
\]
\[+ (-1)^{mn} \sum_{\sigma \in S_{(m,n)}} (-1)^{\sigma} \Xi_P(x_{\sigma(1)}, \cdots, x_{\sigma(m)})(\Xi_Q(x_{\sigma(m+1)}, \cdots, x_{\sigma(n+m)}))(a)
\]
\[+ \sum_{\sigma \in S_{(m,n)}} (-1)^{\sigma} \Xi_Q(x_{\sigma(1)}, \cdots, x_{\sigma(m)})(\Xi_P(x_{\sigma(m+1)}, \cdots, x_{\sigma(m+n)}))(a).
\]
Thus \([P, Q]_{MN} \in \mathcal{D}^{m+n}(L)\).

It was shown in [9] that the Matsushima-Nijenhuis bracket provides a graded Lie algebra structure on the graded vector space \(\oplus_{n \geq 1} \text{Hom}(\Lambda^{n-1}L \otimes L, L)\). Therefore, \((\mathcal{D}^{*}(L), [\cdot, \cdot]_{MN})\) is a graded Lie algebra.

(ii) Let \(\pi \in \mathcal{D}^{1}(L)\), we have
\[
\pi(ax_1, x_2) = a\pi(x_1, x_2), \quad \pi(x_1, ax_2) = a\pi(x_1, x_2) + \Xi_{\pi}(x_1)(a)x_2, \quad \forall \ x_1, x_2 \in L.
\]
In addition, we can easily check that
\[
[\pi, \pi]_{MN}(x_1, x_2, x_3) = 2(\pi(\pi(x_1, x_2), x_3) - \pi(\pi(x_2, x_1), x_3)
\]
\[= -\pi(x_1, \pi(x_2, x_3)) + \pi(x_2, (x_1, x_3))).
\]
Thus \((L, A, \pi, \Xi_{\pi})\) is a left-symmetric Rinehart algebra if and only if \([\pi, \pi]_{MN} = 0\). □

**Remark 5.2.** The cohomology of left-symmetric algebras first appeared in the unpublished paper of Y. Matsushima. Then A. Nijenhuis constructed a graded Lie bracket, which produces the cohomology theory for left-symmetric algebras. Thus the aforementioned graded Lie bracket is usually called the Matsushima–Nijenhuis bracket.
Let \((L, A, \pi, \ell)\) be a left-symmetric Rinehart algebra. According to Theorem 5.1, we have \([\pi, \pi]_{MN} = 0\). Using the graded Jacobi identity, we get a coboundary operator \(\delta : \mathfrak{D}^{n-1}(L) \to \mathfrak{D}^{n}(L)\), by putting
\[
\delta(P) = (-1)^{n-1}[\pi, P]_{MN}, \quad \forall P \in \mathfrak{D}^{n-1}(L).
\] (13)

By straightforward computation, we obtain

**Proposition 5.3.** For any \(P \in \mathfrak{D}^{n-1}(L)\), we have
\[
\delta P(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n}(-1)^{i+1} \pi(x_i, P(x_1, x_2, \cdots, \hat{x}_i, \cdots, x_{n+1})) + \sum_{i=1}^{n}(-1)^{i+1}P(x_1, x_2, \cdots, \hat{x}_i, \cdots, x_n, x_i, x_{n+1}) - \sum_{i=1}^{n}(-1)^{i+1}P(x_1, x_2, \cdots, \hat{x}_i, \cdots, x_n, \pi(x_i, x_{n+1}))
\]
\[
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} P(\pi(x_i, x_j) - \pi(x_j, x_i), x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1})
\] (14)

for all \(x_i \in L, i = 1, 2 \cdots, n + 1\) and \(\Xi_{\delta P}\) is given by
\[
\Xi_{\delta P}(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n}(-1)^{i+1}[\pi(x_i), \Xi_P(x_1, x_2, \cdots, \hat{x}_i, \cdots, x_n)] + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \Xi_P(\pi(x_i, x_j) - \pi(x_j, x_i), x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n)
\]
\[
+ \sum_{i=1}^{n}(-1)^{i+1}\Xi_{\pi}(P(x_1, x_2, \cdots, \hat{x}_i, \cdots, x_n, x_i)).
\] (15)

**Definition 5.4.** The cochain complex \((\mathfrak{D}^{\ast}(L) = \bigoplus_{n \geq 0} \mathfrak{D}^{n}(L), \delta)\) is called the deformation complex of the left-symmetric Rinehart algebra \(L\). The corresponding \(k\)-th cohomology group, which we denote by \(H^k(L)\), is called the \(k\)-th deformation cohomology group.

### 6 Deformation of left-symmetric Rinehart algebra

We investigate in this section a deformation theory of left-symmetric Rinehart algebras. But first let us introduce some notation. For a left-symmetric Rinehart algebra \((L, A, m, \ell)\) we will denote the left-symmetric multiplication \(\cdot \) by \(m\) in the sequel of the paper. Let \(K[[t]]\) be the formal power series ring in one variable \(t\) and coefficients in \(K\). Let \(L[[t]]\) be the set of formal power series whose coefficients are elements of \(L\) (note that \(L[[t]]\) is obtained by extending the coefficients domain of \(K[[t]]\) from \(K\) to \(L\)). Thus, \(L[[t]]\) is a \(K[[t]]\)-module.
6.1 Formal deformations

**Definition 6.1.** A deformation of a left-symmetric Rinehart algebra \((L, A, m, \ell)\) is a \(\mathbb{K}[[t]]\)-bilinear map

\[
m_t : L[[t]] \otimes L[[t]] \to L[[t]]
\]

which is given by \(m_t(x, y) = \sum_{i \geq 0} t^i m_i(x, y)\), where \(m_0 = m\) and the \(m_i \in \mathcal{D}^1(L)\) satisfy the condition \([m_t, m_i]_{MN} = 0\).

Note that \(m_t\) is a 1-degree multiderivation of \(A\) with symbol \(\Xi_{m_t} : L \to \text{Der}(A)\) given by

\[
\Xi_{m_t} = \sum_{i \geq 0} t^i \Xi_{m_i}.
\]

Moreover, since \([m_t, m_i]_{MN} = 0\), it corresponds to a left symmetric Lie Rinehart algebra structure. In particular, it yields a \(t\)-parameterized family of products \(m_t : L \otimes L \to L\) and a family of maps \(\ell_t : L \to \text{Der}(A)\), which satisfy the following identities for all \(x, y \in L\):

\[
m_t(x, y) = x \cdot y + \sum_{i \geq 1} t^i m_i(x, y),
\]

\[
\ell_t(x) = \ell(x) + \sum_{i \geq 1} t^i \Xi_{m_i}(x).
\]

The \(t\)-parametrized family \((L, A, m_t, \ell_t)\) is called a 1-parameter formal deformation of \((L, A, m, \ell)\) generated by \(m_1, \ldots, m_n \in \mathcal{D}^1(L)\).

Let \((L, A, m_t, \ell_t)\) be a deformation of \(m\). Then, for all \(a \in A, x, y, z \in L\)

\[
m_t(m_t(x, y), z) - m_t(x, m_t(y, z)) = m_t(m_t(y, x), z) - m_t(y, m_t(x, z)), \quad (16)
\]

\[
m_t(ax, y) = am_t(x, y), \quad (17)
\]

\[
m_t(x, ay) = am_t(x, y) + \ell_t(x)ay \quad (18)
\]

The identities (17)–(18), mean that \(m_t \in \mathcal{D}^1(L)\). Comparing the coefficients of \(t^n\) for \(n \geq 0\) in equation (16), we get the following:

\[
\sum_{i+j=n} m_i(m_j(x, y), z) - m_i(x, m_j(y, z)) - m_i(m_j(y, x), z) + m_i(y, m_j(x, z)) = 0. \quad (19)
\]

For \(n = 1\), equation (19) implies

\[
m_1(m(x, y), z) + m(m_1(x, y), z) - m_1(x, m(y, z)) - m(x, m_1(y, z))
\]

\[
- m_1(m(y, x), z) - m(m_1(y, x), z) + m_1(y, m(x, z)) + m(y, m_1(x, z)) = 0.
\]

Or equivalently \(\delta(m_1) = [m, m_1]_{MN} = 0\).

The 1-degree multiderivation \(m_1\) is called the infinitesimal of the deformation \(m_t\). More generally, if \(m_i = 0\) for \(1 \leq i \leq n - 1\) and \(m_n\) is non zero 1-degree multiderivation then \(m_n\) is called the \(n\)-infinitesimal of the deformation \(m_t\). By the above discussion, the following proposition follows immediately.
Proposition 6.2. The infinitesimal of the deformation \( m_t \) is a 2-cocycle in \( D^1(L) \). More generally, the \( n \)-infinitesimal is a 2-cocycle.

Now we give a notion of equivalence of two deformations. Let us denote a deformation \((L, A, m, \ell)\) of \((L, A, m, \ell)\) simply by \( L_t \). Let us consider two deformations \( L_t \) and \( L_t' \) of \((L, A, m, \ell)\), generated by \( m_i \) and \( m_i' \), respectively, for \( i \geq 0 \).

Definition 6.3. Two deformations \( L_t \) and \( L_t' \) are said to be equivalent if there exists a formal automorphism \( \Phi_t : L[[t]] \to L[[t]] \) defined as \( \Phi_t = id_L + \sum_{i \geq 1} t^i \phi_i \) where for each \( i \geq 1 \), \( \phi_i : L \to L \) is a \( K \)-linear map such that

\[
m_i'(x, y) = \Phi_t^{-1} m_i(\Phi_t(x), \Phi_t(y)) \quad \text{and} \quad \ell'_i(\Phi_t(x)) = \ell_i(x).
\]

Definition 6.4. Any deformation that is equivalent to the deformation \( m_0 = m \) is said to be a trivial deformation.

Theorem 6.5. The cohomology class of the infinitesimal of a deformation \( m_t \) is determined by the equivalence class of \( m_t \).

Proof. Let \( \Phi_t \) be an equivalence of deformation between \( m_t \) and \( \tilde{m}_t \). Then we get,

\[
\tilde{m}_t(x, y) = \Phi_t^{-1} m_t(\Phi_t(x), \Phi_t(y)).
\]

Comparing the coefficients of \( t \) from both sides of the above equation we have

\[
\tilde{m}_1(x, y) + \Phi_1(m_0(x, y)) = m_1(x, y) + m_0(\Phi_1(x), y) + m_0(x, \Phi_1(y)),
\]

or equivalently,

\[
m_1 - \tilde{m}_1 = \delta(\phi_1).
\]

This establishes the result.

Definition 6.6. A left-symmetric Rinehart algebra is said to be rigid if and only if every deformation of it is trivial.

Theorem 6.7. A non-trivial deformation of a left-symmetric Rinehart algebra is equivalent to a deformation whose \( n \)-infinitesimal is not a coboundary for some \( n \geq 1 \).

Proof. Let \( m_t \) be a deformation of left-symmetric Rinehart algebra with \( n \)-infinitesimal \( m_n \) for some \( n \geq 1 \). Assume that there exists a 2-cochain \( \phi \in C^1(L, L) \) with \( \delta(\phi) = m_n \). Then set

\[
\Phi_t = id_L + \phi t^n \quad \text{and} \quad \tilde{m}_t = \Phi_t \circ m_t \circ \Phi_t^{-1}.
\]
Then by computing the expression and comparing coefficients of $t^n$, we get

$$\bar{m}_n - m_n = -\delta(\phi).$$

So, $\bar{m}_n = 0$. We can repeat the argument to kill off any infinitesimal, which is a coboundary.

**Corollary 6.8.** If $H^2(L, L) = 0$, then all deformations of $L$ are equivalent to a trivial deformation.

### 6.2 Obstructions to the extension theory of deformations

Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra. Now we consider the problem of extending a deformation of $m$ of order $n$ to a deformation of $m$ of order $(n + 1)$. Let $m_t$ and $\ell_t$ be a deformation of order $n$ of $m$ and $\ell$ respectively. That is

$$m_t = \sum_{i=0}^{n} m_i t^i = m + \sum_{i=1}^{n} m_i t^i \quad \text{and} \quad \ell_t = \sum_{i=0}^{n} \ell_i t^i = \ell + \sum_{i=1}^{n} \ell_i t^i,$$

where $m_i \in \mathcal{D}^1(L)$ and $\ell_i : L \to \text{Der}(A)$ a linear map for each $1 \leq i \leq n$ such that

$$m_i(m_j(x, y), z) - m_i(x, m_j(y, z)) = m_i(m_j(y, x), z) - m_i(y, m_j(x, z)), \quad (20)$$

$$m_i(ax, y) = am_i(x, y), \quad (21)$$

$$m_i(x, ay) = am_i(x, y) + \ell_i(x)ay \quad (22)$$

for all $1 \leq i, j \leq n$. If there exists a 2-cochain $m_{n+1} \in \mathcal{D}^1(L)$ and $\ell_{n+1} : L \to \text{Der}(A)$ such that $(L, A, \bar{m}_t, \bar{\ell}_t)$ is a deformation of $(L, A, m, \ell)$ of order $n + 1$, where

$$\bar{m}_t = m_t + m_{n+1} t^{n+1} \quad \text{and} \quad \bar{\ell}_t = \ell_t + \ell_{n+1} t^{n+1}.$$

Then we say that $m_t$ extends to a deformation of order $(n + 1)$. In this case $m_t$ is called extendable.

**Definition 6.9.** Let $m_t$ be a deformation of $m$ of order $n$. Consider the cochain in $C^3(L, L)$ defined as

$$\text{Obs}_L(x, y, z) = \sum_{i+j=n+1; \ i,j \geq 0} \left( m_i(m_j(x, y), z) - m_i(x, m_j(y, z)) - m_i(m_j(y, x), z) + m_i(y, m_j(x, z)) \right), \quad (23)$$

for $x, y, z \in L$. The 3-cochain $\text{Obs}_L$ is called an obstruction cochain for extending the deformation of $m$ of order $n$ to a deformation of order $n + 1$.

A straightforward computation gives the following

**Proposition 6.10.** The obstructions are left-symmetric Rinehart algebra 3-cocycles.
Theorem 6.11. Let \( m_t \) be a deformation of \( m \) of order \( n \). Then \( m_t \) extends to a deformation of order \( n + 1 \) if and only if the cohomology class of \( Obs_L \) vanishes.

Proof. Suppose that a deformation \( m_t \) of order \( n \) extends to a deformation of order \( n + 1 \). Then
\[
\sum_{i,j \geq 0; i+j=n+1} \left( m_i(m_j(x,y), z) - m_i(x, m_j(y, z)) - m_i(m_j(y, x), z) + m_i(y, m_j(x, z)) \right) = 0.
\]
As a result, we get \( Obs_L = \delta(m_{n+1}) \). So, the cohomology class of \( Obs_L \) vanishes.

Conversely, let \( Obs_L \) be a coboundary. Suppose that \( Obs_L = \delta(m_{n+1}) \) for some 2-cochain \( m_{n+1} \). Define a map \( \tilde{m}_t \) as follows
\[
\tilde{m}_t = m_t + m_{n+1}t^{n+1}.
\]
Then for any \( x, y, z \in L \), the map \( \tilde{m}_t \) satisfies the following identity
\[
\sum_{i,j \geq 0; i+j=n+1} \left( m_i(m_j(x,y), z) - m_i(x, m_j(y, z)) - m_i(m_j(y, x), z) + m_i(y, m_j(x, z)) \right) = 0.
\]
This, in turn, implies that \( \tilde{m}_t \) is a deformation of \( m \) extending \( m_t \).

Corollary 6.12. If \( H^3(L,L) = 0 \), then every 2-cocycle in \( C^2(L,L) \) is the infinitesimal of some deformation of \( m \).

6.3 Trivial deformation

We study deformations of left-symmetric Rinehart algebras using the deformation cohomology. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra, and \( m \in C^2(L, L) \). Consider a \( t \)-parameterized family of multiplications \( m_t : L[[t]] \otimes L[[t]] \rightarrow L[[t]] \) and linear maps \( \ell_t : L \rightarrow \text{Der}(A) \) given by
\[
m_t(x, y) = x \cdot y + t m(x, y), \quad \ell_t = \ell + t \Xi_m.
\]
If \( L_t = (L, A, m_t, \ell_t) \) is a left-symmetric Rinehart algebra for all \( t \), we say that \( m \) generates a 1-parameter infinitesimal deformation of \((L, A, \cdot, \ell)\).

Since \( m \) is a 2-cochain, we have
\[
m(ax, y) = am(x, y), \quad \text{and} \quad m(x, ay) = am(x, y) + \Xi_m(x)(a)y,
\]
which implies that conditions (4) and (5) in Definition 3.1 are satisfied for \( m_t \). Then we can deduce that \((L, A, m_t, \ell_t)\) is a deformation of \((L, A, \cdot, \ell)\) if and only if
\[
x \cdot m(y, z) - y \cdot m(x, z) + m(y, x) \cdot z - m(x, y) \cdot z
\]
\[
= m(y, x \cdot z) - m(x, y \cdot z) - m([x, y], z),
\]
\[ (26) \]
and

\[ m(m(x, y), z) - m(x, m(y, z)) = m(m(y, x), z) - m(y, m(x, z)). \]  

Equation (26) means that \( m \) is a 2-cocycle, and equation (27) means that \((L, A, m, \Xi_m)\) is a left-symmetric Rinehart algebra.

Recall that a deformation is said to be trivial if there exists a family of left-symmetric Rinehart algebra isomorphisms \( \text{Id} + tN : L_t \rightarrow L \).

By direct computations, \( L_t \) is trivial if and only if

\[ m(x, y) = x \cdot N(y) + N(x) \cdot y - N(x \cdot y), \]  

\[ Nm(x, y) = N(x) \cdot N(y), \]  

\[ \ell \circ N = \Xi_m. \]  

Again, equation (30) can be obtained from equation (28). It follows from (28) and (29) that \( N \) must satisfy the following integrability condition

\[ N(x) \cdot N(y) - x \cdot N(y) - N(x) \cdot y + N^2(x \cdot y) = 0. \]  

Now we give the following definition.

**Definition 6.13.** An \( A \)-linear map \( N : L \rightarrow L \) is called a Nijenhuis operator on a left-symmetric Rinehart algebra \((L, A, \cdot, \ell)\) if the Nijenhuis condition (31) holds.

Obviously, any Nijenhuis operator on a left-symmetric Rinehart algebra is also a Nijenhuis operator on the corresponding sub-adjacent Lie-Rinehart algebra.

We have seen that a trivial deformation of a left-symmetric Rinehart algebra gives rise to a Nijenhuis operator. In fact, the converse is also true as can be seen from the following theorem.

**Theorem 6.14.** Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \( N \) be a Nijenhuis operator. Then a deformation of \((L, A, \cdot, \ell)\) can be obtained by putting

\[ m(x, y) = \delta N(x, y). \]  

Furthermore, this deformation is trivial.

**Proof.** Since \( m \) is a coboundary, then it is a cocycle, i.e. equation (26) holds. To see that \( m \) generates a deformation, we only need to show that (27) holds, which follows from the Nijenhuis condition (31). At the end, we can easily check that

\[ (\text{Id} + tN)(x \cdot y) = (\text{Id} + tN)(x) \cdot (\text{Id} + tN)(y), \quad \ell \circ (\text{Id} + tN) = \ell_t, \]  

which implies that the deformation is trivial. \( \square \)

**Theorem 6.15.** Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \( N \) be a Nijenhuis operator. Then \((L, A, \cdot_N, \ell_N = \ell \circ N)\) is a left-symmetric Rinehart algebra, where

\[ x \cdot_N y = x \cdot N(y) + N(x) \cdot y - N(x \cdot y), \forall x, y \in L. \]
Proof. It is obvious to show that \((L, \cdot N)\) is a left-symmetric algebra and \(\ell_N\) is a representation of \(L\) on \(\text{Der}(A)\). Evidently, we have

\[\ell_N(ax) = a\ell_N(x), \forall x \in L, a \in A.\]

Furthermore, for any \(x, y \in L\) and \(a \in A\) we have

\[
x \cdot_N (ay) = x \cdot N(ay) + N(x) \cdot (ay) - N(x \cdot (ay)) = a(x \cdot N(y)) + \ell(x)aN(y) + a(N(x) \cdot y) + \ell(N(x))ay - aN(x \cdot y) - N(\ell(x)ay)
\]

\[
= a(x \cdot N(y) + N(x) \cdot y - N(x \cdot y)) + \ell_N(x)ay + N(\ell(x)ay) - N(\ell(x)ay).
\]

Moreover,

\[
(ax) \cdot_N y = (ax) \cdot N(y) + N(ax) \cdot y - N((ax) \cdot y) = a(x \cdot N(y)) + a(N(x) \cdot y) - aN(x \cdot y)
\]

\[
= a(x \cdot N(y) + N(x) \cdot y - N(x \cdot y)) = a(x \cdot_N y).
\]

Then, \((L, A, \cdot_N, \ell_N = \ell \circ N)\) is a left-symmetric Rinehart algebra.

By direct calculations, we have the following corollary.

Lemma 6.16. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \(N\) be a Nijenhuis operator. Then for arbitrary positive \(j, k \in \mathbb{N}\), the following equation holds

\[
N^j(x) \cdot N^k(y) - N^k(N^j(x) \cdot y) - N^j(x \cdot N^k(y)) + N^{j+k}(x \cdot y) = 0, \quad \forall x, y \in L. \quad (32)
\]

If \(N\) is invertible, this formula becomes valid for arbitrary \(j, k \in \mathbb{Z}\).

By direct calculations, we have the following corollary.

Corollary 6.17. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \(N\) a Nijenhuis operator.

(i) For all \(k \in \mathbb{N}\), \((L, A, \cdot_{N^k}, \ell_{N^k} = \ell \circ N^k)\) is a left-symmetric Rinehart algebra.

(ii) For all \(l \in \mathbb{N}\), \(N^l\) is a Nijenhuis operator on the left-symmetric Rinehart algebra \((L, A, \cdot_{N^k}, \ell_{N^k})\).

(iii) The left-symmetric Rinehart algebras \((L, A, (\cdot_{N^k})_{N^l}, \ell_{N^{k+l}})\) and \((L, A, \cdot_{N^{k+l}}, \ell_{N^{k+l}})\) are the same.

(iv) \(N^l\) is a left-symmetric Rinehart algebra homomorphism from \((L, A, \cdot_{N^{k+l}}, \ell_{N^{k+l}})\) to \((L, A, \cdot_{N^k}, \ell_{N^k})\).
Theorem 6.18. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \(N\) be a Nijenhuis operator. Then the operator \(P(N) = \sum_{i=0}^{n} c_i N^i\) is a Nijenhuis operator. If \(N\) is invertible, then \(Q(N) = \sum_{i=-m}^{n} c_i N^i\) is also a Nijenhuis operator.

Proof. According to Lemma 6.16, we obtain, \(\forall x, y \in L,\)

\[
P(N)(x) \cdot P(N)(y) - P(N)(P(N)(x) \cdot y) - P(N)(x \cdot P(N)(y)) + P^2(N)(x \cdot y) = n \sum_{i,j=0}^{n} c_j c_k \left( N^j(x) \cdot N^k(y) - N^k(N^j(x) \cdot y) - N^j(x \cdot N^k(y)) + N^{j+k}(x \cdot y) \right) = 0,
\]

which implies that \(P(N)\) is a Nijenhuis operator. Similarly we can easy check the second statement.

7 \(\mathcal{O}\)-operators and Nijenhuis operators

In this section, we highlight the relationships between \(\mathcal{O}\)-operators and Nijenhuis operators on left-symmetric Rinehart algebras. Moreover, we illustrate some connections between Nijenhuis operators and compatible \(\mathcal{O}\)-operators on left-symmetric Rinehart algebras.

7.1 Relationships between \(\mathcal{O}\)-operators and Nijenhuis operators

We first give the definitions of an \(\mathcal{O}\)-operator and of Rota-Baxter operator.

Definition 7.1. An \(\mathcal{O}\)-operator on a left-symmetric Rinehart algebra \((L, A, \cdot, \ell)\) associated to a representation \((M; \rho, \mu)\) is a linear map \(T : M \rightarrow L\) satisfying

\[
T(au) = aT(u), \quad T(u) \cdot T(v) = T \left( \rho(T(u))(v) + \mu(T(v))(u) \right), \quad \forall u, v \in M, a \in A.
\]

Definition 7.2. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \(R : ker(\ell) \rightarrow L\) a linear operator. If \(R\) satisfies

\[
R(ax) = aR(x), \quad R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)), \quad \forall x, y \in L, a \in A,
\]

then \(R\) is called a Rota-Baxter operator of weight 0 on \(L\).

Notice that a Rota-Baxter operator of weight zero on a left-symmetric Rinehart algebra \(L\) is exactly an \(\mathcal{O}\)-operator associated to the adjoint representation \((L; ad^L, ad^R)\).

The following proposition gives connections between Nijenhuis operators and Rota-Baxter operators.

Proposition 7.3. Let \((L, A, \cdot, \ell)\) be a left-symmetric Rinehart algebra and \(N : L \rightarrow L\) a linear operator.
Proposition 7.4. Let

(i) If $N^2 = 1$, then $N$ is a Nijenhuis operator if and only if $N \pm 1$ is a Rota-Baxter operator of weight $\mp 2$ on $(L, A, \cdot, \ell)$.

(ii) If $N^2 = 0$, then $N$ is a Nijenhuis operator if and only if $N$ is a Rota-Baxter operator of weight zero on $(L, A, \cdot, \ell)$.

(iii) If $N^2 = N$, then $N$ is a Nijenhuis operator if and only if $N$ is a Rota-Baxter operator of weight $-1$ on $(L, A, \cdot, \ell)$.

Proof. For Item (i), for all $x, y \in L$ then we have

\[
(N - \text{Id})(x) \cdot (N - \text{Id})(y) - (N - \text{Id})((N - \text{Id})(x) \cdot y + x \cdot (N - \text{Id})(y)) + 2(N - \text{Id})(x \cdot y)
= N(x) \cdot N(y) - N(N(x) \cdot y + x \cdot N(y)) + x \cdot y
= N(x) \cdot N(y) - N(N(x) \cdot y + x \cdot N(y)) + N^2(x \cdot y).
\]

So $N$ is a Nijenhuis operator if and only if $N - \text{Id}$ is a Rota-Baxter operator of weight 2 on $L$. Similarly, we obtain that $N$ is a Nijenhuis operator if and only if $N + \text{Id}$ is a Rota-Baxter operator of weight $-2$ on $L$.

Items (ii) and (iii) are obvious from the definitions of Nijenhuis operators and Rota-Baxter operator. \qed

Proposition 7.4. Let $(L, A, \cdot, \ell)$ be a left-symmetric Rinehart algebra and $(M; \rho, \mu)$ be a representation on $L$. Let $T : M \to L$ be a linear map. For any $\lambda, T$ is an $O$-operator on $L$ associated to $(M; \rho, \mu)$ if and only if the linear map $R_{T, \lambda} := \begin{pmatrix} 0 & T \\ 0 & -\lambda \text{Id} \end{pmatrix}$ is a Rota-Baxter operator of weight $\lambda$ on the semidirect product left-symmetric Rinehart algebra $(L \oplus M, \cdot_{L \oplus M})$, where the multiplication $\cdot_{L \oplus M}$ is given by (9).

Proof. It is easy to check the equation (35). Let $x_1, x_2 \in L$ and $m_1, m_2 \in M$,

\[
R_{T, \lambda}(x_1 + m_1) \cdot_{L \oplus M} R_{T, \lambda}(x_2 + m_2) = (T(m_1) - \lambda m_1) \cdot_{L \oplus M} (T(m_2) - \lambda m_2)
= T(m_1) \cdot T(m_2) - \lambda \rho(T(m_1)m_2 - \mu(T(m_2))m_1).
\]

On the other hand,

\[
R_{T, \lambda}(R_{T, \lambda}(x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2) + (x_1 + m_1) \cdot_{L \oplus M} R_{T, \lambda}(x_2 + m_2)) +
\lambda R_{T, \lambda}((x_1 + m_1) \cdot_{L \oplus M} (x_2 + m_2))
=R_{T, \lambda}((T(m_1) - \lambda m_1) \cdot_{L \oplus M} (x_2 + m_2) + (x_1 + m_1) \cdot_{L \oplus M} (T(m_2) - \lambda m_2)) +
\lambda R_{T, \lambda}(x_1 \cdot x_2 + \rho(x_1)m_2 + \mu(x_2)m_1)
=R_{T, \lambda}(T(m_1) \cdot x_2 + \rho(T(m_1)m_2 - \lambda \mu(x_2)m_1)
+ x_1 \cdot T(m_2) - \lambda \rho(x_1)m_2 + \mu(T(m_2))m_1)
\]

(37)
\[ + \lambda \left( T(\rho(x_1)m_2) + T(\mu(x_2)m_1) - \lambda(\rho(x_1)m_2 + \mu(x_2)m_1) \right) \]
\[ = T(\rho(T(m_1))m_2 + \mu(T(m_2))m_1) - \lambda T(\mu(x_2)m_1 + \rho(x_1)m_2) \]
\[ - \lambda \left( \rho(T(m_1))m_2 + \mu(T(m_2))m_1 \right) + \lambda^2 \left( \mu(x_2)m_1 + \rho(x_1)m_2 \right) \]
\[ + \lambda \left( T(\rho(x_1)m_2) + T(\mu(x_2)m_1) \right) - \lambda^2 \left( \rho(x_1)m_2 + \mu(x_2)m_1 \right) \]
\[ = T(\rho(T(m_1))m_2 + \mu(T(m_2))m_1) - \lambda \rho(T(m_1))m_2 - \lambda \mu(T(m_2))m_1. \]  

(39)

According to equations (37) and (40), \( R_{T,\lambda} \) is a Rota-Baxter operator of weight \( \lambda \) on the semidirect product left-symmetric Rinehart algebra \( (L \oplus M, \cdot_{L \oplus M}) \) if and only if \( T \) is an \( \mathcal{O} \)-operator on \( (L, A, \cdot, \ell) \) associated to \( (M; \rho, \mu) \).

**Proposition 7.5.** Let \( (L, A, \cdot, \ell) \) be a left-symmetric Rinehart algebra and let \( (M; \rho, \mu) \) be a representation on \( L \). Let \( T : M \to L \) be a linear map. Then the following statements are equivalent.

(i) \( T \) is an \( \mathcal{O} \)-operator on the left-symmetric Rinehart algebra \( (L, A, \cdot, \ell) \).

(ii) \( N_T := \left( \begin{array}{cc} 0 & T \\ 0 & \text{Id} \end{array} \right) \) is a Nijenhuis operator on the left-symmetric Rinehart algebra \( (L \oplus M, \cdot_{L \oplus M}) \).

(iii) \( N_T := \left( \begin{array}{cc} 0 & T \\ 0 & 0 \end{array} \right) \) is a Nijenhuis operator on the left-symmetric Rinehart algebra \( (L \oplus M, \cdot_{L \oplus M}) \).

**Proof.** Note that \( N_T = R_{T,-1} \) and \( (N_T)^2 = N_T \), thus \( N_T \) is a Nijenhuis operator on the left-symmetric Rinehart algebra \( (L \oplus M, \cdot_{L \oplus M}) \), using (iii) in Proposition 7.3.

Similarly \( N_T = R_{T,0} \) and \( (N_T)^2 = 0 \), then \( N_T \) is a Nijenhuis operator on the left-symmetric Rinehart algebra \( (L \oplus M, \cdot_{L \oplus M}) \), according to (ii) in Proposition 7.3.

\[ 7.2 \] **Compatible \( \mathcal{O} \)-operators and Nijenhuis operators**

In this subsection we study compatibility of \( \mathcal{O} \)-operators and Nijenhuis operators. First we start with the following definition.

**Definition 7.6.** Let \( (L, A, \cdot, \ell) \) be a left-symmetric Rinehart algebra and let \( (M; \rho, \mu) \) be a representation. Let \( T_1, T_2 : M \to L \) be two \( \mathcal{O} \)-operators associated to \( (M; \rho, \mu) \). Then \( T_1 \) and \( T_2 \) are called compatible if \( T_1 + T_2 \) is an \( \mathcal{O} \)-operator associated to \( (M; \rho, \mu) \).

Let \( T_1, T_2 : M \to L \) be two \( \mathcal{O} \)-operators on a left-symmetric Rinehart algebra \( (L, A, \cdot, \ell) \) associated to a representation \( (M; \rho, \mu) \) such that

\[ T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v) = T_1 \left( \rho(T_2(u))(v) + \mu(T_2(v))(u) \right) \]
\[ + T_2 \left( \rho(T_1(u))(v) + \mu(T_1(v))(u) \right), \]

(41)

for all \( u, v \in M \).
**Lemma 7.7.** Two operators $T_1$ and $T_2$ are compatible if and only if the equation (41) holds.

*Proof.* For all $u, v \in M$, $a \in A$, we have

$$(T_1 + T_2)(au) = T_1(au) + T_2(au)$$

$$= aT_1(u) + aT_2(u)$$

$$= a(T_1 + T_2)(u).$$

Furthermore,

$$(T_1 + T_2)(u) \cdot (T_1 + T_2)(v) = (T_1 + T_2)\left(\rho((T_1 + T_2)(u))(v) + \mu((T_1 + T_2)(v))(u)\right)$$

$$= T_1(u) \cdot T_1(v) + T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v) + T_2(u) \cdot T_2(v)$$

$$- (T_1 + T_2)\left(\rho(T_1(u))(v) + \rho(T_2(u))(v) + \mu(T_1(v))(u) + \mu(T_2(v))(u)\right)$$

$$= T_1(u) \cdot T_1(v) + T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v) + T_2(u) \cdot T_2(v)$$

$$- T_1\left(\rho(T_1(u))(v) + \mu(T_1(v))(u)\right) - T_1\left(\rho(T_2(u))(v) + \mu(T_2(v))(u)\right)$$

$$- T_2\left(\rho(T_1(u))(v) + \mu(T_1(v))(u)\right) - T_2\left(\rho(T_2(u))(v) + \mu(T_2(v))(u)\right)$$

$$= T_1(u) \cdot T_2(v) + T_2(u) \cdot T_1(v)$$

$$- T_1\left(\rho(T_2(u))(v) + \mu(T_2(v))(u)\right) - T_2\left(\rho(T_1(u))(v) + \mu(T_1(v))(u)\right).$$

Then $T_1 + T_2$ is an $\mathcal{O}$-operator associated to $(M; \rho, \mu)$ if and only if equation (41) holds. □

**Remark 7.8.** Equation (41) implies that for any $k_1, k_2$ the linear combination $k_1T_1 + k_2T_2$ is an $\mathcal{O}$-operator.

There is a close relationship between a Nijenhuis operator and a pair of compatible $\mathcal{O}$-operators as can be seen from the following proposition.

**Proposition 7.9.** Let $T_1, T_2 : M \rightarrow L$ be two $\mathcal{O}$-operators on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$. Suppose that $T_2$ is invertible. If $T_1$ and $T_2$ are compatible, then $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator on the left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$.

*Proof.* For all $x, y \in L$, since $T_2$ is invertible, there exist $u, v \in M$ such that $T_2(u) = x$, $T_2(v) = y$. Hence $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator if and only if the following equation holds:

$$NT_2(u) \cdot NT_2(v) = N(NT_2(u) \cdot T_2(v) + T_2(u) \cdot NT_2(v)) - N^2(T_2(u) \cdot T_2(v)).$$

(42)

Since $T_1 = N \circ T_2$ is an $\mathcal{O}$-operator, the left-hand side of the above equation is

$$NT_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)).$$
Using the fact that $T_2$ and $T_1 = N \circ T_2$ are two compatible $\mathcal{O}$-operators, we get

\[ NT_2(u) \cdot T_2(v) + T_2(u) \cdot NT_2(v) \]

\[ = T_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)) + NT_2(\rho(T_2(u))(v) + \mu(T_2(v))(u)) \]

\[ = T_2(\rho(NT_2(u))(v) + \mu(NT_2(v))(u)) + NT(\rho(T(u))(v) + \mu(T(v))(u)) \cdot N(T(u) \cdot T(v)) \]

Hence, equation (42) holds by acting $N$ on both sides of the last equality.

Using an $\mathcal{O}$-operator and a Nijenhuis operator, we can construct a pair of compatible $\mathcal{O}$-operators.

**Proposition 7.10.** Let $T : M \longrightarrow L$ be an $\mathcal{O}$-operator on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$ and let $N$ be a Nijenhuis operator on $(L, A, \cdot, \ell)$. Then $N \circ T$ is an $\mathcal{O}$-operator on the left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to $(M; \rho, \mu)$ if and only if for all $u, v \in M$, the following equation holds:

\[ N\left(NT(u) \cdot T(v) + T(u) \cdot NT(v)\right) = N\left(T(\rho(NT(u))(v) + \mu(NT(v))(u)) + NT(\rho(T(u))(v) + \mu(T(v))(u))\right) \]

(43)

If in addition $N$ is invertible, then $T$ and $NT$ are compatible. More explicitly, for any $\mathcal{O}$-operator $T$, if there exists an invertible Nijenhuis operator $N$ such that $NT$ is also an $\mathcal{O}$-operator, then $T$ and $NT$ are compatible.

**Proof.** Let $u, v \in M$ and $a \in A$, we have

\[ NT(au) = N(T(au)) = N(aT(u)) = aNT(u). \]

In addition, since $N$ is a Nijenhuis operator and $T$ is an $\mathcal{O}$-operator we have

\[ NT(u) \cdot NT(v) = N\left(NT(u) \cdot T(v) + T(u) \cdot NT(v)\right) - N^2(T(u) \cdot T(v)) \]

\[ = NT\left(\rho(NT(u))(v) + \mu(NT(v))(u)\right) \]

if and only if (43) holds.

If $NT$ is an $\mathcal{O}$-operator and $N$ is invertible, then we have

\[ T(u) \cdot T(v) + T(u) \cdot NT(v) \]

\[ = T(\rho(NT(u))(v) + \mu(NT(v))(u)) + NT(\rho(T(u))(v) + \mu(T(v))(u)), \]

which is exactly the condition that $NT$ and $T$ are compatible.

The following result is an immediate consequence of the last two propositions.

**Corollary 7.11.** Let $T_1, T_2 : M \longrightarrow L$ be two $\mathcal{O}$-operators on a left-symmetric Rinehart algebra $(L, A, \cdot, \ell)$ associated to a representation $(M; \rho, \mu)$. Suppose that $T_1$ and $T_2$ are invertible. Then $T_1$ and $T_2$ are compatible if and only if $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator.
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