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On the free metabelian Novikov and metabelian Lie-admissible algebras

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Abstract. In this paper, we consider Lie-admissible algebras, which are free Novikov and free Lie-admissible algebras with an additional metabelian identity. We construct a linear basis for both free metabelian Novikov and free metabelian Lie-admissible algebras. Additionally, we describe a space of symmetric polynomials for both the free metabelian Novikov algebra and the free metabelian Lie-admissible algebra.

Contents

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1 Introduction

In recent years, algebras with metabelian identity have become popular objects in ring theory. Metabelian identity also can be stated as the solvability of index 2. Various types of classical algebras with metabelian identity, such as Lie, Leibniz, Malcev, Jordan, etc. are considered. The variety of metabelian Lie algebras has attracted significant attention, see [18, 21]. A basis of the free metabelian Lie algebra was constructed in [1]. In an analogical way, a basis of the free metabelian Leibniz algebra was constructed in [7]. Symmetric polynomials in the free metabelian Lie algebras were considered in [8]. The generators of symmetric polynomials in free metabelian Leibniz algebras were found in [24]. Other examples of Lie-admissible algebras are assosymmetric algebras [11]. A basis of the free metabelian Malcev algebra is constructed in [20]. For this reason, we add metabelian identity to a well-known class of algebras which are Novikov and Lie-admissible.

Novikov algebras were introduced in a study of Hamiltonian operators concerning the integrability of certain partial differential equations [12]. Later, they played a significant role in a study of Poisson brackets of hydrodynamic type [2].

It is well-known that, given a commutative algebra A with a derivation D, the space A under the product $x_1 \circ x_2 = D(x_1)x_2$ forms a (right) Novikov algebra. Moreover, every Novikov algebra can be embedded into an appropriate commutative algebra with derivation D [4]. Using rooted trees, the monomial basis of the free Novikov algebra in terms of $\circ$ was constructed in [10]. In terms of Young diagrams, the basis was developed in [9]. By utilizing commutative algebra with derivation D and a well-defined order, an alternative monomial basis of the free Novikov algebra is presented in [15]. The issues of solvability and nilpotency of Novikov algebras were addressed in [23]. In section 2, we construct a basis of the free solvable Novikov algebra with an index of 2. In section 3, we explicitly describe the symmetric polynomials for the multilinear part of the free metabelian Novikov algebra.

Let's shift our focus to metabelian Lie-admissible algebras. In the realm of Lie-admissible algebra theory, additional conditions like flexibility or power-associativity are crucial, leading to numerous noteworthy outcomes in this context, see [3, 19]. Another important direction of the research on Lie-admissible algebras concerns the property that their associator satisfies relations defined by a natural action of the symmetric group of degree 3 [13, 14]. A basis of the free Lie-admissible algebra and the Gröbner-Shirshov base theory for Lie-admissible algebras is given in [5]. In addition, there is given an analogue of PBW-theorem for the pair of Lie and Lie-admissible algebras. A basis of the free Lie-admissible algebra in terms of commutator and anti-commutator is given in [17]. In Section 4, we construct the basis of a free metabelian Lie-admissible algebra in terms of commutator and anti-commutator. In Section 5, we explicitly describe the symmetric polynomials for the multilinear part of the free metabelian Lie-admissible algebra.

We consider all algebras over a field $\mathbb{K}$ of characteristic 0.

2 Free metabelian Novikov algebra

An algebra is called metabelian (right) Novikov if it satisfies the following identities:

$$a(bc) = b(ac), \tag{1}$$

$$(ab)c - a(bc) = (ac)b - a(cb), \tag{2}$$

$$(ab)(cd) = 0 \tag{3}$$

Let $X = \{x_1, x_2 \ldots\}$ be a countable set of generators. We denote by $\mathrm{Nov}\langle X\rangle$ and $\mathrm{M\,Nov}\langle X\rangle$ the free Novikov and free metabelian Novikov algebra, respectively.

Let us denote by $\mathcal{N}_n$ the set of monomials of degree n of the following form:

$$\mathcal{N}_n = \{(\cdots((x_{i_1}x_{i_2})x_{i_3})\cdots)x_{i_n} \mid i_2 \leq i_3 \leq \ldots \leq i_n\},$$

where $n \geq 5$. We set

$$\mathcal{N}_1 = \{x_i\}, \mathcal{N}_2 = \{x_{i_1}x_{i_2}\}.$$

For degree 3, $\mathcal{N}_3$ is defined as follows:

$$\mathcal{N}_3 = \{(x_{i_1}x_{i_2})x_{i_3}, x_{i_3}(x_{i_2}x_{i_1}) \mid i_2 \leq i_3\}.$$

For degree 4, we define the set $\mathcal{N}_4$ as follows:

$$\mathcal{N}_4 = \{((x_{i_1}x_{i_2})x_{i_3})x_{i_4}, x_{j_1}(x_{j_2}(x_{j_3}x_{j_4})) \mid i_2 \leq i_3 \leq i_4,\ j_1 \leq j_2 \leq j_3 \leq j_4\}.$$

Lemma 2.1. *The algebra* $\mathrm{M\,Nov}\langle X\rangle$ *satisfies the following identities:*

$$((x_1(x_2x_3))x_4)x_5 = (x_1((x_2x_3)x_4))x_5 = (x_1(x_2(x_3x_4)))x_5 = x_1(((x_2x_3)x_4)x_5) = x_1((x_2(x_3x_4))x_5) = x_1(x_2((x_3x_4)x_5)) = x_1(x_2(x_3(x_4x_5))) = 0. \tag{4}$$

Proof. Using (1), (2) and (3), one can obtain

$$0 = ((x_1x_2)x_3)(x_4x_5) = x_4(((x_1x_2)x_3)x_5) = (x_1(x_2x_5))(x_4x_3) = x_4((x_1(x_2x_5))x_3) = x_4(x_1((x_2x_5)x_3)) + x_4((x_1x_3)(x_2x_5)) - x_4(x_1(x_3(x_2x_5))) = x_4((x_2x_5)(x_1x_3)) - x_4(x_1(x_3(x_2x_5))) = -x_4(x_1(x_3(x_2x_5))),$$

which gives

$$x_1(((x_2x_3)x_4)x_5) = x_1((x_2(x_3x_4))x_5) = x_1(x_2(x_3(x_4x_5))) = x_1(x_2((x_3x_4)x_5)) = 0.$$

By (1), (2) and (3), we have

$$((x_1(x_2x_3))x_4)x_5 = (x_1((x_2x_3)x_4))x_5 + ((x_1x_4)(x_2x_3))x_5 - (x_1(x_4(x_2x_3)))x_5 = - x_1((x_4(x_2x_3))x_5) - (x_1x_5)(x_4(x_2x_3)) + x_1(x_5(x_4(x_2x_3))) = x_1(x_5(x_4(x_2x_3))) = - (x_4(x_2x_3))(x_1x_5) + x_1(x_5(x_4(x_2x_3))) = x_1(x_5(x_4(x_2x_3))) = 0,$$

which gives

$$((x_1(x_2x_3))x_4)x_5 = (x_1((x_2x_3)x_4))x_5 = (x_1(x_2(x_3x_4)))x_5 = 0.$$

□

Theorem 2.2. *The set* $\bigcup \mathcal{N}_i$ *is a linear basis of the algebra* $\mathrm{M\,Nov}\langle X\rangle$.

Proof. The statement for degrees less than 5 can be verified by direct calculations. For $n \geq 5$, firstly, we show that every monomial of $\mathrm{M\,Nov}\langle X\rangle$ can be written as a sum of monomials from the set $\bigcup_{i\geq 5}\mathcal{N}_i$. By Lemma 2.1, we obtain that every monomial except $(\cdots((x_{i_1}x_{i_2})x_{i_3})\cdots)x_{i_n}$ is equal to 0. By (2) and Lemma 2.1, we have

$$(\cdots((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_{m-1}})x_{i_m})\cdots)x_{i_n} = (\cdots((\cdots(x_{i_1}x_{i_2})\cdots)(x_{i_{m-1}}x_{i_m}))\cdots)x_{i_n}+ (\cdots(((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_m})x_{i_{m-1}})\cdots)x_{i_n} - (\cdots((\cdots(x_{i_1}x_{i_2})\cdots)(x_{i_m}x_{i_{m-1}}))\cdots)x_{i_n} = (\cdots(((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_m})x_{i_{m-1}})\cdots)x_{i_n},$$

i.e., the generators x_{i_2}, $x_{i_3},\ldots,$ x_{i_n} are rearrangeable. From these equations, we get that every monomial of $\mathrm{M\,Nov}\langle X\rangle$ which has a degree greater than 4 can be written as a sum of $\bigcup_{i\geq 5}\mathcal{N}_i$.

Now, we consider an algebra $A\langle X\rangle$ with a basis monomials $\bigcup\mathcal{N}_i$ and multiplication $*$. Let us define a multiplication on monomials $\bigcup\mathcal{N}_i$ in $A\langle X\rangle$ as follows:

$$\begin{cases} X_1 * X_2 = 0 \ \text{ if } X_1, X_2 \in \mathcal{N}_i \text{ and } \deg(X_1), \deg(X_2) > 1;\\ x_j * ((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_n}) = 0;\\ ((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_n}) * x_j = ((\cdots(((\cdots(x_{i_1}x_{i_2})\cdots)x_{i_k})x_j)x_{i_{k+1}})\cdots)x_{i_n}, \end{cases}$$

where $n > 4$ and $i_2 \leq \ldots \leq i_k \leq j \leq i_{k+1} \leq \ldots \leq i_n$. Up to degree 4, we define multiplication in $A\langle X\rangle$ that is consistent with identities (1), (2) and (3). By straightforward calculation, we can check that an algebra $A\langle X\rangle$ satisfies to (1), (2) and (3) identities. It remains to note that $A\langle X\rangle \cong \mathrm{M\,Nov}\langle X\rangle$. □

3 Symmetric polynomials of the free metabelian Novikov algebra

Let $p(x_1, x_2, \ldots, x_n)$ be a polynomial of the free metabelian Novikov algebra generated by a finite set $X = \{x_1, x_2, \ldots, x_n\}$. The polynomial $p(x_1, x_2, \ldots, x_n)$ is called symmetric if it satisfies the following condition:

$$\sigma p(x_1, x_2, \ldots, x_n) = p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = p(x_1, x_2, \ldots, x_n),$$

where $\sigma \in S_n$. Let us define a set of polynomials $\mathcal{P}$ in $\mathrm{M\,Nov}\langle X\rangle$ as follows:

$$p_1 = \sum_i x_i,\ p_2 = \sum_{i\neq j} x_i x_j,$$

$$p_{3,1} = \sum_{i=1}^{n}\sum_{j_1<j_2} x_{j_2}(x_{j_1}x_i),\ p_{3,2} = \sum_{i=1}^{n}\sum_{j_1<j_2}(x_i x_{j_1})x_{j_2},$$

$$p_{4,1} = \sum_{j_1<j_2<j_3<j_4} x_{j_1}(x_{j_2}(x_{j_3}x_{j_4})),\ p_{4,2} = \sum_{i=1}^{n}\sum_{j_1<j_2<j_3}((x_i x_{j_1})x_{j_2})x_{j_3},$$

and

$$p_n = \sum_{i=1}^{n} \sum_{j_1<j_2<\ldots<j_{n-1}} (\cdots((x_i x_{j_1})x_{j_2})\cdots)x_{j_{n-1}},$$

where $n \geq 5$. The multilinear part of the free metabelian Novikov algebra is a space consisting of all elements containing each x_i exactly once.

Example 3.1. For multilinear part of $\mathrm{M\,Nov}\langle X\rangle$, we obtain

$$p_{3,1} = x_2(x_1x_3) + x_3(x_2x_1) + x_3(x_1x_2),\ p_{3,2} = (x_1x_2)x_3 + (x_2x_1)x_3 + (x_3x_1)x_2,$$

$$p_{4,1} = x_1(x_2(x_3x_4)),\ p_{4,2} = ((x_1x_2)x_3)x_4 + ((x_2x_1)x_3)x_4 + ((x_3x_1)x_2)x_4 + ((x_4x_1)x_2)x_3,$$

$$p_n = (\cdots(((x_1x_2)x_3)x_4)\cdots)x_n + (\cdots(((x_2x_1)x_3)x_4)\cdots)x_n + (\cdots(((x_3x_1)x_2)x_4)\cdots)x_n + \ldots + (\cdots(((x_nx_1)x_2)x_3)\cdots)x_{n-1},$$

where $n \geq 5$.

Theorem 3.2. *For the multilinear part of the free metabelian Novikov algebra, the symmetric polynomials have the form $\mathcal{P}$.*

Proof. For $n = 1, 2$, the result is obvious. For $n \geq 3$, we use the fact that every Novikov algebra is embeddable into commutative algebra with derivation and spanning elements of Novikov algebra in commutative algebra with derivation are monomials of weight -1 [4,16]. The weight function $\mathrm{wt}(u) \in \mathbb{Z}$ is defined on monomials of commutative algebra with derivation D by induction as follows,

$$\mathrm{wt}(x) = -1, \quad x \in X;$$
$$\mathrm{wt}(d(u)) = \mathrm{wt}(u) + 1; \quad \mathrm{wt}(uv) = \mathrm{wt}(u) + \mathrm{wt}(v).$$

For simplicity, we denote $D(x)$ and $D^n(x)$ by x' and $(D^{n-1}(x))'$, respectively. For the multilinear part to describe the space of symmetric polynomials of $\mathrm{Nov}\langle X\rangle$ in degree 3, we need to find the space of the symmetric polynomials of the differential commutative algebra of weight -1 in degree 3. This is the monomials of the form

$$x_2'x_1'x_3 + x_3'x_2'x_1 + x_3'x_1'x_2 \text{ and } x_1''x_2x_3 + x_2''x_1x_3 + x_3''x_1x_2.$$

Rewriting the first polynomial by the operation of Novikov algebra, we obtain $p_{3,1}$. Rewriting the second polynomial, we obtain

$$(x_1x_2)x_3 - x_1(x_2x_3) + (x_2x_1)x_3 - x_2(x_1x_3) + (x_3x_1)x_2 - x_3(x_1x_2).$$

Adding to the last expression $p_{3,1}$, we obtain $p_{3,2}$.

For degree 4, we have

$$\mathrm{M\,Nov}\langle X\rangle/\{(ab)(cd)\} \cong \mathrm{Com}\langle X\rangle^{(D)}_{-1}/\{a''b'cd + a'b'c'd\},$$

where $\mathrm{Com}\langle X\rangle^{(D)}_{-1}$ is a space of differential commutative algebra of weight -1. Hence, in degree 4 of $\mathrm{Com}\langle X\rangle^{(D)}_{-1}/\{a''b'cd+a'b'c'd\}$ we rewrite all monomials of the form $x''_{i_1}x'_{i_2}x_{i_3}x_{i_4}$ to $-x'_{i_1}x'_{i_2}x'_{i_3}x_{i_4}$. It remains to note that

$$0 = a''b'cd + a'b'c'd - (a''b'dc + a'b'd'c) = a'b'c'd - a'b'd'c$$

which gives that we rewrite monomial $a'b'd'c$ to $a'b'c'd$. Finally, the remained monomials of weight -1 are $x_{i_1}x_{i_2}x_{i_3}x'''_{i_4}$ and $x_{j_1}x'_{j_2}x'_{j_3}x'_{j_4}$, where $i_1 \leq i_2 \leq i_3$ and $j_1 \leq j_2 \leq j_3 \leq j_4$. By Theorem 2.2, these monomials are linearly independent, and for multilinear part symmetric polynomials of $\mathrm{Com}\langle X\rangle^{(D)}_{-1}/\{a''b'cd+a'b'c'd\}$ are

$$x'_1x'_2x'_3x_4 \text{ and } x'''_1x_2x_3x_4 + x'''_2x_1x_3x_4 + x'''_3x_1x_2x_4 + x'''_4x_1x_2x_3,$$

which correspond to $p_{4,1}$ and $p_{4,2}$, analogically as in degree 3.

By Theorem 2.2, starting from degree 5, all monomials of $\mathrm{Com}\langle X\rangle^{(D)}/\{a''b'cd+a'b'c'd\}$ of weight -1 except $x_i^{(n-1)}x_{j_{n-1}}\dots x_{j_1}$ are equal to 0, where $x_i^{(n-1)} = (x_i^{(n-2)})'$. Hence, starting from degree 5, symmetric polynomials of $\mathrm{Com}\langle X\rangle^{(D)}/\ \{a''b'cd+a'b'c'd\}$ of weight -1 are

$$x_1^{(n-1)}x_2x_3x_4\cdots x_n + x_2^{(n-1)}x_1x_3x_4\cdots x_n + x_3^{(n-1)}x_1x_2x_4\cdots x_n + \ldots + x_n^{(n-1)}x_1x_2x_3\cdots x_{n-1},$$

which correspond to p_n. □

4 Free metabelian Lie-admissible algebra

An algebra is called metabelian Lie-admissible if it satisfies the following identities:

$$(ab)c-(ba)c-c(ab)+c(ba)+(bc)a-(cb)a-a(bc) + a(cb)+(ca)b-(ac)b-b(ca)+b(ac)=0, \quad (5)$$

$$(ab)(cd) = 0. \quad (6)$$

Let us consider the polarization of metabelian Lie-admissible algebra, i.e., we consider an algebra with two operations which is defined on metabelian Lie-admissible algebra as follows:

$$[a.b] = ab - ba,\ \{a,b\} = ab + ba.$$

In this case, the defining identities of the variety of metabelian Lie-admissible algebras become to

$$[a,b] = -[b,a],\ \{a,b\} = \{b,a\},$$

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0,$$

$$[[a,b],[c,d]] = [[a,b],\{c,d\}] = [\{a,b\},\{c,d\}] = \{\{a,b\},\{c,d\}\} = \{\{a,b\},[c,d]\} = \{[a,b],[c,d]\} = 0. \quad (7)$$

As a consequence, we obtain

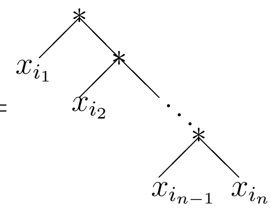
$$[[[a,b],c],d] = [[[a,b],d],c] \quad (8)$$

and

$$[[\{a,b\},c],d] = [[\{a,b\},d],c] \quad (9)$$

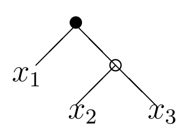
which hold in free metabelian Lie-admissible algebra.

Let us construct a basis of the free metabelian Lie-admissible algebra in terms of binary trees with two types of vertices $\bullet$ and $\circ$. We consider only trees of the following form:

$A =$ 

We place on vertices of the tree $\bullet$ and $\circ$ in all possible ways. On the leaves of the tree, we place generators from the countable set X. This tree in a unique way corresponds to the sequence $(*_{x_{i_1}}, *_{x_{i_2}}, \ldots, *_{x_{i_{n-1}}}, x_{i_n})$.

Example 4.1. If

$A =$ 

then A corresponds to $(\bullet_{x_1}, \circ_{x_2}, x_3)$. We define a set of sequences as follows:

1) If there are several consecutive black dots in a sequence, then all corresponding generators of these vertices are ordered, i.e., for $(\ldots, \circ_{i_{k-1}}, \bullet_{i_k}, \bullet_{i_{k+1}}, \ldots, \bullet_{i_{l-1}}, \circ_{i_l}, \ldots)$, we have $i_k \geq i_{k+1} \geq \ldots \geq i_{l-1}$;

2) If the rightmost vertex is white then the rightmost generator is less than the previous one, i.e., for $(\ldots, \circ_{i_{n-1}}, x_{i_n})$, we have $i_{n-1} \leq i_n$;

3) If a given consecutive sequence of black dots continues to the rightmost vertex and the number of black dots is bigger than 2, then all the generators of these vertices are ordered and the rightmost generator is bigger than the previous one, i.e., for $(\ldots, \bullet_{i_k}, \bullet_{i_{k+1}}, \ldots, \bullet_{i_{n-1}}, x_{i_n})$, we have $i_k \geq i_{k+1} \geq \ldots \geq i_{n-1} < i_n$;

4) In condition 3 if the number of black dots is not bigger than 2, then the generators are ordered as in Lyndon-Shirshov words, i.e., the basis monomials of the free Lie algebra of degrees 2 and 3;

For every such tree, we define a monomial from free metabelian Lie-admissible algebra as follows: the tree with n leaves is a right-normed monomial of degree n, i.e., this is the monomial $x_{i_1} * (x_{i_2} * (\dots (x_{i_{n-1}} * x_{i_n}) \dots))$. The black multiplication $\bullet$ corresponds to the Lie bracket $[\cdot,\cdot]$ and white multiplication $\circ$ corresponds to $\{\cdot,\cdot\}$. We denote by $\mathcal{T}$ a set of trees that satisfy conditions (1), (2), (3) and (4), and we denote by $\mathcal{M}$ the set of monomials which correspond to the trees from $\mathcal{T}$.

Theorem 4.2. *The set of monomials $\mathcal{M}$ is a basis of the free metabelian Lie-admissible algebra.*

Proof. Firstly, we show that any monomial of the free Lie-admissible algebra can be written as a sum of monomials from $\mathcal{M}$. After polarization of Lie-admissible algebra, one obtains that by commutative and anti-commutative identities on $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$, respectively, and by (7), any monomial can be written as a sum of right-normed monomials with multiplications $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$. The condition (1) for right-normed monomials is provided by identities (8) and (9). The condition (3) is provided by the basis of the free metabelian Lie algebra, see [1]. The conditions (2) and (4) are provided by commutativity of $\{\cdot,\cdot\}$ and identities of $[\cdot,\cdot]$, respectively.

Now, let us consider a free algebra $A\langle X\rangle$ with the basis $\mathcal{M}$. The multiplications $\bullet$ and $\circ$ in $A\langle X\rangle$ are defined as follows:

$$\begin{cases} X_1 * X_2 = 0 \ \text{ if } X_1, X_2 \in \mathcal{M}, \deg(X_1), \deg(X_2) > 1 \text{ and } * \text{ is } \bullet \text{ or } \circ; \\ (x_{i_1} * (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots)) \circ x_j = x_j \circ (x_{i_1} * (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots)), \\ (x_{i_1} \circ (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots)) \bullet x_j = -x_j \bullet (x_{i_1} \circ (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots)), \\ \text{where } n \geq 3. \\ (x_{i_1} \bullet (\cdots \bullet (x_{i_k} \bullet (x_{i_{k+1}} \circ (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots))) \cdots)) \bullet x_j = -x_{i_1} \bullet (\cdots \bullet (x_l \\ \bullet (x_j \bullet (x_{l+1} \bullet (\cdots \bullet (x_{i_k} \bullet (x_{i_{k+1}} \circ (\cdots * (x_{i_{n-1}} * x_{i_n}) \cdots))) \cdots)))) \cdots), \\ \text{where } l \leq j \leq l+1 \text{ and } n \geq 3. \end{cases}$$

If the monomials X_1 and X_2 do not involve $\circ$ then we rewrite the product $X_1 \bullet X_2$ according to the multiplication table of free metabelian Lie algebras.

If the multiplications $\bullet$ and $\circ$ correspond to $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$, respectively, then by straightforward calculations, we can check that an algebra $A\langle X\rangle$ satisfies Jacobi identity and identities (7), (8), (9). It remains to note that $A\langle X\rangle \cong \mathrm{M\,Lie\text{-}adm}\langle X\rangle$, where $\mathrm{M\,Lie\text{-}adm}\langle X\rangle$ is a free metabelian Lie-admissible algebra. □

Calculating the dimension of operad M Lie by means of the package [6], we get the following result:

n	1	2	3	4	5	6	7
$\dim(\mathrm{M\,Lie\text{-}adm}(n))$	1	2	11	77	679	7184	88668

We see that the dimension of this operad is growing at a high rate, and this sequence does not coincide with any sequence from OEIS. In [22] was given the dimension of the Lie-admissible operad up to degree 7, which is

n	1	2	3	4	5	6	7
dim(Lie - adm(n))	1	2	11	101	1299	21484	434314

It will be interesting to find a general formula for the dimension of the metabelian Lie-admissible and Lie-admissible operads.

5 Symmetric polynomials of the free metabelian Lie-admissible algebra

For n, let us define the sequence $(*_{x_{i_1}}, *_{x_{i_2}}, \ldots, *_{x_{i_{n-1}}}, x_{i_n})$, where $*$ can be $\circ$ or $\bullet$. For each sequence, we define a space $(*_{x_{i_1}}, *_{x_{i_2}}, \ldots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\ldots,*)}$ as follows:

$$(*_{x_{i_1}}, *_{x_{i_2}}, \ldots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\ldots,*)} = \sum_{i_1,\ldots,i_n} x_{i_1} * (x_{i_2} * (\ldots (x_{i_{n-1}} * x_{i_n}) \ldots)).$$

Example 5.1.

$$(\bullet_{x_{i_1}}, \circ_{x_{i_2}}, x_{i_3})_{(\bullet,\circ)} = \sum_{i_1,i_2,i_3} x_{i_1} \bullet (x_{i_2} \circ x_{i_3}).$$

For each space $(*_{x_{i_1}}, *_{x_{i_2}}, \ldots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\ldots,*)}$, $n \geq 3$, we define a polynomial $p_{(*,*,\ldots,*,n)}$ as follows:

1) We divide this sequence into consecutive vertices of the same colour;

2) For k_i consecutive white vertices, we select k_i generators and write all possible permutations for them;

3) For k_i consecutive black vertices, we select k_i generating ones and write them in descending order;

4) If the sequence ends with k_i vertices of white colour, then for the selected generator we write all possible permutations of k_i generators so that the last two generators are always ordered;

5) If the sequence ends with k_i vertices of black colour, then for the selected generator we write symmetric polynomials of the free metabelian Lie algebra on k_{i+1} generators.

The sum of such monomials gives polynomial $p_{(*,*,\ldots,*,n)}$. For $n = 1, 2$, we set

$$p_{(1)} = x_1 + x_2 + \ldots + x_n \quad \text{and} \quad p_{(2)} = \sum_{i \neq j} x_i \circ x_j.$$

Example 5.2. For multilinear part of M Lie - adm$\langle X \rangle$, let us construct $(\bullet_{x_{i_1}}, \circ_{x_{i_2}}, x_{i_3})_{(\bullet,\circ)}$, $(\circ_{x_{i_1}}, \bullet_{x_{i_2}}, x_{i_3})_{(\bullet,\circ)}$ and $(\circ_{x_{i_1}}, \circ_{x_{i_2}}, x_{i_3})_{(\bullet,\circ)}$.

$$p_{(\bullet,\circ,3)} = x_1 \bullet (x_2 \circ x_3) + x_2 \bullet (x_1 \circ x_3) + x_3 \bullet (x_1 \circ x_2),$$

$$p_{(\circ,\bullet,3)} = x_1 \circ (x_2 \bullet x_3) + x_2 \circ (x_1 \bullet x_3) + x_3 \circ (x_1 \bullet x_2),$$

$$p_{(\circ,\circ,3)} = x_1 \circ (x_2 \circ x_3) + x_2 \circ (x_1 \circ x_3) + x_3 \circ (x_1 \circ x_2).$$

For $(\bullet_{x_{i_1}}, \bullet_{x_{i_2}}, \circ_{x_{i_3}}, x_{i_4})_{(\bullet,\bullet,\circ)}$, we have

$$p_{(\bullet,\bullet,\circ,3)} = x_2 \bullet (x_1 \bullet (x_3 \circ x_4)) + x_3 \bullet (x_1 \bullet (x_2 \circ x_4)) + x_4 \bullet (x_1 \bullet (x_2 \circ x_3)) + \\ x_3 \bullet (x_2 \bullet (x_1 \circ x_4)) + x_4 \bullet (x_2 \bullet (x_1 \circ x_3)) + x_4 \bullet (x_3 \bullet (x_1 \circ x_2)).$$

For each $p_{(*,*,\dots,*,n)}$, we replace $\bullet$ and $\circ$ to $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$, respectively. Finally, we obtain the following result:

Theorem 5.3. *For the multilinear part of the free metabelian Lie-admissible algebra, the symmetric polynomials have the form* $p_{(*,*,\dots,*,n)}$.

Proof. For $n = 1, 2$, the result is obvious. From the multiplication table of the free metabelian Lie-admissible algebra, one obtains

$$\mathrm{M\,Lie\text{-}adm}_{\geq 3}\langle X\rangle = \oplus_{(*,*,\dots,*)}(*_{x_{i_1}}, *_{x_{i_2}}, \dots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\dots,*)}$$

as a vector space, where $\mathrm{M\,Lie\text{-}adm}_{\geq 3}\langle X\rangle$ is a multilinear part of the free metabelian Lie-admissible algebra of degree greater than 2. For example, if $n = 3$ then

$$\mathrm{M\,Lie\text{-}adm}_3\langle X\rangle = (\bullet_{x_{i_1}}, \bullet_{x_{i_2}}, x_{i_3})_{(\bullet,\bullet)} \oplus (\bullet_{x_{i_1}}, \circ_{x_{i_2}}, x_{i_3})_{(\bullet,\circ)} \oplus \\ (\circ_{x_{i_1}}, \bullet_{x_{i_2}}, x_{i_3})_{(\circ,\bullet)} \oplus (\circ_{x_{i_1}}, \circ_{x_{i_2}}, x_{i_3})_{(\circ,\circ)}.$$

Moreover, each monomial of the space $(*_{x_{i_1}}, \dots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\dots,*)}$ under action S_n belongs to the same space. It remains to note that $p_{(*,*,\dots,*,n)}$ is a symmetric polynomial, and for each space $(*_{x_{i_1}}, *_{x_{i_2}}, \dots, *_{x_{i_{n-1}}}, x_{i_n})_{(*,*,\dots,*)}$ there is one unique symmetric polynomial which is $p_{(*,*,\dots,*,n)}$. □

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