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Sharp restriction theory

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Abstract. These are detailed notes for a lecture on "Sharp restriction theory" which I presented as part of my *Agregação em Matemática* in Instituto Superior Técnico, Lisboa, Portugal (9–10 February, 2023).

Contents

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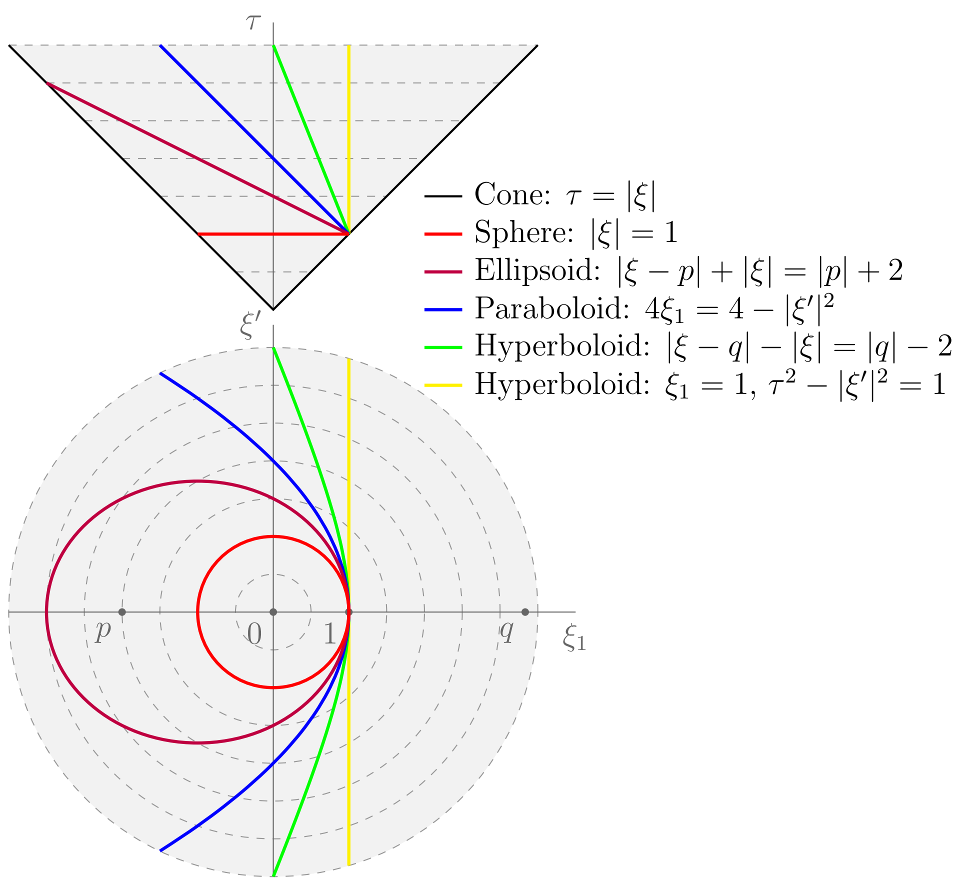

Figure 1: Spheres (§5), paraboloids (§6.2), cones (§6.3), and hyperboloids (§7) can all be obtained as conic sections.

1 Introduction

It has long been known that certain geometric properties related to curvature originate decay of the Fourier transform, and this phenomenon is explained by the behavior of oscillatory integrals. Given the long history of the subject, it is perhaps surprising that the possibility of restricting the Fourier transform to curved submanifolds of Euclidean space was not observed until the late 1960s.

A consequence of the classical Hausdorff–Young inequality in $\mathbb{R}^d$,

$$\|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} \lesssim_{d,p} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}, \text{ if } 1 \leq p \leq 2 \text{ and } \tfrac{1}{p} + \tfrac{1}{p'} = 1, \tag{1}$$

is that the Fourier transform of an L^p-function is defined almost everywhere in $\mathbb{R}^d$ if $1 \leq p \leq 2$. It is a striking observation of Stein from 1967 [126] that, for a special range of p's, the function $\widehat{f}$ can be meaningfully defined on *curved* submanifolds of Euclidean space. The primordial example of such a manifold is the unit sphere $\mathbb{S}^d = \{\omega \in \mathbb{R}^{d+1} : |\omega| = 1\}$, which serves as a model for quite general smooth compact submanifolds of nonvanishing (gaussian) curvature. The simple yet fundamental observation that *curvature causes the Fourier transform to decay* links geometry to analysis, and lies at the core of restriction theory. The celebrated *restriction conjecture* [125, Problem 2] predicts that

$$\|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{S}^d)} \lesssim_{d,p,q} \|f\|_{\mathrm{L}^{q'}(\mathbb{R}^{d+1})}, \text{ if } \tfrac{2}{q'} > \tfrac{d+2}{d+1} \text{ and } \tfrac{d+2}{q'} + \tfrac{d}{p'} \geq d+2, \tag{2}$$

and is remarkable in its numerous connections and applications. It exhibits deep links to Bochner–Riesz summation methods [61, 134] and to decoupling phenomena for the Fourier transform [25], which in turn underpin Bourgain–Demeter–Guth's breakthrough solution of the main conjecture in Vinogradov's mean value theorem [26]. The restriction conjecture is also known to imply the Kakeya conjecture [24] from geometric measure theory which, in its simplest form, predicts that any compact subset of $\mathbb{R}^d$ containing a unit line segment in each direction must have Hausdorff dimension d. Despite the great deal of attention that this circle of questions has received during the past half-century, the restriction conjecture remains an open problem in dimensions $d \geq 2$. Illuminating accounts of the restriction problem can be found in Tao's lecture notes [137] and Stovall's recent survey [131].

By duality, estimate (2) is equivalent to the adjoint restriction, or extension, inequality

$$\int_{\mathbb{R}^{d+1}} |\widehat{f\sigma}(x)|^q \mathrm{d}x \lesssim_{d,p,q} \|f\|^q_{\mathrm{L}^p(\mathbb{S}^d)}. \tag{3}$$

A complete answer for $p = 2$ is given by the celebrated Stein–Tomas inequality [127, 141], which establishes the restriction inequality (2) in the sharp range $q \geq 2 + \frac{4}{d}$. Stein–Tomas is very much related to Strichartz estimates for linear PDE of dispersion type. Let us illustrate this point in one particular instance, that of solutions $u(t, x)$ with $(t, x) \in \mathbb{R}^{1+d}$ to the Schrödinger equation

$$iu_t + \Delta u = 0 \tag{4}$$

with prescribed initial data. Strichartz [133] proved that

$$\|u\|_{\mathrm{L}^{2+\frac{4}{d}}(\mathbb{R}^{1+d})} \lesssim_d \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}, \tag{5}$$

if u is the solution of (4) satisfying $u(0,\cdot) = f$. It turns out that Strichartz estimates for the Schrödinger equation correspond to extension estimates on the paraboloid, an unbounded manifold which exhibits certain scale invariance properties that allow the reduction to the compact setup of the Stein–Tomas inequality.

This lecture focuses on maximizers and optimal constants for sharp forms of restriction and Strichartz-type inequalities. There are at least three reasons why sharp inequalities are worthy of investigation: they typically emerge from deep and beautiful proofs that reveal hidden structure and expose the main obstructions; they can be used to refine existing inequalities and lead to stable versions thereof; and they have striking applications to unexpected fields of mathematics, such as number theory [144], differential geometry [28], and additive combinatorics [79]. The following are natural questions, which can be posed in the particular case of restriction inequalities:

- What is the value of the optimal constant?
- Do maximizers exist?
  - If so, are they unique, possibly after applying the symmetries of the problem?
  - If not, what is the mechanism responsible for this lack of compactness?
- How do maximizing sequences behave?
- What are some qualitative properties of maximizers?
- What are necessary and sufficient conditions for a function to be a maximizer?

Questions of this kind have been asked in a variety of situations, and in the context of classical inequalities from Euclidean harmonic analysis they go back at least to the early work of Beckner [5] on the sharp Hausdorff–Young inequality, see §2, and of Lieb [94] on the sharp Hardy–Littlewood–Sobolev inequality. In comparison, sharp restriction inequalities have a relatively short history, with the first works on the subject going back to Ozawa–Tsutsumi [116], Kunze [92], Hundertmark–Zharnitsky [84], and Foschi [65]. These works concern the sharp form of (5) in the lower dimensional cases $d \in \{1, 2\}$. Being cases for which the Strichartz exponent $2 + \frac{4}{d}$ is an even integer, one can rewrite the left-hand side of (5) as an L^2-norm and invoke Plancherel in order to reduce the problem to a multilinear convolution estimate.

Sharp restriction theory is becoming increasingly more popular, as shown by the large body of work that appeared in the last decade, and in particular in the last few years. We mention a few interesting works that deal with sharp restriction theory on conics (see Figure 1), namely spheres [4, 8, 47, 48, 66, 69, 112, 124], paraboloids [9, 11, 37, 46, 72, 73, 75, 132, 139, 140, 146], cones [18, 19, 33, 118, 121], and hyperboloids [40, 54, 90, 114, 119].

Perturbations of these manifolds have been considered in [22, 59, 86, 87, 104, 106]. Sharp bilinear restriction theory is the subject of [14, 17, 85, 115], whereas other instances of sharp Strichartz inequalities [13, 56], sharp Sobolev–Strichartz inequalities [16, 60, 80] and sharp Airy–Strichartz inequalities [70, 83, 122] have been considered as well. Finally, we mention recent [68, 105] and very recent [103] surveys on sharp restriction theory which may be consulted for information complementary to that on this Introduction, and further references.

And so we begin.

2 Hausdorff–Young, sharp and sharpened

The symmetries of the Hausdorff–Young inequality (1) form the full affine group $\mathrm{Aff}(d)$. For an inequality to have such a high degree of symmetry is quite rare,[1] and this is tied to the fact that refinements of (1) tend to constitute formidable problems in mathematical analysis. In 1961, Babenko [2] found the sharp form of (1) whenever p' is an even integer. Beckner's landmark sharp Hausdorff–Young inequality [5] from 1975 states that (1) holds with optimal constant of the form $\mathbf{B}_p^d$, where $\mathbf{B}_p = p^{1/2p}q^{-1/2q}$ and $q = p' \geq 2$, and is saturated by gaussians. Fifteen years later, Lieb [95] went farther, proving that all maximizers are in fact gaussians. Much more recently, in 2014, Christ [45] obtained a stabler form of uniqueness of maximizers, and a sharper inequality: If $1 < p < 2$ and $d \geq 1$, then there exists $c = c(p, d) > 0$ such that, for any nonzero $f \in \mathrm{L}^p$,

$$\|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} \leq \left[\mathbf{B}_p^d - c\left(\frac{\mathrm{dist}_p(f, \mathfrak{G})}{\|f\|_p}\right)^2\right]\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}, \text{ where } \mathrm{dist}_p(f, \mathfrak{G}) := \inf_{G\in\mathfrak{G}} \|f - G\|_p. \quad (6)$$

Here $\mathfrak{G}$ denotes the set of gaussians $G(x) = a\exp(-Q(x)+x\cdot v)$, with $(a, v) \in \mathbb{C}\times\mathbb{C}^d$ and Q a positive definite real quadratic form. Christ's analysis [45] proceeds by contradiction, and its key step is a non-quantitative concentration-compactness result which relies on inverse theorems of Balog–Szemerédi and Freĭman from additive combinatorics. Consequently, the *stability constant* c in (6) is not obtained in explicit form, nor is it quantified in any way. It would be desirable to obtain some quantitative control over the stability constant in (6).

Figalli's 2014 ICM address [63] highlights the subtlety of constructive stability in an array of positive problems, including the Sobolev, isoperimetric, Gagliardo–Nirenberg, and Brunn–Minkowski inequalities; see also [34, 62]. In settings where symmetrization methods are unavailable, the difficulty in obtaining effective stability constants is illustrated in [55]. The Hausdorff–Young inequality often serves as a good toy model for the more sophisticated restriction inequalities which we consider in this lecture. Further model examples will be discussed in §4.

[1] For instance, the Sobolev inequality $\|f\|_{2^*} \lesssim_d \|\nabla f\|_2$, where $2^* = \frac{2d}{d-2}$, and the closely related isoperimetric inequality both remain invariant if f is replaced by $f \circ \phi$ where $\phi(x) = rR(x) + a$ for some $(R, r) \in \mathrm{SO}(d) \times (\mathbb{R} \setminus \{0\})$, but this is not the case for general $\phi \in \mathrm{Aff}(d)$ if $d > 1$.

3 Restriction theory

The connection between the geometric notion of *curvature* and the analytic notion of *Fourier decay* has been a powerful driving force for the development of Euclidean harmonic analysis in the last several decades. To further illustrate this point, let us consider the following three apparently unrelated questions.

Question 1. In what sense do Fourier series and Fourier integrals converge? This is a very classical question that lies at the heart of harmonic analysis. When considering the higher dimensional problem, one is naturally led to the analysis of the family of multipliers

$$\widehat{S_R^\delta(f)} = S_R^\delta \widehat{f},$$

where $S_R^\delta(\xi) := \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta$ and $x_+ = \max\{x, 0\}$ denotes the positive part of x.

Question 2. For which sets $\Sigma \subset \mathbb{R}^{d+1}$ and exponents p can the Fourier transform of a rough function $f \in \mathrm{L}^p(\mathbb{R}^{d+1})$ be meaningfully restricted to Σ? Asking this question for the unit sphere $\Sigma = \mathbb{S}^d$, one is naturally led to consider the operator

$$\mathcal{R}(f) := \widehat{f}|_{\mathbb{S}^d}. \tag{7}$$

Question 3. What is the smallest area needed to rotate a unit line segment by 180 degrees in the plane? In pursuing higher dimensional analogues of this question, one is naturally led to define the maximal function $f_\delta^* : \mathbb{S}^d \to \mathbb{R}$,

$$f_\delta^*(\omega) := \sup_{a \in \mathbb{R}^{d+1}} \frac{1}{|T_\omega^\delta(a)|} \int_{\mathbb{R}^{d+1}} |f|. \tag{8}$$

Here, $\delta > 0$, $\omega \in \mathbb{S}^{d-1}$, $a \in \mathbb{R}^d$, and $T_\omega^\delta(a)$ denotes the δ-neighborhood of the unit line segment in the direction of ω centered at a.

These questions immediately prompt the following natural problems.

Problem 3.1. For which p and $\delta > 0$ does the following inequality hold?

$$\|S_1^\delta(f)\|_{\mathrm{L}^p(\mathbb{R}^{d+1})} \lesssim_{d,p,\delta} \|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})} \tag{9}$$

Problem 3.2. For which p, q does the following inequality hold?

$$\|\mathcal{R}(f)\|_{\mathrm{L}^q(\mathbb{S}^d)} \lesssim_{d,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})} \tag{10}$$

Problem 3.3. For which p does the following inequality hold?

$$\forall \varepsilon > 0 \ \exists C_\varepsilon < \infty : \|f_\delta^*\|_{\mathrm{L}^p(\mathbb{S}^d)} \lesssim_{d,p} C_\varepsilon \delta^{-\varepsilon} \|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})} \tag{11}$$

Problem 3.1 is known as the *Bochner–Riesz problem*. By scale invariance, it is equivalent to the following question: For which exponents p and parameters $\delta > 0$ do the means $S_R^\delta f$ converge to f in the L^p norm as $R \to \infty$, for every $f \in \mathrm{L}^p(\mathbb{R}^{d+1})$? As $\delta \to 0^+$, the multiplier S_R^δ approaches the ball multiplier $\mathbf{1}_{B(0,R)}$, for which similar questions can be posed. In this case, a complete answer is known. In fact, Fefferman [61] famously disproved the *ball multiplier conjecture*. Both the nonvanishing curvature of the boundary of the ball, and elementary properties of Kakeya sets as described below, played a crucial role in Fefferman's proof. If $\delta > 0$, then the multiplier is smoother at the boundary of the ball. If $\delta < 0$, then the multiplier has an unbounded symbol, and in particular fails to preserve any L^p-space. Therefore a necessary condition for (9) to hold is that $\delta > 0$. Another necessary condition is

$$(d+1)\Big|\frac{1}{2} - \frac{1}{p}\Big| - \frac{1}{2} < \delta, \tag{12}$$

as follows from considering the stationary phase asymptotics of the convolution kernel $K_\delta = ((1 - |\cdot|^2)_+^\delta)^\vee$ of S_1^δ, which belongs to $\mathrm{L}^p(\mathbb{R}^{d+1})$ only if condition (12) is satisfied. For more information on the Bochner–Riesz problem, see [127] and [137, Lecture 5].

Problem 3.2 is known as the *restriction problem*. The Fourier transform of an L^1-function is uniformly continuous, and can therefore be restricted to any subset of $\mathbb{R}^{d+1}$. On the other hand, the Fourier transform of an L^2-function is again in L^2, and in view of Plancherel no better properties can be expected. The interesting question is then what happens for intermediate values of $1 < p < 2$. Necessary conditions for (10) turn out to be

$$1 \leq p < 2\frac{d+1}{d+2} \text{ and } q \leq \frac{dp'}{d+2}. \tag{13}$$

We will come back to this later on in the lecture.

Problem 3.3 is known as the *maximal Kakeya problem*. The name derives from its connection to the Kakeya problem, which we now briefly describe. A *Kakeya set* is a compact set $K \subset \mathbb{R}^{d+1}$ which contains a unit line segment in every direction. It has long been known that, perhaps unintuitively, there exist Kakeya sets in $\mathbb{R}^{d+1}$ with zero Lebesgue measure, as long as $d \geq 1$. In particular, given any $\varepsilon > 0$, there exists a planar set of area at most ε within which a unit line segment can be continuously rotated. A more refined question is whether the Hausdorff dimension of a Kakeya set in $\mathbb{R}^{d+1}$ must necessarily equal $d+1$. This is known to be true if $d = 1$ [52], and is an open problem if $d \geq 2$. An affirmative answer would follow from appropriate bounds for the Kakeya maximal function (8). From this definition, it is clear that (11) holds if $p = \infty$. If $p < \infty$, then there can be no bound of the form $\|f_\delta^*\|_{\mathrm{L}^q} \lesssim \|f\|_{\mathrm{L}^p}$ with a constant independent of δ, as can be seen by considering the indicator function of a δ-neighborhood of a zero measure Kakeya set. Moreover, (11) cannot hold for any $p < d+1$, as can easily be seen by considering the indicator function of the ball centered at the origin of radius δ. Further information on the Kakeya problem and its maximal functional variant can be found in [149, §10–§11].

The following conjectures summarize the expected answers to the problems we just discussed.

Conjecture 3.4 (Bochner–Riesz). *Necessary and sufficient conditions for* (9) *are*

$$\delta > 0 \text{ and } (d+1)\Big|\frac{1}{2} - \frac{1}{p}\Big| - \frac{1}{2} < \delta.$$

Conjecture 3.5 (Restriction). *Necessary and sufficient conditions for* (10) *are*

$$1 \leq p < 2\frac{d+1}{d+2} \text{ and } q \leq \frac{dp'}{d+2}.$$

Conjecture 3.6 (Maximal Kakeya). *Inequality* (11) *holds if* $p = d+1$.

All three conjectures have been given affirmative answers in the lowest dimensional case $d = 1$ a long time ago; see [36] (Bochner–Riesz), [150] (restriction), and [50] (maximal Kakeya). Despite tremendous effort, and very substantial partial progress, they are still open in dimensions $d \geq 2$. These conjectures are not unrelated. In fact, Conjecture 3.4 implies Conjecture 3.5, which in turn implies Conjecture 3.6. The first implication is the content of [134]. For a short proof of the second implication, see [149, §10]. Significant progress in this circle of problems has been accomplished by realizing that under appropriate conditions these implications can be partially reversed. For instance:

- restriction implies Bochner–Riesz for paraboloids [35];
- the *bush* [24] and *hairbrush* [147] constructions (which respectively imply lower bounds of $\frac{d+2}{2}$ and $\frac{d+3}{2}$ for the Hausdorff dimension of a Kakeya set in $\mathbb{R}^{d+1}$) were used to break the Stein–Tomas [127, 141] barrier of restriction exponents;
- elaborate versions of the *polynomial method* (originally used to settle the finite field Kakeya conjecture [57]) were recently employed in order to establish the current world record for restriction and Kakeya exponents in all remaining dimensions; see [76, 77, 81, 82, 145].

For further discussion of the deep connections between Conjectures 3.4, 3.5, 3.6, we refer the reader to the survey [135].

Before introducing the main topic of this lecture, let us consider the restriction problem in more detail. Historically, its starting point goes back to unpublished work of Stein in the late 1960s, see [127, p. 432], which culminated in the celebrated Stein–Tomas inequality:

Theorem 3.7 ([127, 141]). *Estimate* (10) *holds for* $q = 2$ *and* $1 \leq p \leq 2\frac{d+2}{d+4}$.

A few comments are in order. Firstly, the range of exponents is best possible in L^2, as there can be no $\mathrm{L}^p \to \mathrm{L}^2(\sigma)$ restriction estimates on the sphere if $p > 2\frac{d+2}{d+4}$. This is shown via the so-called *Knapp construction*, which amounts to testing the inequality dual to (10) on the indicator function of a small spherical cap. Secondly, the sphere $\mathbb{S}^d$ can be replaced by any smooth compact hypersurface in $\mathbb{R}^{d+1}$, as long as its gaussian curvature never vanishes. Some degree of curvature is essential, since a function supported on a hyperplane exhibits no Fourier decay in the orthogonal direction. However, nonvanishing gaussian curvature is a strong assumption that can be replaced by the nonvanishing of some principal curvatures, at the expense of decreasing the range of admissible exponents p.

Several routes towards Theorem 3.7 exist. If $\mathcal{R}$ denotes the restriction operator defined in (7), then its adjoint $\mathcal{R}^*$, known as the *extension operator*, is given by $\mathcal{R}^*(f) = \widehat{f\sigma}$, where the Fourier transform of the measure $f\sigma$ is defined as

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^d} f(\omega)e^{-ix\cdot\omega}\mathrm{d}\sigma(\omega) \quad (x \in \mathbb{R}^{d+1}). \tag{14}$$

The endpoint Stein–Tomas inequality corresponding to $(p,q) = (2\frac{d+2}{d+4}, 2)$ is equivalent to the extension estimate

$$\|\widehat{f\sigma}\|_{\mathrm{L}^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \lesssim \|f\|_{\mathrm{L}^2(\mathbb{S}^d)}. \tag{15}$$

If $f \in \mathrm{L}^2(\mathbb{S}^d)$, then the composition $(\mathcal{R}^* \circ \mathcal{R})(f)$ is well-defined, and given by

$$(\mathcal{R}^* \circ \mathcal{R})(f) = f * \widehat{\sigma}. \tag{16}$$

Since the operator norms satisfy $\|\mathcal{R}\|^2 = \|\mathcal{R}^*\|^2 = \|\mathcal{R}^* \circ \mathcal{R}\|$, the boundedness of these three operators is equivalent, and we may focus on $\mathcal{R}^* \circ \mathcal{R}$. In view of (16), boundedness of $\mathcal{R}^* \circ \mathcal{R}$ is only ensured if the Fourier transform $\widehat{\sigma}(x)$ exhibits some decay, as $|x| \to \infty$. In turn, this follows from the principle of stationary phase, since the nonvanishing gaussian curvature of the sphere translates into a nondegenerate Hessian for the phase function of the oscillatory integral given by (14) when $f \equiv 1$. This is the starting point for the original argument of Tomas [141], which worked only in the non-endpoint setting but was quickly extended to the endpoint $2 + \frac{4}{d}$ by embedding $\mathcal{R}^* \circ \mathcal{R}$ into an analytic family of operators and invoking Stein's complex interpolation theorem [127].

A second method to prove Stein–Tomas restriction-type estimates goes back to Ginibre–Velo [71]. It consists of introducing a time parameter and treating the extension operator as an evolution operator. Two key ingredients for this approach are the Hausdorff–Young inequality (1) and fractional integration in the form of the Hardy–Littlewood–Sobolev inequality.[2] These methods are more amenable to the needs of the PDE community, and allow to further treat the case of mixed norm spaces; see §4.2 below.

In the rare cases in which the dual exponent is an even integer, one can devise yet another proof of Stein–Tomas that comes from the world of bilinear estimates; see e.g. [67,91]. One rewrites the left-hand side of (15) as an L^2-norm, and appeals to Plancherel's theorem in order to reduce the problem to a multilinear convolution estimate. This simple strategy will be key to sharpening a number of restriction-type estimates, and we shall give several examples in the course of this lecture.

It has long been known that Stein–Tomas restriction-type estimates are related to Strichartz estimates for linear PDE of dispersive type. We illustrate this connection in the

[2] Interestingly, the restriction conjecture on $\mathbb{R}^2$ can be proved via a combination of Hausdorff–Young and Hardy–Littlewood–Sobolev; see [127, pp. 412–414].

context of the Schrödinger equation. The multiplier operator $e^{it\Delta}$ is defined as

$$\widehat{e^{it\Delta}f}(\xi) = e^{-it|\xi|^2}\widehat{f}(\xi), \tag{17}$$

for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$. It is easily seen that $e^{it\Delta}f(x)$ solves the Schrödinger equation $iu_t + \Delta u = 0$ with initial datum $u(0,\cdot) = f$. The Schrödinger propagator $e^{it\Delta}$ extends to a bounded operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^p_t\mathrm{L}^q_x(\mathbb{R}\times\mathbb{R}^d)$ if and only if the triplet (d,p,q) is *Schrödinger-admissible*. This is the content of the following result.

Theorem 3.8 ([88, 133]). *There exists $C = C_{d,p,q} < \infty$ for which*

$$\|e^{it\Delta}f\|_{\mathrm{L}^p_t\mathrm{L}^q_x(\mathbb{R}\times\mathbb{R}^d)} \le C\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}, \tag{18}$$

for every $f \in \mathrm{L}^2(\mathbb{R}^d)$, if and only if

$$p, q \ge 2, \;\; (d,p,q) \neq (2,2,\infty), \;\; \frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{19}$$

The diagonal case $p = q = 2 + \frac{4}{d}$ is due to Strichartz [133], who followed the original arguments of Tomas and Stein, whereas the case $p \neq q$ is fully treated in Keel–Tao [88]. A hint that the diagonal case of Theorem 3.8 might be related to Theorem 3.7 comes from the numerology of the exponents: The Strichartz exponent $2 + \frac{4}{d}$ coincides with the dual endpoint Stein–Tomas exponent. It turns out that Strichartz estimates for the Schrödinger equation correspond to extension estimates on the paraboloid (which is not a compact hypersurface, but by scale invariance it can be treated as such).

The question of what happens to (10) for $q < 2$ is the starting point of the Restriction Conjecture 3.5, which we have already briefly addressed. The necessity of conditions (13) follows from dimensional analysis and Knapp-type examples. Note that the endpoints of this relation are the trivial case $(p,q) = (1,\infty)$, and the case in which $p, q \to 2\frac{d+1}{d+2}$, as depicted in Figure 2 below.

Tools that have led to progress on the restriction conjecture include, in addition to the previously mentioned ones, local restriction estimates, wave packet decompositions, and induction-on-scale arguments. These methods played a prominent role in the proof of Tao's sharp bilinear restriction estimate for paraboloids [136], a deep result which in turn implied further progress on the linear restriction problem. Bennett–Carbery–Tao [15] then established almost sharp *multilinear* restriction estimates together with their multilinear Kakeya counterparts. Surprisingly, at a higher level of multilinearity they turn out to be basically equivalent. Multilinear restriction estimates allowed Bourgain–Guth [27] to make further progress on the restriction problem, and played a prominent role in the seminal work of Bourgain–Demeter [25] on ℓ^2 decoupling, ultimately leading to Bourgain–Demeter–Guth's breakthrough solution of the main conjecture in Vinogradov's mean value theorem [26]. Guth [76, 77] applied these multilinear tools together with the polynomial method to get some further progress on the restriction problem; see [81, 145] for the latest developments along these lines, together with Guth's 2022 plenary ICM address [78].

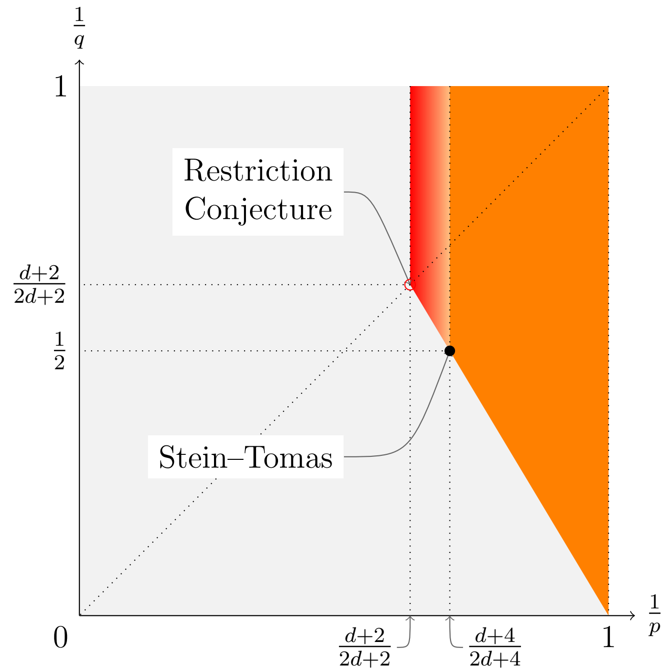

Figure 2: L^p–$\mathrm{L}^q(\sigma)$ boundedness of the restriction operator to $\mathbb{S}^d$.

Inequalities like (10), (15), (18) can be sharpened in a number of ways. In order to clarify what *sharp* means for the purpose of our discussion, we focus on the restriction inequality (10), the adaptation to other cases being straightforward. This inequality can be rewritten as

$$\|\mathcal{R}f\|_{\mathrm{L}^q(\mathbb{S}^d)} \leq \mathbf{C}_{d,p,q}\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})}, \tag{20}$$

with $\mathbf{C}_{d,p,q}$ being the *optimal* – or *best* – constant, defined as

$$\mathbf{C}_{d,p,q} = \sup_{0\neq f\in \mathrm{L}^p} \frac{\|\mathcal{R}f\|_{\mathrm{L}^q(\mathbb{S}^d)}}{\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})}}.$$

A *maximizer* for inequality (20) is a nonzero function $f \in \mathrm{L}^p(\mathbb{R}^{d+1})$ which satisfies

$$\|\mathcal{R}f\|_{\mathrm{L}^q(\mathbb{S}^d)} = \mathbf{C}_{d,p,q}\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})}.$$

A *maximizing sequence* for inequality (20) is a sequence of functions $\{f_n\} \subset \mathrm{L}^p(\mathbb{R}^{d+1})$ satisfying $\|f_n\|_p \leq 1$, such that

$$\|\mathcal{R}f_n\|_{\mathrm{L}^q(\mathbb{S}^d)} \to \mathbf{C}_{d,p,q}, \text{ as } n \to \infty.$$

Several natural questions which fall under the category of *sharp restriction theory*, and which we will address in this lecture, include: What is the value of the optimal constant? Is it attained? If so, what is an example of a maximizer? If not, what is the mechanism responsible for this lack of compactness? In either case, how do maximizing sequences behave? If maximizers exist, are they unique modulo the symmetries of the problem?

An ideal setting to explore these questions is that of the Stein–Tomas inequality in low dimensions $d \in \{1, 2\}$, and its Strichartz counterpart. The endpoint exponent $2 + \frac{4}{d}$ is then an even integer, and the aforementioned convolution trick transforms the original oscillatory problem into a question of geometric integration over a concrete manifold.

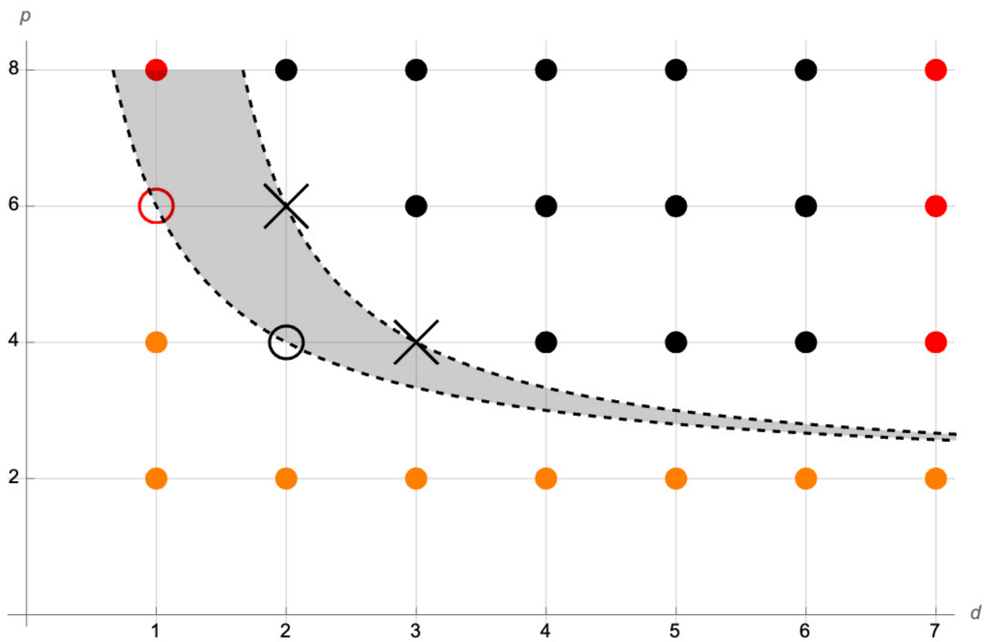

Figure 3: On the horizontal axis, dimension d of $\mathbb{S}^d$. On the vertical axis, Lebesgue exponent p of the $\mathrm{L}^2(\sigma) - \mathrm{L}^p$ extension inequality. Circles ($\circ$) correspond to endpoint Stein–Tomas for spheres and Stein–Tomas for paraboloids, while crosses ($\times$) correspond to Stein–Tomas for cones. The shaded region between the two dashed curves corresponds to hyperboloids. For spheres, black entries correspond to inequalities which are maximized by constants, while red entries correspond to inequalities which are conjecturally maximized by constants; orange entries correspond to situations in which Stein–Tomas does not hold, but other replacement inequalities such as Agmon–Hörmander may be available.

We will see several illustrative examples in the remainder of this lecture, which is organized as follows (see also Figure 3):

- In §4, we present two simple inequalities for the spherical extension operator which we have recently put in sharp/sharpened form [43, 101];
- In §5, we consider spherical Stein–Tomas, and discuss some of our results related to existence [98] and characterization [103, 109, 110] of maximizers, also in the recently introduced weighted setting [39];

- In §6, we focus on paraboloids and cones. In §6.1, we discuss sharp Strichartz inequalities for the Schrödinger and wave equations. In §6.2, we describe our work [30, 108] leading to a powerful *geometric comparison principle* for convolution measures which culminated in the resolution of a dichotomy of Jiang–Pausader–Shao [86]. In §6.3, we present some current ongoing work [102] which in particular establishes the conic analogue of Christ–Quilodrán [46].

- In §7, we discuss hyperboloic sharp restriction in the low [42] and higher [44] dimensional settings. We conclude with some recent progress on the restriction conjecture for the hyperboloic hyperboloid [32].

Along the way we will single out several challenging open problems and comment on possible directions of future research.

4 The inequalities of Agmon–Hörmander and Vega

In this section, we consider two simple estimates for the spherical extension operator which, in non-sharp form, were first established by Agmon–Hörmander [1] and Vega [143] in 1976 and 1988, respectively.

4.1 Agmon–Hörmander

We describe a simple estimate for the extension operator on $\mathbb{S}^d$ which, perhaps surprisingly, is *not always* maximized by constants. For simplicity we restrict attention to the circle $\mathbb{S}^1$, even though we have recently proved analogous results in all dimensions $d \geq 1$; see [101].

Our starting point is the Agmon–Hörmander estimate on the circle,

$$\frac{1}{R}\int_{B_R} |\widehat{f\sigma}(x)|^2 \frac{\mathrm{d}x}{(2\pi)^2} \leq \mathbf{A}_R \int_{\mathbb{S}^1} |f(\omega)|^2 \mathrm{d}\sigma(\omega), \tag{21}$$

where $B_R \subset \mathbb{R}^2$ denotes a ball of arbitrary radius $R > 0$ centered at the origin and σ stands for the usual arclength measure on $\mathbb{S}^1$. Agmon–Hörmander [1] observed that estimate (21) holds with a constant $\mathbf{A}_R$ that approaches $\frac{1}{\pi}$, as $R \to \infty$, but did not investigate its optimal value. The latter is described in terms of the auxiliary quantities

$$\Lambda_R^k := \frac{R}{2} J_k^2(R) - \frac{R}{2} J_{k-1}(R) J_{k+1}(R) \tag{22}$$

in Theorem 4.1 below, where J_n denotes the usual Bessel function. The following result on the optimal constant and the maximizers of (21) holds; see also Figure 4.

Theorem 4.1 ([101]). *For each $R > 0$,*

$$\mathbf{A}_R = \begin{cases} \Lambda_R^0 & \textit{if } (J_0 J_1)(R) \geq 0 \\ \Lambda_R^1 & \textit{if } (J_0 J_1)(R) \leq 0 \end{cases}$$

The corresponding space of maximizers is given by

$$\mathcal{M}_R = \begin{cases} \mathcal{H}_0 & \text{if } (J_0J_1)(R) > 0 \\ \mathcal{H}_1 & \text{if } (J_0J_1)(R) < 0 \\ \mathcal{H}_0 \oplus \mathcal{H}_1 & \text{if } J_0(R) = 0 \\ \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 & \text{if } J_1(R) = 0 \end{cases}$$

where $\mathcal{H}_k \subset \mathrm{L}^2(\mathbb{S}^1)$ denotes the vector space of degree k circular harmonics.

One remarkable feature of Theorem 4.1 is that constants are seen to not always be maximizers, even though they are the unique functions which are invariant under the full rotational symmetry group of (21). In this way Theorem 4.1 identifies an instance of *symmetry breaking*, which to the best of our knowledge had not yet been observed for an estimate involving the spherical extension operator.

The next natural question concerns the *stability* of inequality (21), which can be phrased in terms of lower bounds for the following *deficit functional*:

$$\delta_R[f] := \mathbf{A}_R \|f\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \frac{1}{R}\int_{B_R} |\widehat{f\sigma}(x)|^2 \frac{\mathrm{d}x}{(2\pi)^2}.$$

Clearly, $\delta_R[f] \geq 0$ for every $f \in \mathrm{L}^2(\mathbb{S}^1)$ but, in the spirit of Bianchi–Egnell [21], more can be said; see also Figure 4, and recall that the space $\mathcal{M}_R$ has been defined in Theorem 4.1.

Theorem 4.2 ([101]). *The following sharp two-sided inequality holds:*

$$\mathbf{S}_R \operatorname{dist}^2(f, \mathcal{M}_R) \leq \delta_R[f] \leq \mathbf{A}_R \operatorname{dist}^2(f, \mathcal{M}_R) \tag{23}$$

Equality occurs in the right-hand side inequality of (23) if and only if $f \in \mathcal{M}_R$. Equality occurs in the left-hand side inequality of (23) if and only if $f \in \mathcal{M}_R \oplus \mathcal{E}_R$, where:

$\mathbf{S}_R$	$\mathcal{E}_R$	Conditions
$\Lambda_R^0 - \Lambda_R^1$	$\mathcal{H}_1$	$(J_0J_1)(R) > 0$ *and* $(J_1J_2)(R) > 0$
$=$	$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$	$(J_0J_1)(R) > 0$ *and* $(J_1J_2)(R) = 0$
$\Lambda_R^0 - \Lambda_R^2$	$\mathcal{H}_2$	$(J_0J_1)(R) > 0$ *and* $(J_1J_2)(R) < 0$
$\Lambda_R^1 - \Lambda_R^0$	$\mathcal{H}_0$	$(J_0J_1)(R) < 0$ *and* $(J_0J_1 + J_1J_2 + J_2J_3)(R) > 0$
$=$	$\mathcal{H}_0 \oplus \mathcal{H}_3$	$(J_0J_1)(R) < 0$ *and* $(J_0J_1 + J_1J_2 + J_2J_3)(R) = 0$
$\Lambda_R^1 - \Lambda_R^3$	$\mathcal{H}_3$	$(J_0J_1)(R) < 0$ *and* $(J_0J_1 + J_1J_2 + J_2J_3)(R) < 0$
$\Lambda_R^0 - \Lambda_R^2$	$\mathcal{H}_2$	$J_0(R) = 0$ *and* $(J_2J_3)(R) > 0$
$\Lambda_R^0 - \Lambda_R^3$	$\mathcal{H}_3$	$J_0(R) = 0$ *and* $(J_2J_3)(R) < 0$
$\Lambda_R^0 - \Lambda_R^3$	$\mathcal{H}_3$	$J_1(R) = 0$ *and* $(J_3J_4)(R) > 0$
$\Lambda_R^0 - \Lambda_R^4$	$\mathcal{H}_4$	$J_1(R) = 0$ *and* $(J_3J_4)(R) < 0$

The proofs of Theorems 4.1 and 4.2 rely on two observations. Firstly, by orthogonality of the circular harmonic decomposition $f = \sum_{k\geq 0} Y_k$, we have that

$$\frac{1}{R}\int_{B_R} |\widehat{f\sigma}(x)|^2 \frac{\mathrm{d}x}{(2\pi)^2} = \sum_{k\geq 0} \Lambda_R^k \|Y_k\|^2_{\mathrm{L}^2},$$

where the Λ_R^k have been defined in (22). Secondly, $\mathbf{A}_R = \sup_{k\geq 0} \Lambda_R^k$, where f attains the supremum if and only if $f = \sum Y_{k_j}$, for some $k_j \in \{k \geq 0 : \Lambda_R^k = \sup_h \Lambda_R^h\}$; see also [20]. In our case of interest, we can conveniently rewrite the Λ_R^k in integral form,

$$\Lambda_R^k = \frac{1}{R}\int_0^R J_k^2(r) r \mathrm{d}r,$$

and invoke certain well-known Bessel recursions to start gaining control on both extremal problems corresponding to Theorems 4.1 and 4.2.

The above explicit expressions for the optimal constant $\mathbf{A}_R$ and the stability constant $\mathbf{S}_R$ lead to the following *loss-of-regularity* statements, which have been recently observed in the related setting of the Brascamp–Lieb inequalities [10, 12]. Firstly, $\mathbf{A}_R$ is not a differentiable function of R at each positive zero of $J_0 J_1$; it defines a Lipschitz function on $(0, \infty)$ which is real-analytic between any two consecutive zeros of $J_0 J_1$. Secondly, $\mathbf{S}_R$ has a jump discontinuity at each positive zero of $J_0 J_1$; it defines a piecewise real-analytic function of R between any two consecutive zeros of $J_0 J_1$ which fails to be differentiable at each positive zero of J_2; see [101, Cor. 3].

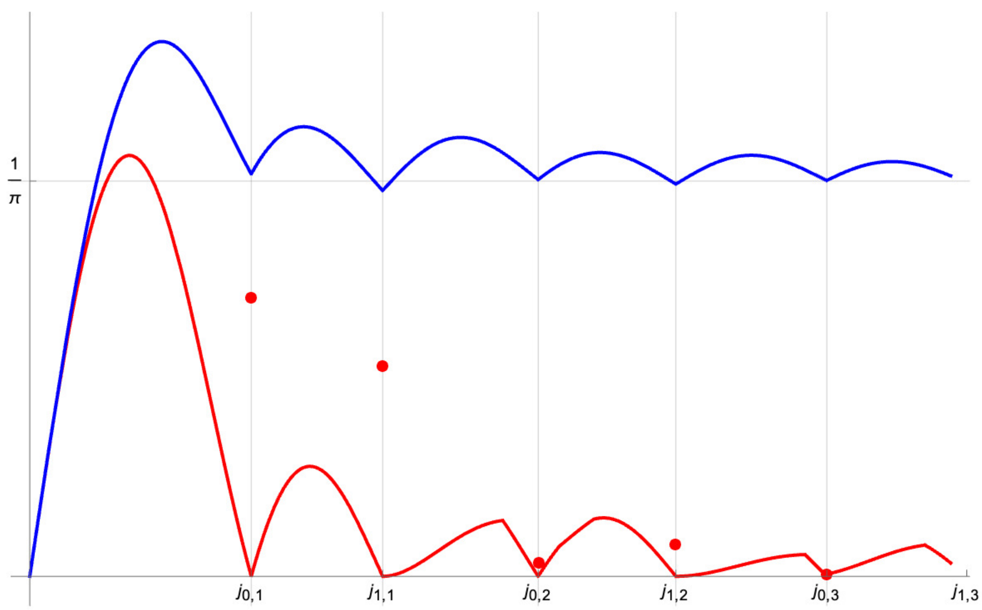

Figure 4: Optimal constant $\mathbf{A}_R$ (blue) and stability constant $\mathbf{S}_R$ (red) for the Agmon–Hörmander estimate on the circle when $0 < R < 10$.

4.2 Vega

If we merely require $\widehat{f\sigma} \in \mathrm{L}^6_{\text{rad}}\mathrm{L}^2_{\text{ang}}(\mathbb{R}^2)$ (weakening the assumption on the target space in (15)), we obtain the following sharp inequality, first noticed in [68, §3.2]:

$$\|\widehat{f\sigma}\|_{\mathrm{L}^6_{\text{rad}}\mathrm{L}^2_{\text{ang}}(\mathbb{R}^2)} \leq \left(\int_0^\infty J_0^6(r) r \mathrm{d}r\right)^{\frac{1}{6}} \|f\|_{\mathrm{L}^2(\mathbb{S}^1)}. \tag{24}$$

By contrast to §4.1, constants are the unique nonnegative maximizers of (24). We reproduce the short proof here. The starting point is the well-known formula

$$\widehat{Y_k\sigma}(x) = i^k J_{\frac{d-2}{2}+k}(|x|)|x|^{\frac{2-d}{2}} Y_k\left(\frac{x}{|x|}\right), \quad (x \in \mathbb{R}^d) \tag{25}$$

which provides a connection between spherical harmonics and Bessel functions in all dimensions. It can be proved by applying the Funk–Hecke formula together with Rodrigues' formula for Gegenbauer polynomials; see [129]. Given $f \in \mathrm{L}^2(\mathbb{S}^1)$, we Fourier expand it, $f = \sum_{k\geq 0} a_k Y_k$, with each $\|Y_k\|_{\mathrm{L}^2} = 1$. Appealing to identity (25) and to the orthogonality of the $\{Y_k\}_{k\geq 0}$,

$$\int_{\mathbb{S}^1} |\widehat{f\sigma}(r\omega)|^2 \mathrm{d}\sigma(\omega) = \sum_{k\geq 0} |a_k|^2 J_k^2(r) r.$$

The left-hand side of inequality (24) is thus equal to

$$\int_0^\infty \Big(\sum_{k\geq 0} |a_k|^2 J_k^2(r)\Big)^3 r\mathrm{d}r, \tag{26}$$

which can be rewritten as

$$\sum_{k,\ell,m\geq 0} |a_k|^2|a_\ell|^2|a_m|^2 I(k,\ell,m), \text{ with } I(k,\ell,m) := \int_0^\infty J_k^2(r)J_\ell^2(r)J_m^2(r) r\mathrm{d}r.$$

The integrals $I(k,\ell,m)$ satisfy a crucial monotonicity property,

$$I(k,\ell,m) \leq I(0,0,0), \tag{27}$$

with equality if and only if $k = \ell = m = 0$. To check this, note that

$$I(k,\ell,m) = \int_{\mathbb{R}^2} \widehat{e_k\sigma}\overline{\widehat{e_k\sigma}}\widehat{e_\ell\sigma}\overline{\widehat{e_\ell\sigma}}\widehat{e_m\sigma}\overline{\widehat{e_m\sigma}}\mathrm{d}x = \int_{(\mathbb{S}^1)^6} (\omega_1\overline{\omega_2})^k(\omega_3\overline{\omega_4})^\ell(\omega_5\overline{\omega_6})^m \mathrm{d}\Sigma(\vec{\omega}),$$

where the function $e_n : \mathbb{S}^1 \to \mathbb{C}$ is defined via $e_n(\omega) = \omega^n$, and the measure Σ is given by

$$\mathrm{d}\Sigma(\vec{\omega}) = \boldsymbol{\delta}\Big(\textstyle\sum_{j=1}^6 \omega_j\Big) \displaystyle\prod_{j=1}^6 \mathrm{d}\sigma(\omega_j). \tag{28}$$

The triangle inequality immediately implies $I(k,\ell,m) \leq I(0,0,0)$, with equality if and only

$$(\omega_1\overline{\omega_2})^k(\omega_3\overline{\omega_4})^\ell(\omega_5\overline{\omega_6})^m = 1,$$

for every $\omega_1,\ldots,\omega_6 \in \mathbb{S}^1$ for which $\sum_{j=1}^6 \omega_j = 0$. It follows that $k = \ell = m = 0$, and the proof of (27) is complete. As a consequence, we obtain the sharp inequality (24) at once:

$$\begin{aligned}\|\widehat{f\sigma}\|^6_{\mathrm{L}^6_{\mathrm{rad}}\mathrm{L}^2_{\mathrm{ang}}(\mathbb{R}^2)} &= \sum_{k,\ell,m\geq 0} |a_k|^2|a_\ell|^2|a_m|^2 I(k,\ell,m) \\ &\leq I(0,0,0) \sum_{k,\ell,m\geq 0} |a_k|^2|a_\ell|^2|a_m|^2 = I(0,0,0)\|f\|^6_{\mathrm{L}^2(\mathbb{S}^1)}.\end{aligned}$$

The characterization of maximizers is now straightforward.

The simple argument presented above admits a vast generalization which we developed in [43]. Let $d \geq 2$ be an integer and let $2d/(d-1) < q \leq \infty$. In his doctoral thesis, Vega [143] proved the following inequality:

$$\|\widehat{f\sigma}\|_{\mathrm{L}^q_{\mathrm{rad}}\mathrm{L}^2_{\mathrm{ang}}(\mathbb{R}^d)} := \left(\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} \left|\widehat{f\sigma}(r\omega)\right|^2 \mathrm{d}\sigma(\omega)\right)^{\frac{q}{2}} r^{d-1}\,\mathrm{d}r\right)^{\frac{1}{q}} \leq \mathbf{V}_{d,q}\,\|f\|_{\mathrm{L}^2(\mathbb{S}^{d-1})}. \quad (29)$$

The example $f \equiv 1$ shows that the requirement $q > 2d/(d-1)$ is necessary for this mixed norm extension inequality to hold. An alternative proof of (29) was recently given by Córdoba [51]. With a view towards putting (29) in sharp form, we define, for each $k \in \{0, 1, 2, \ldots\}$, the Bessel integral

$$\Lambda_{d,q}(k) := \left(\int_0^\infty \left|r^{1-\frac{d}{2}}\, J_{\frac{d}{2}-1+k}(r)\right|^q r^{d-1}\,\mathrm{d}r\right)^{\frac{1}{q}}. \quad (30)$$

Theorem 4.3 ([43]). *Let $d \geq 2$ and $2d/(d-1) < q \leq \infty$. The following statements hold.*

(i) The sequence $\{\Lambda_{d,q}(k)\}_{k\geq 0}$ satisfies $\lim_{k\to\infty}\Lambda_{d,q}(k) = 0$ and we have

$$\mathbf{V}_{d,q} = (2\pi)^{d/2} \max_{k\geq 0} \Lambda_{d,q}(k).$$

Moreover, f is a maximizer of (29) if and only if $f \in \mathcal{H}^d_\ell$, where ℓ is such that

$$\Lambda_{d,q}(\ell) = \max_{k\geq 0} \Lambda_{d,q}(k).$$

(ii) If q is an even integer or $q = \infty$, then

$$\mathbf{V}_{d,q} = (2\pi)^{d/2}\Lambda_{d,q}(0),$$

and the constant functions are the unique maximizers of (29).

(iii) For a fixed d, the set $\mathcal{A}_d = \{q \in (2d/(d-1), \infty] : \Lambda_{d,q}(0) > \Lambda_{d,q}(k) \text{ for all } k \geq 1\}$, for which the constant functions are the unique maximizers of (29), is an open set in the extended topology.

From Theorem 4.3 it is clear that, as in §4.1, the crux of the matter boils down to a hierarchy between the Bessel integrals (30), a nontrivial question of independent interest in the theory of special functions. Numerical simulations indicate that $\Lambda_{d,q}(0)$ is a good candidate to be the largest, and we were able to prove this for all even q. An interesting feature of our proof for $q \in 2\mathbb{N}$ is that it takes a detour from the Bessel world [111], rewriting these integrals using spherical harmonics and relying on a decisive application of *delta calculus*[3] and the theory of orthogonal polynomials. On the other hand, Theorem

[3] See [68, Appendix A] for a short introduction to delta calculus.

4.3 (iii) implies the existence of a neighborhood of exponents around each even integer for which constants continue to be global maximizers. This breaks for the first time the even integer barrier in sharp restriction theory. Moreover, $\mathcal{A}_d$ contains a neighborhood of the point at infinity which, in particular, leads us to define

$$q_0(d) := \inf\{r :\ 2d/(d-1) < r < \infty \text{ and } (r,\infty] \subset \mathcal{A}_d\}. \tag{31}$$

In [43, Theorem 3] we proved that

$$q_0(d) \leq \left(\tfrac{1}{2} + o(1)\right) d \log d. \tag{32}$$

In low dimensions we have the following explicit bounds

$$\begin{array}{lllll} q_0(2) \leq 6.76\ ; & q_0(3) \leq 5.45\ ; & q_0(4) \leq 5.53\ ; & q_0(5) \leq 6.07\ ; & q_0(6) \leq 6.82\ ; \\ q_0(7) \leq 7.70\ ; & q_0(8) \leq 8.69\ ; & q_0(9) \leq 9.78\ ; & q_0(10) \leq 10.95\ ; & \end{array}$$

see also the recent improvements in [49]. It remains an interesting open problem to find a quantitative improvement for (32).

5 Sphere

In this section, we discuss a number of our recent results in sharp spherical restriction. In §5.1, we explain our path towards the unconditional existence of endpoint Stein–Tomas maximizers when additional symmetries are present; in §5.2, we revisit Foschi's argument [66] for the sharp endpoint Stein–Tomas inequality on $\mathbb{S}^2$; in §5.3–§5.5, we describe a bootstrapping procedure in order to fully characterize the real-valued maximizers of $\mathrm{L}^2(\sigma) \to \mathrm{L}^{2k}$ extension inequalities on $\mathbb{S}^d$, $2 \leq d \leq 6$, $k \geq 3$; in §5.6, we address complex-valued maximizers for the same inequalities; in §5.7, we present a sharp extension inequality on $\mathbb{S}^7$.

5.1 Existence

Maximizers for the Stein–Tomas inequality on $\mathbb{S}^d$,

$$\|\widehat{f\sigma}\|_{\mathrm{L}^q(\mathbb{R}^{d+1})} \leq \mathbf{T}_{d,q}\|f\|_{\mathrm{L}^2(\mathbb{S}^d)}, \text{ if } q \geq 2 + \tfrac{4}{d}; \quad \mathbf{T}_{d,q} := \sup_{\mathbf{0}\neq f \in \mathrm{L}^2} \frac{\|\widehat{f\sigma}\|_{\mathrm{L}^q(\mathbb{R}^{d+1})}}{\|f\|_{\mathrm{L}^2(\mathbb{S}^d)}}, \tag{33}$$

are known to exist in the following cases:

- non-endpoint $q > 2 + 4/d$ [59];
- endpoint $(d,q) = (2,4)$ [47] and $(d,q) = (1,6)$ [124];
- endpoint $(d, 2+4/d), d \geq 3$, *conditionally* on Lieb's Conjecture 6.1 [69].

In the non L^2-setting, we refer to the very recent [64] for further existence results.

In [98], we prove new restriction estimates to $\mathbb{S}^d$ on the class of $O(d-k+1)\times O(k)$-symmetric functions, for every $d\geq 3$ and $2\leq k\leq d-1$, and consequently establish the *unconditional* existence of maximizers for the endpoint Stein–Tomas inequality within that class. We also construct examples showing that the range of Lebesgue exponents in our estimates is sharp. Let us make this more precise.

Given a subgroup $G\subset O(d+1)$ of the orthogonal group, a function $f:\mathbb{R}^{d+1}\to\mathbb{C}$ is said to be *G-symmetric in* $\mathbb{R}^{d+1}$ if $f\circ A=f$ holds for every $A\in G$. An especially interesting situation arises when considering the subgroup $G_k:=O(d-k+1)\times O(k)$ for some $k\in\{0,1,\dots,d+1\}$. We are interested in restriction estimates to the unit sphere,

$$\left(\int_{\mathbb{S}^d}|\widehat{f}(\omega)|^q\,\mathrm{d}\sigma(\omega)\right)^{\frac{1}{q}}\leq C(k,d,p,q)\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})},\tag{34}$$

which hold in the class of G_k-symmetric functions, and are led to define the Banach space

$$\mathrm{L}^p_k(\mathbb{R}^{d+1}):=\{f\in\mathrm{L}^p(\mathbb{R}^{d+1}):f\text{ is }G_k\text{-symmetric}\}.$$

The cases $k\in\{0,d+1\}$ correspond to radial functions on $\mathbb{R}^{d+1}$. If $f\in\mathrm{L}^p(\mathbb{R}^{d+1})$ is radial, then $\widehat{f}$ is continuous on $\mathbb{R}^{d+1}\setminus\{0\}$ whenever $1\leq p<2\frac{d+1}{d+2}$; see [128, Prop. 5.1]. In particular, inequality (34) holds for radial functions provided $1\leq p<2\frac{d+1}{d+2}$ and $1\leq q\leq\infty$, and this range is optimal. Thus, the L^p–L^q mapping properties of the restriction operator in the radial cases $k\in\{0,d+1\}$ are completely understood. The cases $k\in\{1,d\}$ are likewise special. Since Knapp's construction in $\mathbb{R}^{d+1}$ is rotationally invariant with respect to d variables, G_k-symmetry does not allow for a larger range of Lebesgue exponents on which restriction estimates can hold when $k\in\{1,d\}$. For this reason, we focus on the situation when $k\in\{2,3,\dots,d-1\}$ and present the following main result; see also Figure 5, which should be compared with Figure 2.

Theorem 5.1 ([98]). *Let $d\geq 3, k\in\{2,3,\dots,d-1\}$, and $m:=\min\{d-k+1,k\}$. Then*

$$\left(\int_{\mathbb{S}^d}|\widehat{f}(\omega)|^2\,\mathrm{d}\sigma(\omega)\right)^{\frac{1}{2}}\leq C(k,d,p)\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})}\tag{35}$$

holds for every G_k-symmetric function $f:\mathbb{R}^{d+1}\to\mathbb{C}$ if $1\leq p\leq 2\frac{d+m+1}{d+m+3}$.

Given that $2\frac{d+m+1}{d+m+3}$ is strictly larger than the Stein–Tomas exponent $2\frac{d+2}{d+4}$, Theorem 5.1 improves upon (33). This result is new whenever $d\geq 3$, and leads to the precompactness of maximizing sequences for the constrained optimization problem

$$\mathbf{T}_{d,k}(p):=\sup_{0\neq f\in\mathrm{L}^p_k(\mathbb{R}^{d+1})}\frac{\left(\int_{\mathbb{S}^d}|\widehat{f}(\omega)|^2\mathrm{d}\sigma(\omega)\right)^{\frac{1}{2}}}{\|f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})}},$$

and consequently to the unconditional existence of maximizers for $\mathbf{T}_{d,k}(p)$.

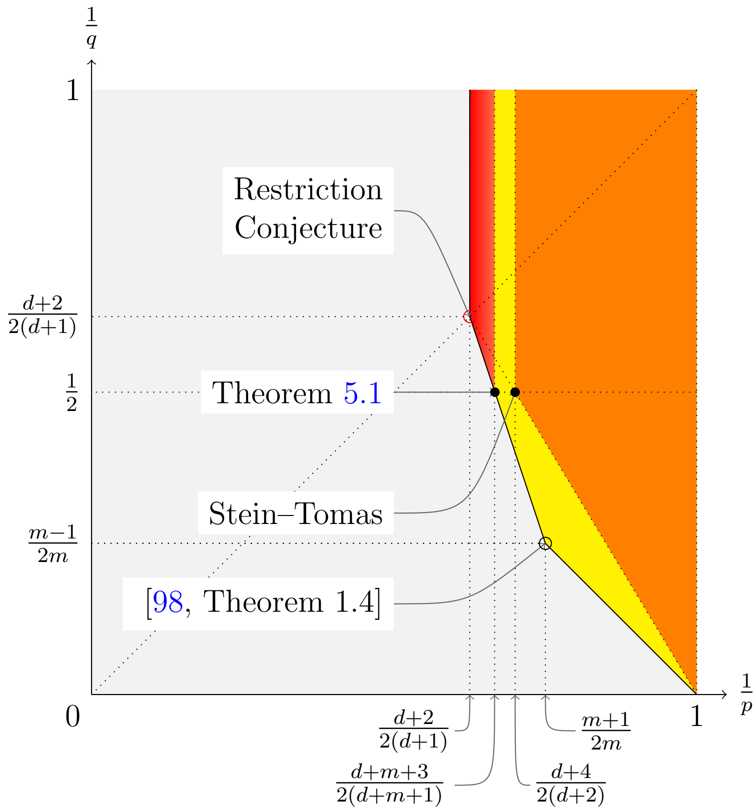

Figure 5: Riesz diagram for the G_k-symmetric restriction problem to $\mathbb{S}^d$. Estimates in the orange region follow from the Stein–Tomas inequality (33), estimates in the yellow region follow from our results [98], and no estimates within the grey region are possible. The possibility of estimates in the red region remains an open problem.

Theorem 5.2 ([98]). *Let* $d \geq 3$, $k \in \{2, 3, \ldots, d-1\}$, $m := \min\{d-k+1, k\}$, *and* $1 \leq p < 2\frac{d+m+1}{d+m+3}$. *Maximizing sequences for* $\mathbf{T}_{d,k}(p)$*, normalized in* $\mathrm{L}^p(\mathbb{R}^{d+1})$*, are precompact in* $\mathrm{L}^p_k(\mathbb{R}^{d+1})$*. In particular, maximizers for* $\mathbf{T}_{d,k}(p)$ *exist.*

The set of all maximizers for $\mathbf{T}_{d,k}(p)$ is itself compact as long as $1 \leq p < 2\frac{d+m+1}{d+m+3}$. Theorem 5.2 thus implies the unconditional existence of maximizers for the classical Stein–Tomas inequality within the class of G_k-symmetric functions. In contrast to the conclusion of [69, Theorem 1.1], precompactness of complex-valued maximizing sequences is *not* expected to hold modulo symmetries only, since G_k-symmetry eliminates the loss of compactness due to translations. There is still the danger that a maximizing sequence might converge weakly to zero. To show that this is not the case, the proof of Theorem 5.2 makes use of a decay property of the Fourier transform which is only available in the G_k-symmetric setting for $2 \leq k \leq d-1$; see [98, Prop. 2.4 and Cor. 2.5]. The endpoint case of Theorem 5.2 remains an interesting open question.

We now come to the key question of what the maximizers actually look like. Despite the partial progress past the even integer barrier which we reported in §4.2, the following fundamental problem remains open in its full generality.

Conjecture 5.3 (Stein). *Constants maximize the endpoint inequality (33), for every $d \geq 1$.*

Conjecture 5.3 is a central and elusive question that has attracted attention from the community since Stein formulated the restriction conjecture in the 1970s. A breakthrough of Foschi [66] established the case of $\mathbb{S}^2$ via a remarkable geometric argument; see §5.2. Even though connections with Conjecture 6.1 exist, significant additional challenges are present. Contrary to the paraboloid, the sphere possesses no exact scaling symmetries. Furthermore, the existence of antipodal pairs on the sphere complicates the picture, and gives rise to non-local phenomena which are absent in the Strichartz setting.

5.2 The 2-dimensional sphere $\mathbb{S}^2$

We recall Foschi's argument [66] in a form reflecting more recent insights, developed with G. Negro and C. Thiele while writing the survey [103]. We aim to show

$$\|\widehat{f\sigma}\|_4^4 \leq \|\widehat{\sigma}\|_4^4 \tag{36}$$

for all f that have the same $\mathrm{L}^2(\sigma)$ norm as the constant function $\mathbf{1}$. Basic steps that are recalled in §5.4.2 allow us to restrict attention to f that are real and antipodally symmetric, so that also $u := \widehat{f\sigma}$ is real and even. As $\widehat{u}$ is supported on the unit sphere, u satisfies the Helmholtz equation,

$$u + \Delta u = 0.$$

This equation is used in the first key step that is sometimes referred to as the *magic identity*. It expresses the left-hand side of (36) in a way that later mollifies a singularity of $\sigma * \sigma$ at the origin. The derivation of the magic identity uses partial integration, which is justified because u and all its derivatives are in L^4; see [39, Prop. 6]. Replacing one factor of u with Helmholtz, we obtain for the left-hand side of (36)

$$-\int_{\mathbb{R}^3} (\Delta u)u^3 = \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u^3) = 3\int_{\mathbb{R}^3} |\nabla u|^2 u^2 = \frac{3}{4}\int_{\mathbb{R}^3} |\nabla u^2|^2 = -\frac{3}{4}\int (\Delta u^2)u^2.$$

Expressing this again in terms of f, and doing analogously for the constant function, we reduce (36) to

$$\Lambda(f,f,f,f) \leq \Lambda(1,1,1,1)$$

where the 4-linear form Λ is defined by

$$\Lambda(f_1,f_2,f_3,f_4) = \int_{(\mathbb{R}^3)^4} |x_1+x_2|^2\, \boldsymbol{\delta}\big(\textstyle\sum_{k=1}^4 x_k\big) \displaystyle\prod_{j=1}^4 f_j\sigma(x_j)\mathrm{d}x_j.$$

The second key step is a positivity argument reminiscent of the Cauchy–Schwarz inequality. We use the inequality

$$2ab \leq a^2 + b^2 \tag{37}$$

for real numbers a and b chosen to be $f(x_1)f(x_2)$ and $f(x_3)f(x_4)$ at every point x_1, x_2, x_3, x_4 and obtain by positivity of the integral kernel

$$2\Lambda(f,f,f,f) \le \Lambda(f^2,f^2,1,1) + \Lambda(1,1,f^2,f^2). \tag{38}$$

Using symmetry of Λ thanks to $|x_1+x_2| = |x_3+x_4|$, we are reduced to showing

$$\Lambda(f^2,f^2,1,1) \le \Lambda(1,1,1,1).$$

We write $f^2 = 1+g$, where $g\sigma$ has integral zero. In the integral expression for $\Lambda(1,1,1,g)$, the measure $g\sigma$ is integrated against a radially symmetric function and thus this integral is zero. Using expansion and symmetry of Λ again, we are reduced to showing the third key step, which is the inequality

$$\Lambda(g,g,1,1) \le 0. \tag{39}$$

This relies on the important calculation, see e.g. [109] and Lemma 5.6 below, that $\sigma * \sigma(x)$ is a positive scalar multiple of

$$|x|^{-1}\mathbf{1}_{|x|\le 2}.$$

This convolution appears in the expression for $\Lambda(g,g,1,1)$ by integrating out the last two variables. Thanks to the support of the remaining functions, we may omit the indicator $\mathbf{1}_{|x|\le 2}$ and reduce (39) to

$$\int_{(\mathbb{R}^3)^2} \frac{|x_1+x_2|^2}{|x_1+x_2|} g\sigma(x_1) g\sigma(x_2) \mathrm{d}x_1 \mathrm{d}x_2 \le 0. \tag{40}$$

It is here that as a result of the magic identity the singularity at $|x_1+x_2| = 0$ is cancelled.

Define the analytic family h_s of tempered distributions on $\mathbb{R}^3$ by

$$h_s(\phi) = \Gamma((s+3)/2)^{-1} \int_{\mathbb{R}^3} |x|^s \phi(x) \mathrm{d}x. \tag{41}$$

The integral (41) is well-defined for $\Re(s) > -3$ and positive if ϕ is a gaussian. The inequality (40) is then equivalent to

$$h_1(g\sigma * g\sigma) \le 0. \tag{42}$$

The distribution h_s is rotationally symmetric and homogeneous of degree s, and it is uniquely determined by these symmetries and positivity on the gaussian, up to a positive scalar. As the Fourier transform $\widehat{h}_s$ also has rotational symmetry and dilation symmetry with degree of homogeneity $-3-s$, we have for $-3 < \Re(s) < 0$

$$a\widehat{h}_s = h_{-3-s}, \tag{43}$$

for some positive constant a. Analytic continuation with (41) and (43) allows to define h_s for all complex numbers s. By unique continuation, $h_s(\phi)$ is expressed by (41) whenever ϕ vanishes of sufficiently high order at 0 so that the integral is absolutely integrable.

By Plancherel, we reduce (42) to

$$h_{-4}((\widehat{g\sigma})^2) \leq 0. \tag{44}$$

As $(\widehat{g\sigma})^2$ vanishes of second order at the origin, the pairing with h_{-4} is given by the expression (41). Inequality (44) follows, because $(\widehat{g\sigma})^2$ is nonnegative and $\Gamma(-1/2) < 0$. This concludes our discussion of sharp endpoint Stein–Tomas inequality on $\mathbb{S}^2$.

5.3 The d-dimensional spheres $\mathbb{S}^d$

In this section, we consider spheres $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ equipped with the usual surface measure σ. The five black $\mathrm{L}^2 - \mathrm{L}^4$ entries on Figure 3 correspond to cases for which constants are global maximizers; see [41]. Above them lie infinitely many $\mathrm{L}^2 - \mathrm{L}^{2k}$ estimates which we have recently put in sharp form. Defining the functional

$$\Phi_{d,p}[f] = \|\widehat{f\sigma}\|_p^p \|f\|_{\mathrm{L}^2(\mathbb{S}^d)}^{-p},$$

the following is the main result in [109, 110].

Theorem 5.4 ([109, 110]). *Let $d \in \{2,3,4,5,6\}$ and $p \geq 6$ be an even integer. Then constant functions are the unique real-valued maximizers of the functional $\Phi_{d,p}$. The same conclusion holds for $d = 1$ and even $p > 6$ if constants maximize $\Phi_{1,6}$.*

For the remainder of this section, our goal is four-fold. Firstly, we briefly discuss the proof of Theorem 5.4 in the particular but representative case when $(d,p) = (2,6)$. Secondly, we describe the extra ingredients which are needed in order to obtain sharp $\mathrm{L}^2 - \mathrm{L}^{2k}$ estimates for higher $k \geq 4$. Thirdly, we fully characterize the complex-valued maximizers of $\Phi_{d,p}$, for d,p in the range of Theorem 5.4. Finally, we present a sharp extension inequality on $\mathbb{S}^7$, which is the lowest dimension to which Theorem 5.4 does not apply.

5.4 The representative case $(d,p) = (2,6)$

We abbreviate notation by writing $\Phi_p := \Phi_{2,p}$ and

$$\mathbf{T}_p := \sup\{\Phi_p[f]^{1/p} : 0 \neq f \in \mathrm{L}^2(\mathbb{S}^2)\}.$$

The proof naturally splits into five steps, which use tools from the calculus of variations, symmetrization, operator theory, Lie theory, and probability. We present them next, and then see how they all come together nicely in the end.

5.4.1 Calculus of variations

The existence of maximizers for Φ_6 is ensured by [59]. Let f be one such maximizer, normalized so that $\|f\|_{\mathrm{L}^2} = 1$. Consider the extension operator $\mathcal{E}(f) := \widehat{f\sigma}$ with adjoint given by $\mathcal{E}^*(g) := g^\vee|_{\mathbb{S}^2}$. Then the operator norm can be estimated as follows:

$$\begin{aligned}
\|\mathcal{E}\|_{\mathrm{L}^2\to\mathrm{L}^6}^6 &= \|\mathcal{E}(f)\|_{\mathrm{L}^6(\mathbb{R}^3)}^6 = \langle |\mathcal{E}(f)|^4\mathcal{E}(f), \mathcal{E}(f)\rangle = \langle \mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f)), f\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \\
&\leq \|\mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f))\|_{\mathrm{L}^2(\mathbb{S}^2)} \leq \|\mathcal{E}^*\|_{\mathrm{L}^{6/5}\to\mathrm{L}^2} \||\mathcal{E}(f)|^4\mathcal{E}(f)\|_{\mathrm{L}^{6/5}(\mathbb{R}^3)} \\
&= \|\mathcal{E}^*\|_{\mathrm{L}^{6/5}\to\mathrm{L}^2} \|\mathcal{E}(f)\|_{\mathrm{L}^6(\mathbb{R}^3)}^5 = \|\mathcal{E}\|_{\mathrm{L}^2\to\mathrm{L}^6}^6.
\end{aligned}$$

Thus, all inequalities are equalities, and the particular equality in the Cauchy–Schwarz step above yields the *Euler–Lagrange equation*,

$$(|\widehat{f\sigma}|^4\widehat{f\sigma})^\vee|_{\mathbb{S}^2} = \lambda f,$$

which, in convolution form, reads as follows:

$$(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma * f\sigma)|_{\mathbb{S}^2} = (2\pi)^{-3}\lambda f, \tag{45}$$

where $f_\star := \overline{f(-\cdot)}$. A bootstrapping procedure can then be used to show that f, and indeed any L^2-solution of (45), is C^∞-smooth. We omit the details and refer the interested reader to [110], which extends the main result of [48] to the higher dimensional setting of even exponents.

5.4.2 Symmetrization

Since $p = 6$ is an even integer, the problem is inherently positive, in the sense that nonnegative maximizers exist. In fact,

$$\|f\sigma * f\sigma * f\sigma\|_{\mathrm{L}^2(\mathbb{R}^3)} \leq \| \, |f|\sigma * |f|\sigma * |f|\sigma \, \|_{\mathrm{L}^2(\mathbb{R}^3)}.$$

Defining $f_\sharp := \sqrt{\frac{|f|^2+|f_\star|^2}{2}}$, we also have the following monotonicity under antipodal symmetrization which can be readily verified via a creative application of the Cauchy–Schwarz inequality in the spirit of (38):

$$\|f\sigma * f\sigma * f\sigma\|_{\mathrm{L}^2(\mathbb{R}^3)} \leq \|f_\sharp\sigma * f_\sharp\sigma * f_\sharp\sigma\|_{\mathrm{L}^2(\mathbb{R}^3)}. \tag{46}$$

Interestingly, this inequality has a variant in the non-algebraic case when $p \notin 2\mathbb{N}$; see [30, Prop. 6.7] and our discussion in §6.2. We conclude that

$$\mathbf{T}_6 = \max\{\Phi_6[f]^{1/6} : 0 \neq f \in C^\infty(\mathbb{S}^2), f \text{ is nonnegative and even}\},$$

an important simplification which will be crucial in the sequel.

5.4.3 Operator Theory

We now explore some of the compactness inherent to the problem. Given $f \in \mathrm{L}^2(\mathbb{S}^2)$, consider the integral operator $T_f : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ defined by

$$T_f(g)(\omega) := (g * K_f)(\omega) = \int_{\mathbb{S}^2} g(\nu)K_f(\omega-\nu)\mathrm{d}\sigma(\nu), \tag{47}$$

which acts on functions $g \in \mathrm{L}^2(\mathbb{S}^2)$ by convolution with the kernel

$$K_f(\xi) := (|\widehat{f\sigma}|^4)^\vee(\xi) = (2\pi)^3(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma)(\xi).$$

The relevance of the operator T_f becomes apparent once one realizes that the Euler–Lagrange equation (45) boils down to the eigenfunction equation $T_f(f) = \lambda f$. The kernel

K_f defines a bounded, continuous function on $\mathbb{R}^3$ which satisfies $K_f(\xi) = \overline{K_f(-\xi)}$, for all ξ, and crucially $K_f(0) = \|\widehat{f\sigma}\|_4^4$. Correspondingly, the operator T_f is self-adjoint and positive definite. In fact, one can check that T_f is trace-class and that its trace is given by

$$\operatorname{tr}(T_f) = 4\pi\|\widehat{f\sigma}\|_4^4. \tag{48}$$

This is a consequence of Mercer's theorem, which is the infinite-dimensional analogue of the well-known statement that any positive semidefinite matrix is the Gram matrix of a certain set of vectors.

5.4.4 Lie Theory

We proceed to discuss the symmetries of the problem. The set of 3×3 orthogonal matrices with unit determinant form the special orthogonal group SO(3), with Lie algebra $\mathfrak{so}(3)$. As a preliminary observation, we note that the exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, $A \mapsto e^A$, is surjective onto SO(3), and that the functional Φ_6 is rotation- and modulation-invariant. In other words,

$$\Phi_6[f \circ e^{tA}] = \Phi_6[f] = \Phi_6[e^{i\xi\cdot} f],$$

for all $(t, A) \in \mathbb{R} \times \mathfrak{so}(3)$ and $\xi \in \mathbb{R}^3$. As we shall now see, these symmetries give rise to new eigenfunctions for the operator T_f defined in (47) in a natural way. Consider the vector field ∂_A acting on sufficiently smooth functions $f : \mathbb{S}^2 \to \mathbb{C}$ via

$$\partial_A f := \frac{\partial}{\partial t}\Big|_{t=0}(f \circ e^{tA}).$$

We have the following key lemma, where we write $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$, and by $\omega_j f$ we mean the function defined via $(\omega_j f)(\omega) = \omega_j f(\omega)$.

Lemma 5.5 ([109]). *Let $f : \mathbb{S}^2 \to \mathbb{R}$ be non-constant, such that $f_\star = f \in C^1(\mathbb{S}^2)$ and $\|f\|_{L^2} = 1$. Assume $T_f(f) = \lambda f$. Then*

$$T_f(\partial_A f) = \tfrac{\lambda}{5}\partial_A f, \quad \text{for every } A \in \mathfrak{so}(3);$$
$$T_f(\omega_j f) = \tfrac{\lambda}{5}\omega_j f, \quad \text{for every } j \in \{1, 2, 3\}.$$

Moreover, there exist $A, B \in \mathfrak{so}(3)$, such that the set $\{\partial_A f, \partial_B f, \omega_1 f, \omega_2 f, \omega_3 f\}$ is linearly independent over $\mathbb{C}$.

The proof of Lemma 5.5 hinges on the fact that the codimension of a proper, nontrivial subalgebra of $\mathfrak{so}(3)$ equals 2. The linear independence of the set $\{\partial_A f, \partial_B f, \omega_1 f, \omega_2 f, \omega_3 f\}$ follows from the fact that $\partial_A f, \partial_B f$ are real *even* functions, whereas $\omega_1 f, \omega_2 f, \omega_3 f$ are real *odd* functions.

5.4.5 Uniform random walks in $\mathbb{R}^3$

We will need explicit expressions for various convolution measures σ^{*k}. These can be interpreted in terms of random walks, and as such are sometimes available in the probability theory literature. More precisely, consider i.i.d. random variables X_1, X_2, X_3, taking values on $\mathbb{S}^2$ with uniform distribution. Then $Y_3 = X_1 + X_2 + X_3$ is the *uniform 3-step random walk* in $\mathbb{R}^3$. If p_3 denotes the probability density of $|Y_3|$, then a straightforward computation in polar coordinates reveals that $(\sigma * \sigma * \sigma)(r) = \sigma(\mathbb{S}^2)^2 p_3(r) r^{-2}$. Such considerations quickly lead to the following formulae for spherical convolutions.

Lemma 5.6 ([109]). *The following identities hold:*

$$(\sigma * \sigma)(\xi) = \frac{2\pi}{|\xi|}, \ \text{if } |\xi| \le 2, \tag{49}$$

$$(\sigma * \sigma * \sigma)(\xi) = \begin{cases} 8\pi^2, & \text{if } |\xi| \le 1, \\ 4\pi^2\Big(\frac{3}{|\xi|} - 1\Big), & \text{if } 1 \le |\xi| \le 3. \end{cases} \tag{50}$$

Corollary 5.7. $\Phi_6[\mathbf{1}] = 2\pi\Phi_4[\mathbf{1}]$.

Indeed, from Lemma 5.6 we have that

$$\Phi_4[\mathbf{1}] = (2\pi)^3 \|\mathbf{1}\|_{\mathrm{L}^2(\mathbb{S}^2)}^{-4} \|\sigma * \sigma\|_2^2 = 16\pi^4, \text{ and}$$
$$\Phi_6[\mathbf{1}] = (2\pi)^3 \|\mathbf{1}\|_{\mathrm{L}^2(\mathbb{S}^2)}^{-6} \|\sigma * \sigma * \sigma\|_2^2 = 32\pi^5.$$

5.4.6 The conclusion of the proof of Theorem 5.4 when $(d, p) = (2, 6)$

By §5.4.1, it suffices to check that any non-constant *critical point* (i.e. an L^2-solution of the Euler–Lagrange equation (45)) $f : \mathbb{S}^2 \to \mathbb{C} \in C^1(\mathbb{S}^2)$ of Φ_6 satisfies $\Phi_6[f] < \Phi_6[\mathbf{1}]$. By §5.4.2, we may further assume that $f_\star = f$ is real-valued, and that $\|f\|_{\mathrm{L}^2} = 1$. From $T_f(f) = \lambda f$, one checks that $\lambda = \Phi_6[f]$. Thus, by §5.4.3–§5.4.4,

$$\Phi_6[f] = \lambda = \tfrac{1}{2}(\lambda + 5 \times \tfrac{\lambda}{5}) < \frac{1}{2}\mathrm{tr}(T_f) = 2\pi\|\widehat{f\sigma}\|_4^4, \tag{51}$$

where the strict inequality is a consequence of Lemma 5.5 together with the fact that all eigenvalues of T_f are strictly positive, and the last identity has been observed in (48). But

$$2\pi\|\widehat{f\sigma}\|_4^4 = 2\pi\Phi_4[f] \le 2\pi\Phi_4[\mathbf{1}] = \Phi_6[\mathbf{1}] \tag{52}$$

where the inequality follows from the result (36) of Foschi [66] reviewed in §5.2, and the last identity is Corollary 5.7. From (51) and (52) it follows that $\Phi_6[f] < \Phi_6[\mathbf{1}]$, and this concludes the sketch of the proof of the case $(d, p) = (2, 6)$ of Theorem 5.4.

5.5 Higher dimensions and exponents

The special case $(d, p) = (2, 6)$ of Theorem 5.4 discussed in §5.4, while illustrative of the general scheme, relies on several crucial simplifications which made the proof sketch fit in just a few pages. In order to deal with general even exponents and different dimensions, further ideas and techniques are needed. These turn out to be broadly connected with the following areas:

- *Non-commutative algebra.* When trying to generalize Lemma 5.5 to higher dimensions, one is naturally led to the following question: What is the minimal codimension of a proper subalgebra of $\mathfrak{so}(d)$? The answer is known and reveals an interesting difference that occurs in the four-dimensional case: the minimal codimension of a proper subalgebra of $\mathfrak{so}(d)$ equals $d-1$ if $d \geq 3, d \neq 4$, but equals 2 if $d = 4$. In group theoretical terms, the group $\mathrm{SO}(4)/\{\pm I\}$ is *not* simple, whereas all other groups $\mathrm{SO}(d)$ are simple (after modding out by $\{\pm I\}$ if d is even).

- *Combinatorial geometry.* When trying to extend the relevant estimates from Corollary 5.7 to the multilinear setting of $(p/2)$-fold spherical convolutions, one faces certain variants of the *cube slicing problem*: Given $0 < k < d$, what is the maximal volume of the intersection of the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ with a k-dimensional subspace of $\mathbb{R}^d$? The cube slicing problem has been intensely studied, but a complete solution remains out of reach. Fortunately, the methods that have been developed for this problem can be adapted to fulfil our needs.

- *Analytic number theory.* The rather direct approach we took in §5.4.5 needs to be refined in order to tackle other dimensions. Uniform random walks in $\mathbb{R}^d$ are lurking in the background and, despite being a classic topic in probability theory, a complete answer in even dimensions remains a fascinating, largely unsolved problem, which exhibits some deep connections to analytic number theory via the theory of hypergeometric functions and modular forms [23]. In view of this, we combine known formulae for uniform random walks with rigorous numerical integration and asymptotic analysis for a certain family of weighted integrals in order to complete our task.

5.6 Complex-valued maximizers

Once real-valued maximizers have been identified, one can proceed to characterize *all* complex-valued maximizers.

Theorem 5.8 ([109]). *Let $d \geq 1$ and $p \geq 2 + \frac{4}{d}$ be an even integer. Then each complex-valued maximizer of the functional $\Phi_{d,p}$ is of the form*

$$ce^{i\xi\cdot\omega}F(\omega),$$

for some $\xi \in \mathbb{R}^{d+1}$, some $c \in \mathbb{C}\setminus\{0\}$, and some nonnegative maximizer F of $\Phi_{d,p}$ satisfying $F(\omega) = F(-\omega)$, for every $\omega \in \mathbb{S}^d$.

Our next result is an immediate consequence of Theorems 5.4, 5.8 and [41, 66].

Corollary 5.9. *Let $d \in \{2,3,4,5,6\}$ and $p \geq 4$ be an even integer. Then all complex-valued maximizers of the functional $\Phi_{d,p}$ are given by*

$$f(\omega) = ce^{i\xi\cdot\omega},$$

for some $\xi \in \mathbb{R}^{d+1}$ and $c \in \mathbb{C}\setminus\{0\}$. The same conclusion holds for $d = 1$ and even integers $p \geq 8$, provided that constants maximize $\Phi_{1,6}$.

5.7 A sharp extension inequality on $\mathbb{S}^7$

The study of sharp *weighted* spherical extension estimates is naturally linked to the question of stability of such estimates, and was very recently inaugurated in [39]. In particular, the sharp weighted extension inequality from [39, Theorem 1] leads to the following result, which is the first instance of a sharp extension inequality on $\mathbb{S}^7$.

Theorem 5.10 ([39]). *For every $a > \frac{2^{25}\pi^2}{5^2 7^2 11}$, the following sharp inequality holds:*

$$\int_{\mathbb{R}^8} |\widehat{f\sigma}(x)|^4 \,\mathrm{d}x + a \left| \int_{\mathbb{S}^7} f(\omega)\,\mathrm{d}\sigma(\omega) \right|^4 \leq \mathbf{P}_a \left(\int_{\mathbb{S}^7} |f(\omega)|^2 \,\mathrm{d}\sigma(\omega) \right)^2 \tag{53}$$

with optimal constant given by

$$\mathbf{P}_a = \int_{\mathbb{R}^8} \widehat{\sigma}(x)^4 \frac{\mathrm{d}x}{\sigma(\mathbb{S}^7)^2} + a\sigma(\mathbb{S}^7)^2$$

Equality in (53) occurs if and only if f is constant on $\mathbb{S}^7$.

We emphasize that constants are the unique *complex-valued* maximizers for (53), in contrast to the situation considered in §5.6. An interesting problem is whether the value of the threshold $\frac{2^{25}\pi^2}{5^2 7^2 11}$ can be lowered, hopefully all the way down to 0.

6 Paraboloid and Cone

The *Schrödinger equation* describes the evolution of a physical system in which quantum effects, such as wave-particle duality, are significant. Given its dispersive nature (i.e., different frequencies propagate at different velocities), certain estimates quantifying the size of the solutions of the Schrödinger equation in terms of the size of the initial datum are a direct manifestation of restriction theory, and play a key role in quantum mechanics. To make this more precise, consider the initial-value problem $u_t = i\Delta u$, $u(0,\cdot) = f \in \mathrm{L}^2(\mathbb{R}^d)$, whose solution is given in terms of the Fourier transform by

$$e^{it\Delta} f(x) := \int_{\mathbb{R}^d} \exp(it|\xi|^2 + ix\cdot\xi)\widehat{f}(\xi)\,\mathrm{d}\xi, \text{ or } \widetilde{u}(\tau,\xi) = \boldsymbol{\delta}\big(\tau - |\xi|^2\big)\widehat{f}(\xi). \tag{54}$$

Here, $\widetilde{u}$ stands for the space-time Fourier transform of u, and $\boldsymbol{\delta}$ denotes the Dirac delta distribution on the real line. From representation (54), it follows that all the action in frequency space takes place on the paraboloid $\{(\tau,\xi) \in \mathbb{R}^{1+d} : \tau = |\xi|^2\}$ – a non-compact, homogeneous hypersurface with nonvanishing gaussian curvature – which is thus referred to as the associated manifold to the Schrödinger equation. Even though the mass is conserved, i.e., $\|e^{it\Delta} f\|_2 = \|f\|_2$ for all t, solutions decay in time due to spreading in different directions. This is yet another manifestation of the destructive interference, cancellation, and decay phenomena described in connection to (15). Strichartz [133] observed in 1977 that the scale invariance of the endpoint Stein–Tomas argument leads to the following estimate for the solution of the Schrödinger equation:

$$\|e^{it\Delta}f\|_{\mathrm{L}^{2+\frac{4}{d}}(\mathbb{R}^{1+d})} \leq \mathbf{S}_d\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}; \text{ here, } \mathbf{S}_d := \sup_{\mathbf{0}\neq f\in\mathrm{L}^2} \frac{\|e^{it\Delta}f\|_{\mathrm{L}^{2+\frac{4}{d}}(\mathbb{R}^{1+d})}}{\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}}. \tag{55}$$

The *wave equation* describes the propagation of light in a vacuum, and constitutes the simplest and most fundamental model for waves and vibrations in arbitrary dimensions. The wave equation is only partly dispersive: the frequency of a wave determines the direction of propagation, but all frequencies move with the same speed. The solution of the initial-value problem with initial Sobolev data,

$$u_{tt} = \Delta u, \ (u, u_t)(0,\cdot) = (f,g) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^{2d}),$$

is

$$\mathcal{S}(f,g)(t,x) := \cos(t\sqrt{-\Delta})f(x) + \tfrac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g(x),$$

or equivalently

$$\widetilde{u}(\tau,\xi) = \boldsymbol{\delta}\big(|\tau| - |\xi|\big)\Big(\tfrac{1}{2}\widehat{f}(\xi) + \tfrac{\operatorname{sgn}(\tau)}{2i|\xi|}\widehat{g}(\xi)\Big).$$

Thus, the light-cone $\{(\tau,\xi) \in \mathbb{R}^{1+d} : \tau^2 = |\xi|^2\}$ is the associated manifold to the wave equation. This is a non-compact, homogeneous hypersurface whose gaussian curvature *vanishes identically*, which in principle could be problematic from the restriction point of view. However, all but one principal curvature of the light-cone are nonzero, and the endpoint Stein–Tomas argument can be rescued. Strichartz [133] thus obtained

$$\|\mathcal{S}(f,g)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{1+d})} \leq \mathbf{W}_d\|(f,g)\|_{\mathcal{H}^{\frac{1}{2}}}, \tag{56}$$

where $\mathbf{W}_d$ denotes the optimal constant, and

$$\|(f,g)\|^2_{\mathcal{H}^s} := \int_{\mathbb{R}^d} \Big(|\xi|^{2s}|\widehat{f}(\xi)|^2 + |\xi|^{2(s-1)}|\widehat{g}(\xi)|^2\Big)\,\mathrm{d}\xi. \tag{57}$$

The norm $\|\mathcal{S}(f,g)(t,\cdot)\|_{\mathcal{H}^s} = \|(f,g)\|_{\mathcal{H}^s}$ is conserved for all s, the case $s = 1/2$ from (57) corresponding to the *relativistic kinetic energy*.

Symmetries are key to a deeper understanding of the Strichartz estimates for the Schrödinger and the wave equations, which share a number of geometric invariance properties: space-time translations, dilations, scaling, space rotations, phase shifts. Additionally, inequality (55) remains invariant under Galilean transformations, whereas (56) does not change under the action of Lorentz transformations. The groups of symmetries of the Strichartz estimates for the Schrödinger and wave equations are the so-called Galilean and Poincaré groups, respectively. Mixed-norm Strichartz variants of (55) have been intensely investigated; see (58) below, and [138, §2.3]. Even though various $(\mathcal{H}^s \to \mathrm{L}^r_t\mathrm{L}^q_x)$ estimates for the solution of the wave equation have been established, only (56), corresponding to $s = 1/2$ and $q = r = 2\frac{d+1}{d-1}$, is *conformally invariant* (i.e., invariant under the full Poincaré group).

6.1 Sharp Strichartz inequalities

What about maximizers and optimal constants? In contrast to the sphere (§5.1), the unconditional existence of maximizers has been settled for both the paraboloid [7] and the light-cone [121] in all dimensions. It is their *shape* that has been the subject of the most remarkable contributions, of which we highlight the following. Firstly, gaussians are known to maximize the mixed-norm partner of the Strichartz inequality (55) guaranteed by Theorem 3.8,[4]

$$\|e^{it\Delta}f(x)\|_{\mathrm{L}^r_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \leq \mathbf{S}_{q,r}\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}, \text{ if } q,r\geq 2 \text{ and } \tfrac{d}{q}+\tfrac{2}{r}=\tfrac{d}{2}, \tag{58}$$

provided $(d,q,r)\in\{(1,6,6),(1,4,8),(2,4,4)\}$; see [37, 65, 116]. All maximizers are in fact given by initial data corresponding to the orbit of the Schrödinger propagator of the standard gaussian $\exp(-|\cdot|^2)$ under the Galilean group of symmetries. Secondly, the Strichartz inequality for the wave equation (56) is saturated by the pair $(f_\star := (1+|\cdot|^2)^{(1-d)/2}, \mathbf{0})$ if $d=3$; see [65]. All maximizers are then obtained by letting the Poincaré group act on the wave propagator of $(f_\star,\mathbf{0})$. Different proofs of these facts relying on heat flow monotonicity [9], orthogonal polynomials [72], and representation formulae [84] are available, but *they all ultimately hinge on the Lebesgue exponents in question being even integers.* In this case, one can invoke Plancherel's identity in order to reduce the problem to a simpler multilinear convolution estimate. On the other hand, the following fundamental problems remain open in their full generality.

Conjecture 6.1 (Lieb)**.** *Gaussians maximize* (58) *for all admissible* q,r,d.

Conjecture 6.2 (Foschi)**.** *The pair* $((1+|\cdot|^2)^{\frac{1-d}{2}},\mathbf{0})$ *maximizes* (56) *for every* $d\geq 2$.

The diagonal case $q=r=2+\frac{4}{d}$ of Conjecture 6.1 has been implicitly raised in the 1977 seminal work of Strichartz [133]. It was precisely formulated by Lieb [95] in 1990, and has appeared in several papers since then; see e.g. [9, 84], and [72] for the general form of this longstanding conjecture. Despite considerable effort and promising partial progress (in particular during the last decade), the problem remains open in arbitrary dimensions. Conjecture 6.1 has generated a great deal of interest since its first appearance, for a number of reasons. Firstly, it is a very natural question. Gaussians are known to maximize (58) in the lower dimensional cases $d\in\{1,2\}$, provided q is an even integer dividing r. Is this an isolated fact? Or does it hint at some deeper truth? A historically similar situation surrounded the epic breakthroughs of Beckner [5] and Lieb [94] for non-even instances of the sharp Hausdorff–Young and Hardy–Littlewood–Sobolev inequalities, respectively. Moreover, if gaussians were known to maximize (58), then the unconditional existence of maximizers for the endpoint Stein–Tomas inequality (33) would follow; see [69].

Foschi's conjecture, first formulated in [65, Conj. 1.11], can be viewed as a hybrid between the conjectures of Stein and Lieb. In fact, the light-cone and the paraboloid are both homogeneous hypersurfaces, which implies an *exact* scale invariance that is a mere

[4] As in the statement of Theorem 3.8, the endpoint case $(d,p,q)=(2,2,\infty)$ should be excluded.

approximate symmetry in the case of the sphere; on the other hand, the existence of pairs of antipodal points on the light-cone – as in the sphere – complicates the picture, and gives rise to non-local phenomena which are absent in the Strichartz–Schrödinger setting. Thus, Conjecture 6.2 is a privileged testing ground for several ideas which may prove useful to tackle Conjectures 5.3 and 6.1. G. Negro disproved Conjecture 6.2 whenever $d \geq 2$ is *even* [99]. Strikingly, he discovered that in those cases the pair $(f_\star, \mathbf{0})$ is not even a critical point of the functional associated to inequality (56). Nevertheless, $(f_\star, \mathbf{0})$ turned out to locally maximize (56) whenever $d \geq 3$ is odd [74]. Both proofs rely on the *Penrose transform*, a well-known conformal map of Minkowski spacetime $\mathbb{R}^{1+d}$ which is a powerful tool to investigate subtle local properties of the solution map to the wave equation; see §6.3.

6.2 Perturbed paraboloid

Given a sufficiently nice function $\phi : \mathbb{R}^d \to \mathbb{R}$, consider the hypersurface in $\mathbb{R}^{d+1}$

$$\Sigma_\phi = \{(|y|^2 + \phi(y), y) : y \in \mathbb{R}^d\}, \tag{59}$$

equipped with projection measure $\mathrm{d}\nu(s, y) = \boldsymbol{\delta}\big(s - |y|^2 - \phi(y)\big)\mathrm{d}s\mathrm{d}y$. The following *geometric comparison principle* is the main result from [108]. It holds in all dimensions $d \geq 1$, and generalizes [107, Theorem 1.3] to n-fold convolutions.

Theorem 6.3 ([108]). *For $d \geq 1$, let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, continuously differentiable, strictly convex function, let $\psi = |\cdot|^2 + \phi$, and let ν_0, ν denote the projection measures on the hypersurfaces Σ_0, Σ_ϕ, respectively. Then, for any integer $n \geq 2$,*

$$\nu^{*n}(\tau, \xi) \leq \nu_0^{*n}(\tau - n\phi(\xi/n), \xi), \tag{60}$$

for every $\xi \in \mathbb{R}^d$ and $\tau > n\psi(\xi/n)$. Moreover, this inequality is strict at every point in the interior of the support of the measure ν^{*n}.

Under the assumptions of the theorem, the support of the convolution measure ν^{*n} is contained in that of ν_0^{*n}. Moreover, both measures define continuous functions inside their supports and, as $\tau \to n\psi(\xi/n)^+$, the left- and right-hand sides of (60) approach the boundary values of ν^{*n} and ν_0^{*n}, respectively; see [108, Prop. 2.1]. We emphasize that, at least when $(d, n) \neq (1, 2)$, inequality (60) is stronger than the mere claim

$$\nu^{*n}(\tau, \xi) \leq \nu_0^{*n}(\tau, \xi), \text{ for every } \xi \in \mathbb{R}^d \text{ and } \tau > n\psi(\xi/n)$$

since the function $\tau \mapsto \nu_0^{*n}(\tau, \xi)$ is non-decreasing; see [108, Remark 2.2].

The geometric comparison principle encoded in Theorem 6.3 has immediate implications towards sharp restriction theory and sharp Strichartz estimates. We present three of the more recent applications from [30], and refer to [105] for a survey of the original motivation. Firstly, we resolve the dichotomy from [86] concerning the existence of maximizers for the Strichartz inequality for the fourth order Schrödinger equation in one spatial dimension. This is related to the extension problem on the planar curve $s = y^4$.

Secondly, we study similar questions in the more general setting of the extension problem on the curve $s = |y|^\alpha$, for arbitrary $\alpha > 1$. We also consider *odd* curves $s = y|y|^{\alpha-1}$, $\alpha > 1$, the case $\alpha = 3$ relating to the Airy–Strichartz inequality [70, 122]. Thirdly, we study super-exponential decay and analyticity of the corresponding extremizers and their Fourier transform via a bootstrapping procedure. We now turn to the details.

Jiang–Pausader–Shao [86] considered the fourth order Schrödinger equation

$$\begin{cases} iu_t - \mu u_{xx} + u_{xxxx} = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,\cdot) = f \in \mathrm{L}^2_x(\mathbb{R}), \end{cases} \tag{61}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and $\mu \in \{0,1\}$. The solution of the Cauchy problem (61) is given by

$$u(t,x) = e^{it(\partial_x^4 - \mu\partial_x^2)} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{it(\xi^4 + \mu\xi^2)} \widehat{f}(\xi) \mathrm{d}\xi,$$

which disperses as $|t| \to \infty$. Consequently, the following Strichartz inequality due to Kenig–Ponce–Vega [89, Theorem 2.1] holds:

$$\|D_\mu^{\frac{1}{3}} e^{it(\partial_x^4 - \mu\partial_x^2)} f\|_{\mathrm{L}^6_{t,x}(\mathbb{R}^{1+1})} \lesssim \|f\|_{\mathrm{L}^2(\mathbb{R})}. \tag{62}$$

The main result of [86] is a linear profile decomposition for (61), which uses a refinement of the Strichartz inequality (62) in the scale of Besov spaces, together with improved localized restriction estimates. As a consequence, the authors of [86] establish a dichotomy result for the existence of maximizers for (62) when $\mu = 0$, which can be summarized as follows. Consider the sharp inequality in multiplier form

$$\|D_0^{\frac{1}{3}} e^{it\partial_x^4} f\|_{\mathrm{L}^6_{t,x}(\mathbb{R}^{1+1})} \leq \mathbf{M} \|f\|_{\mathrm{L}^2(\mathbb{R})}, \tag{63}$$

with optimal constant given by

$$\mathbf{M} := \sup_{0 \neq f \in \mathrm{L}^2} \frac{\|D_0^{\frac{1}{3}} e^{it\partial_x^4} f\|_{\mathrm{L}^6_{t,x}(\mathbb{R}^{1+1})}}{\|f\|_{\mathrm{L}^2(\mathbb{R})}}. \tag{64}$$

Then [86, Theorem 1.8] states that either a maximizer for (63) exists, or there exist a sequence $\{a_n\} \subset \mathbb{R}$ satisfying $|a_n| \to \infty$, as $n \to \infty$, and a function $f \in \mathrm{L}^2$, such that

$$\mathbf{M} = \lim_{n \to \infty} \frac{\|D_0^{\frac{1}{3}} e^{it\partial_x^4} (e^{ia_n x} f)\|_{\mathrm{L}^6_{t,x}(\mathbb{R}^{1+1})}}{\|f\|_{\mathrm{L}^2(\mathbb{R})}}.$$

In the latter case, one necessarily has $\mathbf{M} = \mathbf{S}_1$, where $\mathbf{S}_1$ denotes the optimal constant defined in (55). The first main result from [30] resolves this dichotomy.

Theorem 6.4 ([30]). *Maximizers for* (63) *exist.*

Theorem 6.4 follows from a more general result which we now introduce. As noted in [89, §2], the operator $D_0^{1/3} e^{it\partial_x^4}$ is nothing but a constant multiple of the Fourier transform of the singular measure

$$\mathrm{d}\sigma_4(s,y) = \boldsymbol{\delta}\big(s - y^4\big)|y|^{\frac{1}{3}}\mathrm{d}s\mathrm{d}y \tag{65}$$

at the point $(-t,-x) \in \mathbb{R}^2$. As in [107, §6.4], one is naturally led to consider generic power curves $s = |y|^\alpha$. The corresponding inequality is then

$$\|\mathcal{M}_\alpha(f)\|_{\mathrm{L}^6_{t,x}(\mathbb{R}^{1+1})} \leq \mathbf{M}_\alpha \|f\|_{\mathrm{L}^2(\mathbb{R})}, \tag{66}$$

where the multiplier operator $\mathcal{M}_\alpha$ is defined as $\mathcal{M}_\alpha(f)(t,x) = D_0^{\frac{\alpha-2}{6}} e^{it|\partial_x|^\alpha} f(x)$. Inequality (66) can be equivalently restated as the extension inequality

$$\|\mathcal{E}_\alpha(f)\|_{\mathrm{L}^6(\mathbb{R}^2)} \leq \mathbf{E}_\alpha \|f\|_{\mathrm{L}^2(\mathbb{R})}, \tag{67}$$

or in convolution form as

$$\|f\sigma_\alpha * f\sigma_\alpha * f\sigma_\alpha\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \mathbf{C}_\alpha^3 \|f\|^3_{\mathrm{L}^2(\mathbb{R})}. \tag{68}$$

Here, the singular measure σ_α is defined in accordance with (65) by

$$\mathrm{d}\sigma_\alpha(s,y) = \boldsymbol{\delta}\big(s - |y|^\alpha\big)|y|^{\frac{\alpha-2}{6}}\mathrm{d}s\mathrm{d}y, \tag{69}$$

and the extension operator $\mathcal{E}_\alpha(f) = \widehat{f\sigma_\alpha}(-\cdot)$ is given by

$$\mathcal{E}_\alpha(f)(t,x) = \int_{\mathbb{R}} e^{ixy} e^{it|y|^\alpha} |y|^{\frac{\alpha-2}{6}} f(y)\mathrm{d}y, \tag{70}$$

so that $6^{\frac{\alpha-2}{12}}\mathcal{E}_\alpha(\widehat{f}) = 2\pi\mathcal{M}_\alpha(f)$. If f maximizes (67), then f likewise maximizes (68), and $f^\vee$ maximizes (66). Thus, these three existence problems are essentially equivalent. The convolution form (68) implies that the search for maximizers can be restricted to the class of nonnegative functions. An application of Plancherel further shows that

$$\mathbf{E}_\alpha^6 = (2\pi)^2\mathbf{C}_\alpha^6 = (2\pi)^3 6^{1-\frac{\alpha}{2}}\mathbf{M}_\alpha^6.$$

The next result extends the dichotomy proved in [86, Theorem 1.8] to the case of arbitrary exponents $\alpha > 1$. It states that one of two possible scenarios occurs, *compactness* or *concentration* at a point. We make the latter notion precise.

Definition 6.5. A sequence of functions $\{f_n\} \subset \mathrm{L}^2(\mathbb{R})$ *concentrates at a point* $y_0 \in \mathbb{R}$ if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\int_{|y-y_0|\geq\rho} |f_n(y)|^2\mathrm{d}y < \varepsilon\|f_n\|^2_{\mathrm{L}^2(\mathbb{R})}.$$

We phrase the second main result from [30] in terms of (68) because (71) has a simple geometric meaning in terms of the boundary value of the relevant 3-fold convolution measure.

Theorem 6.6 ([30]). *Let $\alpha > 1$. If*

$$\mathbf{C}_\alpha^6 > \frac{2\pi}{\sqrt{3}\alpha(\alpha-1)}, \tag{71}$$

then any maximizing sequence of nonnegative functions in $\mathrm{L}^2(\mathbb{R})$ for (68) is precompact, after normalization and scaling. In this case, maximizers for (68) exist. If instead equality holds in (71) then, given any $y_0 \in \mathbb{R}$, there exists a maximizing sequence for (68) which concentrates at y_0.

A few remarks may help. Firstly, if $\alpha = 1$, then the curve $s = |y|$ has no curvature, and no non-trivial extension estimate can hold. Secondly, if equality holds in (71), then Theorem 6.6 does *not* guarantee that maximizers do not exist. Indeed, $\mathbf{C}_2^6 = \pi/\sqrt{3}$, and gaussians are known to maximize (68) when $\alpha = 2$; recall our discussion in §6.1. Various results of a similar flavour to that of Theorem 6.6 have appeared in the recent literature. They are typically derived from concentration-compactness techniques [47, 124], a full profile decomposition [53,86,87,123], or the "missing mass method" [69,70]. In [30] we introduced a new variant which follows the spirit of the celebrated works of Lieb [29, 94] and Lions [96, 97], but seems more elementary and may be easier to adapt to other manifolds. The proof of Theorem 6.6 involves a variant of Lions' concentration-compactness lemma [96], a variant of the corollary of the Brézis–Lieb lemma from [59], bilinear extension estimates, and a refinement of (67) over a suitable *cap space*. In a range of exponents that includes the case $\alpha = 4$, we were then able to resolve the dichotomy posed by Theorem 6.6.

Theorem 6.7 ([30]). *There exists $\alpha_0 > 4$ such that, for every $\alpha \in (1, \alpha_0) \setminus \{2\}$, the strict inequality (71) holds. In particular, if $\alpha \in (1, \alpha_0)$, then there exists a maximizer for (68).*

Our method yields $\alpha_0 \approx 4.803$ with 3 decimal places, and effectively computes arbitrarily good lower bounds for the ratio of L^2-norms in (68) via expansions of suitable trial functions in the orthogonal basis of Legendre polynomials. We remark that the value $\alpha_0 \approx 4.803$ is suboptimal, in the sense that a natural refinement of our argument allows to increase this value to $\alpha_0 \approx 5.485$; see [30, §4C].

Once the existence of maximizers has been established, certain qualitative properties can often be deduced from the associated Euler–Lagrange equation; recall §5.4.1. Following this paradigm, we proved that any maximizer of (67) decays super-exponentially in L^2, which reflects the analyticity of its Fourier transform. This is the content of the next result.

Theorem 6.8 ([30]). *Let $\alpha > 1$. If f maximizes (67), then there exists $\mu_0 > 0$, such that*

$$x \mapsto e^{\mu_0 |x|^\alpha} f(x) \in \mathrm{L}^2(\mathbb{R}).$$

In particular, its Fourier transform $\widehat{f}$ can be extended to an entire function on $\mathbb{C}$.

Note that the exponent μ_0 necessarily depends on the maximizer itself; see [48, p. 964]. The proof relies on a bootstrapping argument that found similar applications in [48,58,83,124].

Our methods are able to partially handle the case of the planar *odd* curves of the form $s = y|y|^{\alpha-1}$, $\alpha > 1$. Defining the singular measure

$$\mathrm{d}\mu_\alpha(s,y) = \boldsymbol{\delta}\big(s - y|y|^{\alpha-1}\big)|y|^{\frac{\alpha-2}{6}}\mathrm{d}s\mathrm{d}y, \tag{72}$$

the associated extension operator $\mathcal{S}_\alpha(f) = \widehat{f\mu_\alpha}(-\cdot)$ satisfies the estimate

$$\|\mathcal{S}_\alpha(f)\|_{\mathrm{L}^6} \lesssim \|f\|_{\mathrm{L}^2}.$$

In sharp convolution form, this can be rewritten as

$$\|f\mu_\alpha * f\mu_\alpha * f\mu_\alpha\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \mathbf{Q}_\alpha^3 \|f\|_{\mathrm{L}^2(\mathbb{R})}^3, \tag{73}$$

where $\mathbf{Q}_\alpha$ denotes the optimal constant. Odd curves are of independent interest, in particular because a new phenomenon emerges: caps centered around points with parallel tangents interact strongly, regardless of separation between the points. This mechanism was discovered in [47], and further explored in [38, 66, 69, 70, 124]. Some of these works include a symmetrization step which relies on the convolution structure of the underlying inequality. In the present case, we were able to show that the search for maximizers can be further restricted to the class of even functions, but interestingly *our symmetrization argument does not depend on the convolution structure*. This may be of independent interest since it applies to other extension inequalities where some additional symmetry is present; see [30, §6A]. The following versions of Theorems 6.6 and 6.7 hold for odd curves.

Theorem 6.9 ([30]). *Let $\alpha > 1$. If*

$$\mathbf{Q}_\alpha^6 > \frac{5\pi}{\sqrt{3}\alpha(\alpha-1)}, \tag{74}$$

then any maximizing sequence of nonnegative even functions in $\mathrm{L}^2(\mathbb{R})$ for (73) is precompact, after normalization and scaling. In this case, maximizers for (73) exist. If instead equality holds in (74) then, given any $y_0 \in \mathbb{R}$, there exists a maximizing sequence for (73) which concentrates at the pair $\{-y_0, y_0\}$.

The case $\alpha = 3$ of Theorem 6.9 coincides with a special case of [70, Theorem 1], which was obtained by different methods.

Theorem 6.10 ([30]). *If $\alpha \in (1,2)$, then the strict inequality (74) holds and, in particular, there exists a maximizer for (73).*

We believe that maximizers do not exist if $\alpha \geq 2$; see [30, Conj. 6.6].

6.3 Cone

The restriction conjecture for the (one-sheeted) cone in $\mathbb{R}^{1+d}$, $d \geq 2$, which should be compared to Conjecture 3.5 on $\mathbb{S}^d$, predicts that

$$\|\widehat{g\mu}\|_{\mathrm{L}^q(\mathbb{R}^{1+d})} \lesssim_{p,q} \|g\|_{\mathrm{L}^p(\mathrm{d}\mu)} \tag{75}$$

if and only if $q > \frac{2d}{d-1}$ and $\frac{d+1}{q} = \frac{d-1}{p'}$; here, $d\mu(\tau,\xi) = \boldsymbol{\delta}\big(\tau - |\xi|\big)|\xi|^{-1}\mathrm{d}\tau\mathrm{d}\xi$. Letting

$$\widehat{g\mu} = e^{it\sqrt{-\Delta}}f, \quad f(x) = \mathcal{F}_{\xi\to x}\left(\frac{g(|\xi|,\xi)}{|\xi|}\right), \tag{76}$$

so that f is now a function on $\mathbb{R}^d$, estimate (75) reads as follows:

$$\|e^{it\sqrt{-\Delta}}f\|_{\mathrm{L}^q(\mathbb{R}^{1+d})} \lesssim_{p,q} \|\widehat{f}\|_{\mathrm{L}^p(\mathbb{R}^d,|\xi|^{p-1}\,\mathrm{d}\xi)}. \tag{77}$$

The choice $(p,q) = (2, 2\frac{d+1}{d-1})$ yields the *conformal* (i.e., Stein–Tomas) estimate

$$\|e^{it\sqrt{-\Delta}}f\|_{\mathrm{L}^{2\frac{d+1}{d-1}}(\mathbb{R}^{1+d})} \lesssim_d \|f\|_{\dot{H}^{1/2}(\mathbb{R}^d)}.$$

Very recently, we proved in [102] that maximizers for (75) exist in the full range of exponents for which the estimate does hold; however, if $p \neq 2$, then they are not what one would naïvely expect from Conjecture 6.2. In order to make this precise, consider the extension operator associated to the one-sheeted cone in $\mathbb{R}^{1+d}$, $d \geq 2$,

$$\mathcal{E}g(t,x) := \int_{\mathbb{R}^d} e^{i(t,x)\cdot(|\xi|,\xi)} g(\xi)\,\tfrac{\mathrm{d}\xi}{|\xi|},$$

initially defined on smooth functions with compact support in $\mathbb{R}^d \setminus \{0\}$. It is conjectured that $\mathcal{E}$ extends from $\mathrm{L}^p(|\xi|^{-1}\mathrm{d}\xi)$ to $\mathrm{L}^q(\mathbb{R}^{1+d})$ for $q = \frac{d+1}{d-1}p' =: q(p)$, as a bounded linear functional, for all $1 \leq p < \frac{2d}{d-1} < q \leq \infty$. In other words,

$$\|\mathcal{E}g\|_{\mathrm{L}^q(\mathbb{R}^{1+d})} \lesssim_{p,q} \|g\|_{\mathrm{L}^p(|\xi|^{-1}\mathrm{d}\xi)}, \tag{78}$$

which is just a restatement of (75) with slightly different notation. The case $(p,q) = (1,\infty)$ is elementary, the case $d = 2$ is due to Barcelo [3], the case $d = 3$ is due to Wolff [148], the case $d = 4$ is due to Ou–Wang [113], and the the question is open in all higher dimensions, with the current world record due to Ou–Wang [113].

We assume *a priori* that $\mathcal{E}$ extends as a bounded linear operator from $\mathrm{L}^{\tilde{p}}(|\xi|^{-1}\mathrm{d}\xi)$ to $\mathrm{L}^{q(\tilde{p})}(\mathbb{R}^{1+d})$ for all $\tilde{p}$ in some neighborhood of p. We let $\mathbf{C}_{p,q}$ denote the operator norm $\|\mathcal{E}\|_{\mathrm{L}^p(|\xi|^{-1}\mathrm{d}\xi)\to\mathrm{L}^q}$, that is, the smallest constant for which (78) holds.

Theorem 6.11 ([102]). *Let $d \geq 2$, let $1 < p < 2d/(d-1)$, and assume that $\mathcal{E}$ extends as a bounded linear operator from $\mathrm{L}^{\tilde{p}}(|\xi|^{-1}\mathrm{d}\xi)$ to $\mathrm{L}^{q(\tilde{p})}(\mathbb{R}^{1+d})$, for all $\tilde{p}$ in some neighborhood of p. Then there exist nonzero functions $f \in \mathrm{L}^p(|\xi|^{-1}\mathrm{d}\xi)$ such that $\|\mathcal{E}f\|_q = \mathbf{C}_{p,q}\|f\|_p$. Furthermore, if $\{f_n\} \subseteq \mathrm{L}^p(|\xi|^{-1}\mathrm{d}\xi)$ is any norm-one sequence with $\lim_{n\to\infty}\|\mathcal{E}f_n\|_q = \mathbf{C}_{p,q}$, then a subsequence of $\{f_n\}$ converges modulo symmetries to a maximizer of $\mathcal{E}$.*

Theorem 6.11 results from ensuring that, after passing to a subsequence and applying the symmetries of the operator, any maximizing sequence for (78) has a subsequence with good frequency *and* space localization properties. We postpone a more detailed discussion of possible concentration mechanisms in the related context of the one-sheeted hyperboloid to §7.1, and proceed to discuss the second main result of [102].

Theorem 6.12 ([102]). *Let $d \geq 2$, let $1 < p < 2d/(d-1)$, and set $q = q(p)$. Foschians are critical points for the $\mathrm{L}^p \to \mathrm{L}^q$ extension inequality (77) if and only if $p = 2$.*

The analogue of Theorem 6.12 for paraboloids[5] is due to Christ–Quilodrán [46], see also [40], but their methods are complex analytic and tied to the fact that gaussians define holomorphic functions. *Foschians*, defined in (80) below, do not. Our approach towards Theorem 6.12 makes use of the Penrose transform, whose relevance was realized in [74,99, 100]. The Euler–Lagrange equation for (77) is satisfied by some function $f_\star$ if and only if

$$\Re \int_{\mathbb{R}^{1+d}} |e^{it\sqrt{-\Delta}} f_\star(x)|^{q-2} \overline{e^{it\sqrt{-\Delta}} f_\star}(x) e^{it\sqrt{-\Delta}} f(x)\, \mathrm{d}t\mathrm{d}x \\ = \lambda_{pq} \Re \int_{\mathbb{R}^d} |\widehat{f}_\star(\xi)|^{p-2} \overline{\widehat{f}_\star(\xi)} \widehat{f}(\xi) |\xi|^{p-1}\, \mathrm{d}\xi, \quad (79)$$

with λ_{pq} independent of the arbitrary test function f. Considering the Foschian given by

$$\widehat{f}_\star(\xi) = |\xi|^{-1} e^{-|\xi|}, \quad (80)$$

we then have $f_\star = (1 + |\cdot|^2)^{\frac{1-d}{2}}$; recall Conjecture 6.2. The fact that estimate (77) is conformal if and only if $p = 2$ lies at the heart of our approach towards Theorem 6.12. The Penrose transform [117] is a conformal map $\mathcal{P} : \mathbb{R}^{1+d} \to \mathbb{S}^1 \times \mathbb{S}^d$ which compactifies spacetime, and on the initial time slice $\{t = 0\}$ coincides with the usual stereographic projection. It further maps a solution $u = u(t, x)$ of the wave equation $u_{tt} = \Delta u$ into a solution $U = U(T, X)$ of the hyperbolic equation

$$U_{TT} - \Delta_{\mathbb{S}^d} U + (\tfrac{d-1}{2})^2 U = 0 \quad \text{on the } \textit{Penrose diagram } \mathcal{P}(\mathbb{R}^{1+d}) \subsetneq \mathbb{S}^1 \times \mathbb{S}^d. \quad (81)$$

Crucially, the Penrose transform maps $(f_\star = (1 + |\cdot|^2)^{\frac{1-d}{2}}, \mathbf{0})$ on $\mathbb{R}^d$ into the constant data $(\mathbf{1}, \mathbf{0})$ on $\mathbb{S}^d$. This enables the exact computation of the relevant first and second variations, and leads to the proof of Theorem 6.12.

Surprisingly, for the two-sheeted cone, $\{(\tau, \xi) \in \mathbb{R}^{1+d} : \tau^2 = |\xi|^2\}$, the effectiveness of the Penrose transform is affected by dimension parity. If d is odd, then there is a *hidden symmetry*: the solution U to (81) satisfies $U(T + \pi, -X) = U(T, X)$, for every $(T, X) \in \mathbb{S}^1 \times \mathbb{S}^d$. This turns estimate (56) into one where the left-hand integration is over the whole Cartesian product $\mathbb{S}^1 \times \mathbb{S}^d$, instead of the less symmetric and somewhat awkward Penrose diagram $\mathcal{P}(\mathbb{R}^{1+d})$. If d is even, then this simplification is unavailable, and the lack of a hidden symmetry is ultimately responsible for $(f_\star, \mathbf{0})$ not being a maximizer [99]. It would be interesting to establish analogues of Theorems 6.11 and 6.12 for the two-sheeted cone.

[5]For spheres, the situation is different, as one easily checks that constants are *always* critical points.

7 Hyperboloid

Hyperboloids locally look like paraboloids, with largest curvature at the origin, and globally resemble cones. As such, sharp restriction theory on hyperboloids shares features from both paraboloids and cones and serves as a natural bridge between them. On the other hand, genuinely new phenomena emerge, as we shall see.

7.1 Two-sheeted hyperboloid

Consider the upper sheet of the two-sheeted hyperboloid,

$$\mathbb{H}^d = \{(\tau,\xi)\in\mathbb{R}^{1+d} : \tau = \langle\xi\rangle\},\quad \langle\xi\rangle := \sqrt{1+|\xi|^2},$$

and equip it with the Lorentz invariant measure

$$\mathrm{d}\sigma(\tau,\xi) = \boldsymbol{\delta}\big(\tau-\langle\xi\rangle\big)\langle\xi\rangle^{-1}\mathrm{d}\tau\mathrm{d}\xi. \tag{82}$$

The extension operator on $\mathbb{H}^d$ is given by

$$\mathcal{E}f(t,x) = \int_{\mathbb{R}^d} e^{i(t,x)\cdot(\langle\xi\rangle,\xi)} f(\xi)\frac{\mathrm{d}\xi}{\langle\xi\rangle},$$

and since the 1977 work of Strichartz [133] it is known that[6]

$$\|\mathcal{E}f\|_{\mathrm{L}^p(\mathbb{R}^{d+1})} \le \mathbf{H}_{d,p}\|f\|_{\mathrm{L}^2(\mathbb{H}^d)}; \tag{83}$$

$$2+\frac{4}{d} \le p \le 2+\frac{4}{d-1} \text{ if } d\ge 1. \tag{84}$$

Note that the endpoints in the latter range correspond to Stein–Tomas exponents for the paraboloid and cone, respectively; recall Figure 3. $\mathbf{H}_{d,p}$ denotes the optimal constant in inequality (83) and, in light of (82), $\|f\|^2_{\mathrm{L}^2(\mathbb{H}^d)} = \int_{\mathbb{R}^d}|f(\xi)|^2\frac{\mathrm{d}\xi}{\langle\xi\rangle}$.

The extension operator on $\mathbb{H}^d$ naturally relates to the Klein–Gordon equation, given by $u_{tt} = \Delta u - u$. This connection comes via the Klein–Gordon propagator,

$$e^{it\sqrt{1-\Delta}}g(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(t,x)\cdot(\langle\xi\rangle,\xi)}\widehat{g}(\xi)\mathrm{d}\xi,$$

together with the observation that $\mathcal{E}f(t,x) = (2\pi)^d e^{it\sqrt{1-\Delta}}g(x)$ as long as $\widehat{g}(\xi) = \langle\xi\rangle^{-1}\widehat{f}(\xi)$. This relation implies that (83) can be equivalently rewritten as

$$\|e^{it\sqrt{1-\Delta}}g\|_{\mathrm{L}^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \le (2\pi)^{-d}\mathbf{H}_{d,p}\|g\|_{\mathrm{H}^{1/2}(\mathbb{R}^d)}$$

where $\|\cdot\|_{\mathrm{H}^{1/2}(\mathbb{R}^d)}$ denotes the usual nonhomogeneous Sobolev norm.

[6] With the caveat that the endpoint case $p=\infty$ has to be excluded when $d=1$.

Quilodrán [119] investigated some sharp instances of inequality (83), all corresponding to algebraic endpoints of the range (84), and proved that

$$\mathbf{H}_{2,4} = 2^{\frac{3}{4}}\pi,\ \mathbf{H}_{2,6} = (2\pi)^{\frac{5}{2}},\ \mathbf{H}_{3,4} = (2\pi)^{\frac{5}{4}},$$

even though maximizers do not exist. He also asked about the value of $\mathbf{H}_{1,6}$, which corresponds to the last algebraic endpoint question, and whether maximizers exist in the non-endpoint case in all dimensions.[7] In [42, 44] we answered both of these questions.

Theorem 7.1 ([42]). $\mathbf{H}_{1,6} = 3^{-\frac{1}{12}}(2\pi)^{\frac{1}{2}}$. *Maximizers for* (83) *do not exist if* $(d,p) = (1,6)$.

Theorem 7.2 ([42, 44]). *Maximizers for* (83) *exist when* $6 < p < \infty$*, if* $d = 1$*; and when* $2 + \frac{4}{d} < p < 2 + \frac{4}{d-1}$*, if* $d \geq 2$*. In fact, given any maximizing sequence* $\{f_n\}$*, there exist symmetries* $\mathcal{S}_n$ *such that* $\{\mathcal{S}_n f_n\}$ *converges in* $\mathrm{L}^2(\mathbb{H}^d)$ *to a maximizer* f*, after passing to a subsequence.*

The proof of Theorem 7.1 relies on two ingredients. Firstly, by Lorentz invariance it suffices to study the convolution measure $(\sigma * \sigma * \sigma)(\tau, \xi)$ along the axis $\xi = 0$. We verify that $\tau \mapsto (\sigma * \sigma * \sigma)(\tau, 0)$ defines a continuous function on the half-line $\tau > 3$, which extends continuously to the boundary of its support, so that

$$\sup_{\tau>3}(\sigma * \sigma * \sigma)(\tau, 0) = (\sigma * \sigma * \sigma)(3, 0) = \frac{2\pi}{\sqrt{3}}, \tag{85}$$

and that this global maximum is strict. By an application of the Cauchy–Schwarz inequality, which is similar but simpler than the one for the sphere discussed in §5.2, it follows that $\mathbf{H}_{1,6} \leq 3^{-1/12}(2\pi)^{1/2}$. For the reverse inequality, one checks that $f_n = \exp(-n\langle\cdot\rangle)$ forms a maximizing sequence for (83), in the sense that

$$\lim_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma * f_n\sigma\|^2_{\mathrm{L}^2(\mathbb{R}^2)}}{\|f_n\|^6_{\mathrm{L}^2(\mathbb{H}^1)}} = \frac{2\pi}{\sqrt{3}}.$$

This crucially relies on the fact that the strict global maximum of $\sigma * \sigma * \sigma$ occurs at the *boundary* of the support of the convolution; recall (85). For higher-order convolutions σ^{*k}, $k \geq 4$, the global maximum occurs in the interior of the support, and this provided the initial hint towards Theorem 7.2. The proof of the latter is more involved, and crucially relies on a *sharpened Strichartz estimate*. If $d \in \{1, 2\}$, corresponding to ranges whose endpoints are even integers, then this can be obtained via elementary methods, such as the Hausdorff–Young and Hardy–Littlewood–Sobolev inequalities. For instance, on $\mathbb{H}^1$ we proved in [42, Cor. 13], for each $6 \leq p < \infty$, the existence of $C_p < \infty$ for which

$$\|\mathcal{E}f\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq C_p \sup_{k\in\mathbb{Z}} \|f_k\|^{1/3}_{\mathrm{L}^2(\mathbb{H}^1)} \|f\|^{2/3}_{\mathrm{L}^2(\mathbb{H}^1)}. \tag{86}$$

[7] The methods of [60] do not apply directly due to the lack of exact scaling invariance.

The decomposition $f = \sum_{k\in\mathbb{Z}} f_k$ is such that $f_k = f\mathbf{1}_{\mathcal{C}_k}$, where the family of *hyperbolic caps* $\{\mathcal{C}_k\}_{k\in\mathbb{Z}} \subset \mathbb{H}^1$ is given by

$$\mathcal{C}_k := \{(\tau,\xi) \in \mathbb{H}^1 : \sinh(k-\tfrac{1}{2}) \leq \xi \leq \sinh(k+\tfrac{1}{2})\}.$$

Inequality (86) allows us to start gaining control over arbitrary maximizing sequences. In particular, it can be used to show the existence of a *distinguished cap* which contains a positive proportion of the total mass; possibly after a Lorentz boost, the distinguished cap can be assumed to coincide with $\mathcal{C}_0$; see Figure 6. This rules out the possibility of mass concentration at infinity, which had been previously identified in [119] as the main obstruction to the precompactness of maximizing sequences modulo symmetries. If $d \geq 3$, then the sharpened Strichartz estimate follows from bilinear restriction theory; see [44, Theorem 5.1]. We omit the technical details, and refer the interested reader to [44, §5].

It would be interesting to understand whether the two-sheeted hyperboloid shares similar features to the ones of the two-sheeted cone; see the end of §6.3.

Quilodrán [120] recently showed that maximizers *exist* for the endpoint $L^2 \to L^4$ extension inequality on the *one-sheeted* hyperboloid in $\mathbb{R}^{1+3}$. In the final §7.2, we turn our attention to the current status of the restriction conjecture to the one-sheeted hyperboloid in $\mathbb{R}^{1+2}$.

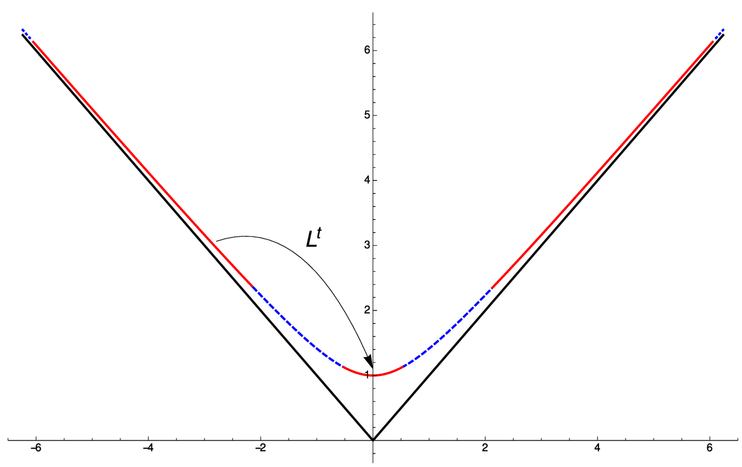

Figure 6: The one-dimensional cap movement: a carefully chosen Lorentz boost L^t : $(\tau,\xi) \mapsto (1-t^2)^{-1/2}(\tau+t\xi, \xi+t\tau)$ interchanges the caps. Here, $L^t(\mathcal{C}_{-2}) = \mathcal{C}_0$ for $t = \tanh(2)$.

7.2 One-sheeted hyperboloid

Sharpened Strichartz estimates, bilinear restriction theory and decoupling tools led to some recent progress on the restriction conjecture for the one-sheeted hyperboloid. We shall finish this lecture by summarizing our results from [32] on the boundedness of the restriction operator associated to the one-sheeted hyperboloid in $\mathbb{R}^{1+2}$,

$$\Gamma := \{(\tau,\xi) \in \mathbb{R}^{1+2} : 1+\tau^2 = |\xi|^2\}.$$

This surface is invariant under the Lorentz transformations

$$L_\nu : (\tau, \xi) \mapsto (\langle \nu \rangle \tau - \nu \cdot \xi, \xi^\perp + \langle \nu \rangle \xi^\parallel - \nu \tau), \qquad \nu \in \mathbb{R}^2, \tag{87}$$

where $\xi^\perp, \xi^\parallel$ are the perpendicular and parallel components of ξ with respect to ν. We endow the surface with the unique (up to scalar multiples) Lorentz-invariant measure σ, which coincides with what is known as the affine surface measure,

$$\int_\Gamma f \, \mathrm{d}\sigma = \int_{\{|\xi|>1\}} (f(-\langle\!\langle \xi \rangle\!\rangle, \xi) + f(\langle\!\langle \xi \rangle\!\rangle, \xi)) \, \tfrac{\mathrm{d}\xi}{\langle\!\langle \xi \rangle\!\rangle}, \quad \text{where} \quad \langle\!\langle \xi \rangle\!\rangle := \sqrt{|\xi|^2 - 1}, \ |\xi| \geq 1.$$

Various geometric features of Γ make it potentially interesting from the perspective of restriction theory. Even though the gaussian curvature is nonvanishing, the principal curvatures have different signs, which presents challenges at all scales because the restriction theory for hyperbolic surfaces is much less well-developed than that for elliptic surfaces.

One of our main contributions was an adaptation of the techniques of [93, 130, 142] to establish unconditional, global restriction inequalities in the bilinear range. In particular, we established the first extension inequalities on the parabolic scaling line $q = 2p'$ beyond the Stein–Tomas range (i.e. with $p > 2$) for any negatively curved surface that is not the hyperbolic paraboloid; see [130]. The above-mentioned techniques are directly applicable in the low frequency region $\{|\xi| \lesssim 1\}$, but at high frequencies, the surface is asymptotic to the cone, presenting some additional complications. In this region, we used conic decoupling and interpolation with bilinear inequalities to prove a conditional result that boosts local restriction inequalities on the low frequency region to global ones in a range that is non-optimal but, nevertheless, offers the possibility of improvement over that obtainable directly from bilinear restriction. Our explorations of the conic region also suggest possible future applications of some – surprisingly, still open – questions about the restriction operator associated to the cone in $\mathbb{R}^{1+2}$.

We turn now to statements of our main results, given in terms of the extension operator $\mathcal{E}f := \widehat{f\mathrm{d}\sigma}$, and its local version $\mathcal{E}_0 f := \mathcal{E}(\mathbf{1}_{\{|\xi|\lesssim 1\}} f)$. We say that $\mathcal{R}^*(p \to q)$ holds if there exists a universal constant $C < \infty$ such that $\|\mathcal{E}f\|_{\mathrm{L}^q(\mathbb{R}^3)} \leq C \|f\|_{\mathrm{L}^p(\Gamma)}$, for all $f \in C_0^\infty(\mathbb{R}^3)$; we say that $\mathcal{R}_0^*(p \to q)$ holds when the analogous statement holds with $\mathcal{E}_0$ in place of $\mathcal{E}$.

Theorem 7.3 ([32]). *For $(p, q) \neq (4, 4)$ obeying $2p' \leq q \leq 3p'$, $q \geq p$, and $q > \frac{10}{3}$, $\mathcal{R}^*(p \to q)$ holds. Moreover, for $3 < q_0 < \frac{10}{3}$, $\mathcal{R}_0^*((\frac{q_0}{2})' \to q_0)$ implies $\mathcal{R}^*(p \to q)$ for all exponent pairs obeying $q_0 < q \leq \frac{10}{3}$, $(\frac{q}{2})' \leq p \leq q$, and*

$$\frac{1}{p} > \frac{2}{5} \cdot \frac{1/q - 3/10}{1/q_0 - 3/10} + \frac{1}{10}.$$

It was proved in [31] that $\mathcal{R}_0^*((\frac{q_0}{2})' \to q_0)$ holds for $q_0 > 3.25$, and so our conditional result implies that $\mathcal{R}^*(p \to q)$ holds for $q \leq \frac{10}{3}$, $(\frac{q}{2})' \leq p \leq q$, and

$$\frac{1}{p} > \frac{52}{q} - \frac{31}{2}.$$

(The upper line segment of this region has endpoints $(\frac{1}{p}, \frac{1}{q}) = (\frac{31}{102}, \frac{31}{102})$ and $(\frac{7}{18}, \frac{11}{36})$.) Because of the loss in the range of q, we expect the conditionality in Theorem 7.3 not to be optimal. However, improvements to the range of $\mathrm{L}^p \times \mathrm{L}^p \to \mathrm{L}^q$ bilinear extension inequalities for the cone in $\mathbb{R}^3$ may suggest a means of improving the range in our conditional result. By contrast with Theorem 7.3, we note the following negative result.

Theorem 7.4 ([32]). *For $(p, q) \in \{(3,3), (4,4)\}$ and for (p^{-1}, q^{-1}) lying outside of the triangle*

$$\{(p^{-1}, q^{-1}) : 2p' \leq q \leq 3p',\ q \geq p\},$$

$\mathcal{R}^*(p \to q)$ fails.

Even though the argument is fairly simple, we had not expected to find any exponent pairs along the diagonal $q = p$ at which $\mathcal{R}^*(p \to q)$ holds; see Figure 7. The Kakeya-like example of [6] rules out even a restricted weak-type inequality at the endpoint $(3, 3)$, but we were not able to exclude the possibility that some weaker inequality might be valid at the endpoint $(4, 4)$. In fact, the analogous question for the extension operator associated to the cone also seems to be open.

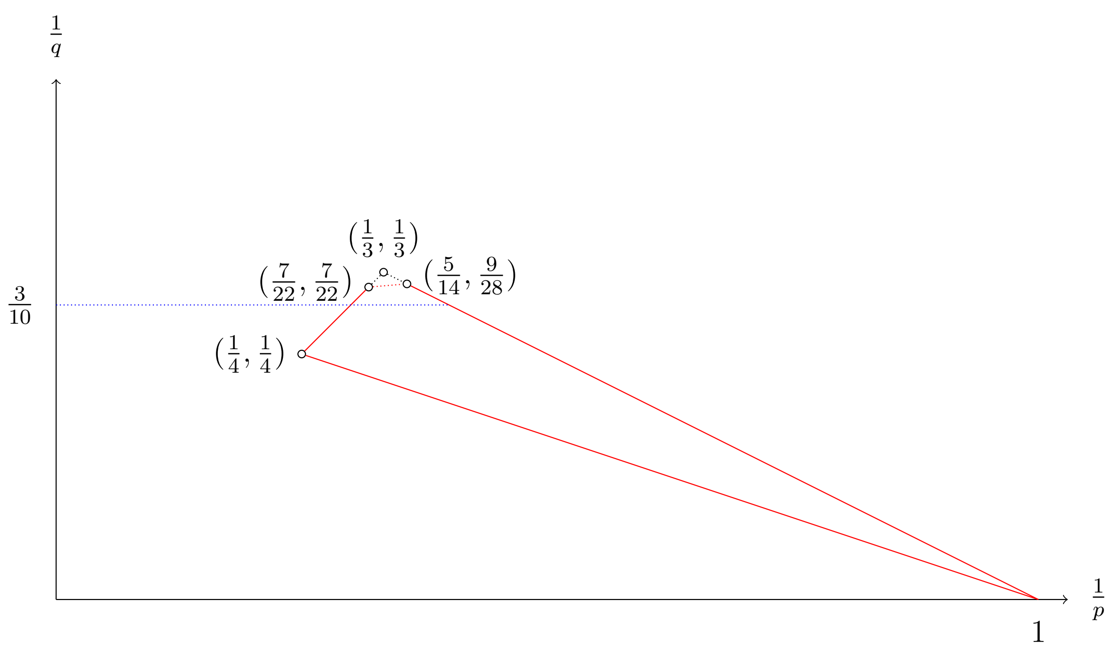

Figure 7: By Theorem 7.3, the full restriction conjecture for the low frequency region would imply global restriction estimates for exponent pairs (p^{-1}, q^{-1}) within the red quadrilateral. Unconditional estimates hold in the bilinear range $q > \frac{10}{3}$.

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References

[1] S. Agmon and L. Hörmander. Asymptotic properties of solutions of differential equations with simple characteristics. *J. Analyse Math.*, 30:1–38, 1976.

[2] K. I. Babenko. An inequality in the theory of Fourier integrals. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:531–542, 1961.

[3] B. Barcelo Taberner. On the restriction of the Fourier transform to a conical surface. *Trans. Amer. Math. Soc.*, 292(1):321–333, 1985.

[4] J. Barker, C. Thiele, and P. Zorin-Kranich. Band-limited maximizers for a Fourier extension inequality on the circle, II. *Exp. Math.*, 32(2):280–293, 2023.

[5] W. Beckner. Inequalities in Fourier analysis. *Ann. of Math. (2)*, 102(1):159–182, 1975.

[6] W. Beckner, A. Carbery, S. Semmes, and F. Soria. A note on restriction of the Fourier transform to spheres. *Bull. London Math. Soc.*, 21(4):394–398, 1989.

[7] P. Bégout and A. Vargas. Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.*, 359(11):5257–5282, 2007.

[8] D. Beltran and L. Vega. Bilinear identities involving the k-plane transform and Fourier extension operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(6):3349–3377, 2020.

[9] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark. Heat-flow monotonicity of Strichartz norms. *Analysis & PDE*, 2(2):147–158, 2009.

[10] J. Bennett, N. Bez, M. G. Cowling, and T. C. Flock. Behaviour of the Brascamp-Lieb constant. *Bull. Lond. Math. Soc.*, 49(3):512–518, 2017.

[11] J. Bennett, N. Bez, T. C. Flock, S. Gutiérrez, and M. Iliopoulou. A sharp k-plane Strichartz inequality for the Schrödinger equation. *Trans. Amer. Math. Soc.*, 370(8):5617–5633, 2018.

[12] J. Bennett, N. Bez, T. C. Flock, and S. Lee. Stability of the Brascamp-Lieb constant and applications. *Amer. J. Math.*, 140(2):543–569, 2018.

[13] J. Bennett, N. Bez, and M. Iliopoulou. Flow monotonicity and Strichartz inequalities. *International Mathematics Research Notices*, 2015(19):9415–9437, 12 2014.

[14] J. Bennett, N. Bez, C. Jeavons, and N. Pattakos. On sharp bilinear Strichartz estimates of Ozawa-Tsutsumi type. *J. Math. Soc. Japan*, 69(2):459–476, 2017.

[15] J. Bennett, A. Carbery, and T. Tao. On the multilinear restriction and Kakeya conjectures. *Acta Math.*, 196(2):261–302, 2006.

[16] N. Bez and C. Jeavons. A sharp Sobolev-Strichartz estimate for the wave equation. *Electron. Res. Announc. Math. Sci.*, 22:46–54, 2015.

[17] N. Bez, C. Jeavons, and T. Ozawa. Some sharp bilinear space-time estimates for the wave equation. *Mathematika*, 62(3):719–737, 2016.

[18] N. Bez, C. Jeavons, T. Ozawa, and H. Saito. A conjecture regarding optimal Strichartz estimates for the wave equation. In *New trends in analysis and interdisciplinary applications*, Trends Math. Res. Perspect., pages 293–299. Birkhäuser/Springer, Cham, 2017.

[19] N. Bez and K. M. Rogers. A sharp Strichartz estimate for the wave equation with data in the energy space. *J. Eur. Math. Soc. (JEMS)*, 15(3):805–823, 2013.

[20] N. Bez and M. Sugimoto. Optimal constants and extremisers for some smoothing estimates. *J. Anal. Math.*, 131:159–187, 2017.

[21] G. Bianchi and H. Egnell. A note on the Sobolev inequality. *J. Funct. Anal.*, 100(1):18–24, 1991.

[22] C. Biswas and B. Stovall. Existence of extremizers for Fourier restriction to the moment curve. *Trans. Amer. Math. Soc.*, 376(5):3473–3492, 2023.

[23] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin. Densities of short uniform random walks. *Canad. J. Math.*, 64(5):961–990, 2012. With an appendix by Don Zagier.

[24] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis. *Geom. Funct. Anal.*, 1(2):147–187, 1991.

[25] J. Bourgain and C. Demeter. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[26] J. Bourgain, C. Demeter, and L. Guth. Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. *Ann. of Math. (2)*, 184(2):633–682, 2016.

[27] J. Bourgain and L. Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[28] S. Brendle. The isoperimetric inequality for a minimal submanifold in Euclidean space. *J. Amer. Math. Soc.*, 34(2):595–603, 2021.

[29] H. Brézis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.

[30] G. Brocchi, D. Oliveira e Silva, and R. Quilodrán. Sharp Strichartz inequalities for fractional and higher-order Schrödinger equations. *Analysis & PDE*, 13:477–526, 03 2020.

[31] B. B. Bruce. Local extension estimates for the hyperbolic hyperboloid in three dimensions. *J. Fourier Anal. Appl.*, 28(5):Paper No. 78, 32, 2022.

[32] B. B. Bruce, D. Oliveira e Silva, and B. Stovall. Restriction inequalities for the hyperbolic hyperboloid. *J. Math. Pures Appl. (9)*, 149:186–215, 2021.

[33] A. Bulut. Maximizers for the Strichartz inequalities for the wave equation. *Differential Integral Equations*, 23(11-12):1035–1072, 2010.

[34] K. J. Böröczky, A. Figalli, and J. P. G. Ramos. A quantitative stability result for the Prékopa-Leindler inequality for arbitrary measurable functions, 2022. To appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

[35] A. Carbery. Restriction implies Bochner-Riesz for paraboloids. *Math. Proc. Cambridge Philos. Soc.*, 111(3):525–529, 1992.

[36] L. Carleson and P. Sjölin. Oscillatory integrals and a multiplier problem for the disc. *Studia Math.*, 44:287–299, 1972.

[37] E. Carneiro. A sharp inequality for the Strichartz norm. *Int. Math. Res. Not. IMRN*, (16):3127–3145, 2009.

[38] E. Carneiro, D. Foschi, D. Oliveira e Silva, and C. Thiele. A sharp trilinear inequality related to Fourier restriction on the circle. *Rev. Mat. Iberoam.*, 33(4):1463–1486, 2017.

[39] E. Carneiro, G. Negro, and D. Oliveira e Silva. Stability of sharp Fourier restriction to spheres, 2022.

[40] E. Carneiro, L. Oliveira, and M. Sousa. Gaussians never extremize Strichartz inequalities for hyperbolic paraboloids. *Proc. Amer. Math. Soc.*, 150(8):3395–3403, 2022.

[41] E. Carneiro and D. Oliveira e Silva. Some sharp restriction inequalities on the sphere. *Int. Math. Res. Not. IMRN*, (17):8233–8267, 2015.

[42] E. Carneiro, D. Oliveira e Silva, and M. Sousa. Extremizers for Fourier restriction on hyperboloids. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(2):389–415, 2019.

[43] E. Carneiro, D. Oliveira e Silva, and M. Sousa. Sharp mixed norm spherical restriction. *Adv. Math.*, 341:583–608, 2019.

[44] E. Carneiro, D. Oliveira e Silva, M. Sousa, and B. Stovall. Extremizers for adjoint Fourier restriction on hyperboloids: the higher-dimensional case. *Indiana Univ. Math. J.*, 70(2):535–559, 2021.

[45] M. Christ. A sharpened Hausdorff-Young inequality, 2014.

[46] M. Christ and R. Quilodrán. Gaussians rarely extremize adjoint Fourier restriction inequalities for paraboloids. *Proc. Amer. Math. Soc.*, 142(3):887–896, 2014.

[47] M. Christ and S. Shao. Existence of extremals for a Fourier restriction inequality. *Anal. PDE*, 5(2):261–312, 2012.

[48] M. Christ and S. Shao. On the extremizers of an adjoint Fourier restriction inequality. *Adv. Math.*, 230(3):957–977, 2012.

[49] V. Ciccone and M. Sousa. Global and local maximizers for some Fourier extension estimates on the sphere, 2023.

[50] A. Córdoba. The Kakeya maximal function and the spherical summation multipliers. *Amer. J. Math.*, 99(1):1–22, 1977.

[51] A. Córdoba. Singular integrals, maximal functions and Fourier restriction to spheres: the disk multiplier revisited. *Adv. Math.*, 290:208–235, 2016.

[52] R. O. Davies. Some remarks on the Kakeya problem. *Proc. Cambridge Philos. Soc.*, 69:417–421, 1971.

[53] B. Di and D. Yan. Extremals for α-Strichartz inequalities. *J. Geom. Anal.*, 33(4):Paper No. 136, 46, 2023.

[54] B. Dodson, J. L. Marzuola, B. Pausader, and D. P. Spirn. The profile decomposition for the hyperbolic Schrödinger equation. *Illinois J. Math.*, 62(1-4):293–320, 2018.

[55] J. Dolbeault, M. J. Esteban, and M. Loss. Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces. *Invent. Math.*, 206(2):397–440, 2016.

[56] T. Duyckaerts, F. Merle, and S. Roudenko. Maximizers for the Strichartz norm for small solutions of mass-critical NLS. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(2):427–476, 2011.

[57] Z. Dvir. On the size of Kakeya sets in finite fields. *J. Amer. Math. Soc.*, 22(4):1093–1097, 2009.

[58] M. B. Erdoğan, D. Hundertmark, and Y.-R. Lee. Exponential decay of dispersion managed solitons for vanishing average dispersion. *Math. Res. Lett.*, 18(1):11–24, 2011.

[59] L. Fanelli, L. Vega, and N. Visciglia. On the existence of maximizers for a family of restriction theorems. *Bull. Lond. Math. Soc.*, 43(4):811–817, 2011.

[60] L. Fanelli, L. Vega, and N. Visciglia. Existence of maximizers for Sobolev-Strichartz inequalities. *Adv. Math.*, 229(3):1912–1923, 2012.

[61] C. Fefferman. The multiplier problem for the ball. *Ann. of Math. (2)*, 94:330–336, 1971.

[62] A. Figalli. Stability in geometric and functional inequalities. In *European Congress of Mathematics*, pages 585–599. Eur. Math. Soc., Zürich, 2013.

[63] A. Figalli. Quantitative stability results for the Brunn-Minkowski inequality. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 237–256. Kyung Moon Sa, Seoul, 2014.

[64] T. C. Flock and B. Stovall. On extremizing sequences for adjoint Fourier restriction to the sphere, 2022.

[65] D. Foschi. Maximizers for the Strichartz inequality. *J. Eur. Math. Soc. (JEMS)*, 9(4):739–774, 2007.

[66] D. Foschi. Global maximizers for the sphere adjoint Fourier restriction inequality. *J. Funct. Anal.*, 268(3):690–702, 2015.

[67] D. Foschi and S. Klainerman. Bilinear space-time estimates for homogeneous wave equations. *Ann. Sci. École Norm. Sup. (4)*, 33(2):211–274, 2000.

[68] D. Foschi and D. Oliveira e Silva. Some recent progress on sharp Fourier restriction theory. *Anal. Math.*, 43(2):241–265, 2017.

[69] R. L. Frank, E. H. Lieb, and J. Sabin. Maximizers for the Stein-Tomas inequality. *Geom. Funct. Anal.*, 26(4):1095–1134, 2016.

[70] R. L. Frank and J. Sabin. Extremizers for the Airy-Strichartz inequality. *Math. Ann.*, 372(3-4):1121–1166, 2018.

[71] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *Journal of Functional Analysis*, 32:1–32, 1979.

[72] F. Gonçalves. Orthogonal polynomials and sharp estimates for the Schrödinger equation. *Int. Math. Res. Not. IMRN*, (8):2356–2383, 2019.

[73] F. Gonçalves. A sharpened Strichartz inequality for radial functions. *J. Funct. Anal.*, 276(6):1925–1947, 2019.

[74] F. Gonçalves and G. Negro. Local maximizers of adjoint Fourier restriction estimates for the cone, paraboloid and sphere. *Anal. PDE*, 15(4):1097–1130, 2022.

[75] F. Gonçalves and D. B. Zagier. Strichartz estimates with broken symmetries. *Rev. Mat. Iberoam.*, 38(5):1709–1722, 2022.

[76] L. Guth. A restriction estimate using polynomial partitioning. *J. Amer. Math. Soc.*, 29(2):371–413, 2016.

[77] L. Guth. Restriction estimates using polynomial partitioning II. *Acta Math.*, 221(1):81–142, 2018.

[78] L. Guth. Decoupling estimates in Fourier analysis, 2022. To appear in ICM Proceedings 2022.

[79] L. Guth and N. H. Katz. On the Erdős distinct distances problem in the plane. *Ann. of Math. (2)*, 181(1):155–190, 2015.

[80] W. Han. The sharp Strichartz and Sobolev-Strichartz inequalities for the fourth-order Schrödinger equation. *Math. Methods Appl. Sci.*, 38(8):1506–1514, 2015.

[81] J. Hickman and K. M. Rogers. Improved Fourier restriction estimates in higher dimensions. *Camb. J. Math.*, 7(3):219–282, 2019.

[82] J. Hickman, K. M. Rogers, and R. Zhang. Improved bounds for the Kakeya maximal conjecture in higher dimensions. *Amer. J. Math.*, 144(6):1511–1560, 2022.

[83] D. Hundertmark and S. Shao. Analyticity of extremizers to the Airy-Strichartz inequality. *Bull. Lond. Math. Soc.*, 44(2):336–352, 2012.

[84] D. Hundertmark and V. Zharnitsky. On sharp Strichartz inequalities in low dimensions. *Int. Math. Res. Not.*, pages Art. ID 34080, 18, 2006.

[85] C. Jeavons. A sharp bilinear estimate for the Klein-Gordon equation in arbitrary space-time dimensions. *Differential Integral Equations*, 27(1-2):137–156, 2014.

[86] J.-C. Jiang, B. Pausader, and S. Shao. The linear profile decomposition for the fourth order Schrödinger equation. *J. Differ. Equations*, 249(10):2521–2547, 2010.

[87] J.-C. Jiang, S. Shao, and B. Stovall. Linear profile decompositions for a family of fourth order Schrödinger equations, 2017.

[88] M. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[89] C. E. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[90] R. Killip, B. Stovall, and M. Visan. Scattering for the cubic Klein-Gordon equation in two space dimensions. *Trans. Amer. Math. Soc.*, 364(3):1571–1631, 2012.

[91] S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. *Comm. Pure Appl. Math.*, 46(9):1221–1268, 1993.

[92] M. Kunze. On the existence of a maximizer for the Strichartz inequality. *Comm. Math. Phys.*, 243(1):137–162, 2003.

[93] S. Lee. Bilinear restriction estimates for surfaces with curvatures of different signs. *Trans. Amer. Math. Soc.*, 358(8):3511–3533, 2006.

[94] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[95] E. H. Lieb. Gaussian kernels have only Gaussian maximizers. *Invent. Math.*, 102(1):179–208, 1990.

[96] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[97] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[98] R. Mandel and D. Oliveira e Silva. The Stein-Tomas inequality under the effect of symmetries. *J. Anal. Math.*, 150(2):547–582, 2023.

[99] G. Negro. A sharpened Strichartz inequality for the wave equation, 2022. To appear in Ann. Sci. Éc. Norm. Supér.

[100] G. Negro. A sharpened energy-Strichartz inequality for the wave equation. *Bulletin of the London Mathematical Society*, 55(6):3063–3076, 2023.

[101] G. Negro and D. Oliveira e Silva. Intermittent symmetry breaking and stability of the sharp Agmon-Hörmander estimate on the sphere. *Proc. Amer. Math. Soc.*, 151(1):87–99, 2023.

[102] G. Negro, D. Oliveira e Silva, B. Stovall, and J. Tautges. Exponentials rarely maximize Fourier extension inequalities for cones, 2023.

[103] G. Negro, D. Oliveira e Silva, and C. Thiele. When does $e^{-|\tau|}$ maximize Fourier extension for a conic section? In *Harmonic analysis and convexity*, volume 9 of *Adv. Anal. Geom.*, pages 391–426. De Gruyter, Berlin, 2023.

[104] D. Oliveira e Silva. Extremizers for Fourier restriction inequalities: convex arcs. *J. Anal. Math.*, 124:337–385, 2014.

[105] D. Oliveira e Silva. Sharp Fourier Restriction Theory, 2017. Habilitation thesis, Universität Bonn.

[106] D. Oliveira e Silva. Nonexistence of extremizers for certain convex curves. *Math. Res. Lett.*, 25(3):973–987, 2018.

[107] D. Oliveira e Silva and R. Quilodrán. On extremizers for Strichartz estimates for higher order Schrödinger equations. *Transactions of the American Mathematical Society*, 370(10):6871–6907, 2018.

[108] D. Oliveira e Silva and R. Quilodrán. A comparison principle for convolution measures with applications. *Math. Proc. Cambridge Philos. Soc.*, 169(2):307–322, 2020.

[109] D. Oliveira e Silva and R. Quilodrán. Global maximizers for adjoint Fourier restriction inequalities on low dimensional spheres. *J. Funct. Anal.*, 280(7):Paper No. 108825, 73, 2021.

[110] D. Oliveira e Silva and R. Quilodrán. Smoothness of solutions of a convolution equation of restricted type on the sphere. *Forum Math. Sigma*, 9:Paper No. e12, 40, 2021.

[111] D. Oliveira e Silva and C. Thiele. Estimates for certain integrals of products of six Bessel functions. *Rev. Mat. Iberoam.*, 33(4):1423–1462, 2017.

[112] D. Oliveira e Silva, C. Thiele, and P. Zorin-Kranich. Band-limited maximizers for a Fourier extension inequality on the circle. *Exp. Math.*, 31(1):192–198, 2022.

[113] Y. Ou and H. Wang. A cone restriction estimate using polynomial partitioning. *J. Eur. Math. Soc. (JEMS)*, 24(10):3557–3595, 2022.

[114] T. Ozawa and K. M. Rogers. Sharp Morawetz estimates. *J. Anal. Math.*, 121:163–175, 2013.

[115] T. Ozawa and K. M. Rogers. A sharp bilinear estimate for the Klein-Gordon equation in $\mathbb{R}^{1+1}$. *Int. Math. Res. Not. IMRN*, (5):1367–1378, 2014.

[116] T. Ozawa and Y. Tsutsumi. Space-time estimates for null gauge forms and nonlinear Schrödinger equations. *Differential Integral Equations*, 11(2):201–222, 1998.

[117] R. Penrose. Republication of: Conformal treatment of infinity. *General Relativity and Gravitation*, 43(3):901–922, Mar 2011.

[118] R. Quilodrán. On extremizing sequences for the adjoint restriction inequality on the cone. *J. Lond. Math. Soc. (2)*, 87(1):223–246, 2013.

[119] R. Quilodrán. Nonexistence of extremals for the adjoint restriction inequality on the hyperboloid. *J. Anal. Math.*, 125:37–70, 2015.

[120] R. Quilodrán. Existence of extremals for a Fourier restriction inequality on the one-sheeted hyperboloid, 2022.

[121] J. Ramos. A refinement of the Strichartz inequality for the wave equation with applications. *Adv. Math.*, 230(2):649–698, 2012.

[122] S. Shao. The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy–Strichartz inequality. *Anal. PDE*, 2(1):83–117, 2009.

[123] S. Shao. Maximizers for the Strichartz and the Sobolev-Strichartz inequalities for the Schrödinger equation. *Electron. J. Differ. Equ.*, 2009:13, 2009. Id/No 03.

[124] S. Shao. On existence of extremizers for the Tomas-Stein inequality for S^1. *J. Funct. Anal.*, 270(10):3996–4038, 2016.

[125] E. M. Stein. Some problems in harmonic analysis. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, pages 3–20. Amer. Math. Soc., Providence, RI, 1979.

[126] E. M. Stein. Oscillatory integrals in Fourier analysis. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 307–355. Princeton Univ. Press, Princeton, NJ, 1986.

[127] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[128] E. M. Stein and R. Shakarchi. *Functional analysis. Introduction to further topics in analysis*, volume 4 of *Princeton Lect. Anal.* Princeton, NJ: Princeton University Press, 2011.

[129] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, NJ, 1971.

[130] B. Stovall. Scale-invariant Fourier restriction to a hyperbolic surface. *Anal. PDE*, 12(5):1215–1224, 2019.

[131] B. Stovall. Waves, spheres, and tubes: a selection of Fourier restriction problems, methods, and applications. *Notices Amer. Math. Soc.*, 66(7):1013–1022, 2019.

[132] B. Stovall. Extremizability of Fourier restriction to the paraboloid. *Adv. Math.*, 360:106898, 18, 2020.

[133] R. S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[134] T. Tao. The Bochner-Riesz conjecture implies the restriction conjecture. *Duke Math. J.*, 96(2):363–375, 1999.

[135] T. Tao. From rotating needles to stability of waves: emerging connections between combinatorics, analysis, and PDE. *Notices Amer. Math. Soc.*, 48(3):294–303, 2001.

[136] T. Tao. A sharp bilinear restrictions estimate for paraboloids. *Geom. Funct. Anal.*, 13(6):1359–1384, 2003.

[137] T. Tao. Some recent progress on the restriction conjecture. In *Fourier analysis and convexity*, Appl. Numer. Harmon. Anal., pages 217–243. Birkhäuser Boston, Boston, MA, 2004.

[138] T. Tao. *Nonlinear dispersive equations: local and global analysis.* Number 106. American Mathematical Soc., 2006.

[139] J. Tautges. Extremizers for adjoint restriction to pairs of translated paraboloids. *Journal of Mathematical Analysis and Applications*, 513(2):126208, 2022.

[140] J. Tautges. Extremizers for adjoint restriction to a pair of reflected paraboloids. *J. Funct. Anal.*, 286(3):Paper No. 110207, 2024.

[141] P. A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.*, 81:477–478, 1975.

[142] A. Vargas. Restriction theorems for a surface with negative curvature. *Math. Z.*, 249(1):97–111, 2005.

[143] L. Vega. Restriction theorems and the Schrödinger multiplier on the torus. In *Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990)*, volume 42 of *IMA Vol. Math. Appl.*, pages 199–211. Springer, New York, 1992.

[144] M. S. Viazovska. The sphere packing problem in dimension 8. *Ann. of Math. (2)*, 185(3):991–1015, 2017.

[145] H. Wang. A restriction estimate in $\mathbb{R}^3$ using brooms. *Duke Math. J.*, 171(8):1749–1822, 2022.

[146] C. E. Wayne and V. Zharnitsky. Critical points of Strichartz functional. *Exp. Math.*, 30(2):235–257, 2021.

[147] T. Wolff. An improved bound for Kakeya type maximal functions. *Rev. Mat. Iberoamericana*, 11(3):651–674, 1995.

[148] T. Wolff. A sharp bilinear cone restriction estimate. *Ann. of Math. (2)*, 153(3):661–698, 2001.

[149] T. H. Wolff. *Lectures on harmonic analysis*, volume 29 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003. With a foreword by Charles Fefferman and a preface by Izabella Łaba, Edited by Łaba and Carol Shubin.

[150] A. Zygmund. On Fourier coefficients and transforms of functions of two variables. *Studia Math.*, 50:189–201, 1974.

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