

# Hochschild cohomology of the universal associative conformal envelope of the Virasoro Lie conformal algebra with coefficients in all finite modules

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**Abstract.** In this paper, we find the Hochschild cohomology groups of the universal associative conformal envelope  $U(3)$  of the Virasoro Lie conformal algebra with respect to associative locality  $N = 3$  on the generator with coefficients in all finite modules. In order to obtain this result, we construct the Anick resolution via the algebraic discrete Morse theory and Gröbner–Shirshov basis.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Review of conformal algebras</b>	<b>3</b>
2.1	Conformal algebra . . . . .	3
2.2	Formal power series and Conformal algebra . . . . .	3
2.3	The coefficient algebra . . . . .	4
2.4	Universal enveloping of Virasoro Lie conformal algebras . . . . .	4
2.5	Conformal module . . . . .	5
2.6	Hochschild cohomology for associative algebras . . . . .	6

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2.7	Hochschild cohomology for associative conformal algebras . . . . .	6
2.8	Anick resolution for associative algebras . . . . .	7
<b>3</b>	<b>Anick resolution for <math>\mathcal{A}_+(U(3))</math></b>	<b>8</b>
3.1	Anick resolution for differential algebras . . . . .	13
<b>4</b>	<b>Application of Anick resolution on <math>\mathcal{A}_+(U(3))</math></b>	<b>14</b>

## 1 Introduction

Conformal algebra is introduced in [8] as an algebraic tool to study the singular part of the operator product expansion (OPE) of chiral fields in 2-dimensional conformal field theory.

The study of universal structures for conformal algebras was initiated in [15]. The classical theory of finite-dimensional Lie algebras often needs universal constructions like free algebras and universal enveloping algebras. This was a motivation for the development of combinatorial issues in the theory of conformal algebras [5].

For example, consider the Virasoro conformal algebra  $\text{Vir}$ . One may fix a natural number  $N$  and construct the associative conformal algebra  $U(N)$  generated by a single element  $v$  such that  $(v \binom{v}{n} v) = 0$  for  $n \geq N$ , and the commutation relations of  $\text{Vir}$  hold. Obviously,  $U(1) = 0$ ; the algebra  $U(2)$  is known as the Weyl conformal algebra (also denoted  $\text{Cend}_{1,x}$  [6]). The structure of  $U(3) = U(4)$  was studied in [12] by means of the Gröbner–Shirshov bases method.

For every conformal algebra  $C$  one may construct an “ordinary” algebra  $\mathcal{A}(C)$  called a coefficient algebra of  $C$ , which inherits many properties of  $C$  (associative, commutative, Lie, Jordan, etc.). Every conformal module over  $C$  is also a module over  $\mathcal{A}(C)$ . Moreover, it was proved in [4] that the (reduced) cohomology of a conformal algebra  $C$  may be calculated via the corresponding cochain complex of its coefficient  $\mathcal{A}(C)$ .

It was shown in [13] that the second Hochschild cohomology groups  $H^2(U(2), M)$  are trivial for every conformal (bi-)module  $M$ , but for higher Hochschild cohomology the direct computation becomes too complicated.

In [2] was applied discrete algebraic Morse theory to evaluate the dimensions of the Hochschild cohomology groups with scalar coefficients of the universal associative conformal envelope  $U(3)$  of the Virasoro Lie conformal algebra relative to the associative locality bound  $N = 3$  on the generator.

The Hochschild cohomology groups of the Weyl associative conformal algebra  $U(2)$  with coefficients in all finite modules was found in [1].

The motivation to study  $U(3)$  is that if  $M$  is a finite irreducible  $\text{Vir}$ -module then  $M$  is a finite irreducible  $U(\text{Vir}, N = 3)$ -module,  $U(3)$  is a more adequate associative conformal envelope of  $\text{Vir}$  than the Weyl conformal algebra  $U(2)$ .

In this paper we find the higher Hochschild cohomology groups  $H^n(U(3), M)$  of the universal associative conformal envelope  $U(3)$  of the Virasoro Lie conformal algebra with

coefficients in all finite modules  $M$ . In order to obtain this result we construct the Anick resolution for its coefficient algebra via the algebraic discrete Morse theory.

## 2 Review of conformal algebras

Throughout the paper,  $\mathbb{k}$  is a field of characteristic zero and  $\mathbb{Z}_+$  is the set of nonnegative integers.

### 2.1 Conformal algebra

A left  $H = \mathbb{k}[\partial]$ -module is called a *conformal algebra* [8] if it is equipped with a linear operator  $\partial : C \rightarrow C$  and with  $\mathbb{k}$ -bilinear product  $a_{(n)}b$  such that the following axioms hold

(C1) for every  $a, b \in C$  there exists  $N = N(a, b) \in \mathbb{Z}_+$  such that  $(a_{(n)}b) = 0$  for all  $n \geq N$ ,

(C2)  $(\partial a_{(n)}b) = -n(a_{(n-1)}b)$ ,

(C3)  $(a_{(n)}\partial b) = \partial(a_{(n)}b) + n(a_{(n-1)}b)$ .

The structure of a conformal algebra on an  $H$ -module  $C$  may be expressed by means of a single polynomial-valued map called  $\lambda$ -product:

$$\begin{aligned} (\cdot_{(\lambda)} \cdot) : C \otimes C &\longrightarrow C[\lambda], \\ (a_{(\lambda)}b) &= \sum_{n=0}^{N(a,b)-1} \frac{\lambda^n}{n!} (a_{(n)}b), \end{aligned}$$

where  $\lambda$  is a formal variable, satisfying the following axioms:

$$(\partial a_{(\lambda)}b) = -\lambda(a_{(\lambda)}b), \tag{1}$$

$$(a_{(\lambda)}\partial b) = (\partial + \lambda)(a_{(\lambda)}b). \tag{2}$$

The number  $N = N(a, b)$  is said to be a *locality level* of  $a, b \in C$ .

### 2.2 Formal power series and Conformal algebra

The previous definition is just a characterization of the following construction. Let  $A$  be an algebra. One can consider formal power series  $a(z), b(z) \in A[[z, z^{-1}]]$  and define their  $n$ -product by the formula [9]:

$$(a_{(n)}b)(z) = \text{Res}_w a(z)b(z)(w-z)^n; \quad n \geq 0. \tag{3}$$

where  $\text{Res}_w$  is the coefficient of  $w^{-1}$  in the formal power series in  $z$  and  $w$ . Then the locality condition (C1) is equivalent to the relation

$$a(z)b(z)(w-z)^N = 0$$

and relation (C2) holds for  $\partial = \frac{d}{dz}$ . In the case of Lie algebras, the family of  $n$ -products  $a_{(n)}b$  describes the singular part of the operator product expansion of two local series  $a(z)$  and  $b(z)$ .

### 2.3 The coefficient algebra

To any conformal algebra  $C$  there corresponds an “usual” algebra  $A = \mathcal{A}(C)$ , and it is called *the coefficient algebra* such that  $C$  lies in  $A[[z, z^{-1}]]$  as a subspace of local series with the  $n$ -th products given by (3), which is constructed in the following way.

Consider the space of Laurent series  $C[t^{-1}, t]$  in an independent variable  $t$  with coefficients in  $C$ . For  $a \in C$ , denote  $a(n) = at^n$ . As a linear space  $A$  is isomorphic to the quotient of  $C[t^{-1}, t]$  over the subspace generated by the vectors  $(\partial a)(n) + na(n-1)$  for  $a \in C$ . The formula for the product in  $A$  is derived from

$$a(n) \cdot b(m) = \sum_{s \geq 0} \binom{n}{s} (a_{(s)} b)(n+m-s).$$

Note that the sum here is finite due to (C1). The space  $\mathcal{A}_+(C)$  spanned by all  $a(n)$ ,  $n \in \mathbb{Z}_+$ ,  $a \in C$ , is a subalgebra of  $\mathcal{A}(C)$  which is closed under the derivation  $\partial$ .

There is a correspondence between the identities in a conformal algebra and the identities in its coefficient algebra. For example,  $A(C)$  is associative if and only if

$$a_{(n)} (b_{(m)} c) = \sum_{s \geq 0} \binom{n}{s} (a_{(n-s)} b)_{(m+s)} c \quad (4)$$

for all  $a, b, c \in C, n, m \in \mathbb{Z}_+$ . In terms of the  $\lambda$ -product, the last relation may be expressed by a single formula

$$a_{(\lambda)} (b_{(\mu)} c) = (a_{(\lambda)} b)_{(\lambda+\mu)} c, \quad a, b, c \in C,$$

where  $\lambda$  and  $\mu$  are independent commuting variables [9]. As in the world of ordinary algebras, an associative conformal algebra  $C$  turns into a Lie one (denoted by  $C^{(-)}$ ) relative to new operations

$$[a_{(\lambda)} b] = (a_{(\lambda)} b) - (b_{(-\partial-\lambda)} a), \quad a, b \in C.$$

**Example 2.1.** The Lie conformal algebra with one generator  $v$  as free  $H$ -module and  $\lambda$ -product

$$(v_{(\lambda)} v) = (\partial + 2\lambda)v$$

is called the Virasoro conformal algebra. The coefficient algebra of  $\text{Vir}$  is the Witt algebra  $\mathcal{A}(\text{Vir}) = \text{Der } \mathbb{k}[t, t^{-1}]$ , and  $\mathcal{A}_+(\text{Vir}) = \text{Der } \mathbb{k}[t]$ .

### 2.4 Universal enveloping of Virasoro Lie conformal algebras

Given an integer  $N \geq 2$ , one may construct an associative conformal envelope  $U(N)$  for the Virasoro Lie conformal algebra  $\text{Vir}$  with a generator  $v$  which is universal in the class of all such envelopes  $C$  that  $N_C(v, v) \leq N$ . For example,  $U(2) = U(\text{Vir}; N = 2)$  is the Weyl conformal algebra  $\text{Cend}_{1,x}$ ; the structure of  $U(3) = U(\text{Vir}; N = 3)$  is more complicated, and it was studied in [11]. We have that  $U(3) = U(\text{Vir}; N = 3)$  is an associative conformal algebra generated by  $v$  relative to

$$\deg_\lambda(v_{(\lambda)} v) < N = 3, \quad (v_{(\lambda)} v) - (v_{(-\partial-\lambda)} v) = (\partial + 2\lambda)v.$$

The structure of the universal enveloping associative conformal algebra  $U(3)$  of the Virasoro Lie conformal algebra  $\text{Vir}$  relative to the locality bound  $N = 3$  may be expressed by means  $(\cdot(n)\cdot)$ , is generated by a single element  $v$  such that  $v_{(n)}v = 0$  for  $n \geq 3$ . The remaining defining relation of  $U(3)$  is

$$2v_{(1)}v - \partial(v_{(2)}v) = 2v. \quad (5)$$

## 2.5 Conformal module

A left conformal module  $M$  over an associative conformal algebra  $C$  is a  $\mathbb{k}[\partial]$ -module endowed with a  $\mathbb{k}$ -bilinear map  $C \otimes M \rightarrow M[\lambda]$  satisfying the following axioms

$$(\partial a_{(\lambda)}m) = -\lambda(a_{(\lambda)}m), \quad (a_{(\lambda)}\partial m) = (\partial + \lambda)(a_{(\lambda)}m), \quad (6)$$

$$(a_{(\lambda)}(b_{(\mu)}m)) = ((a_{(\lambda)}b)_{(\mu+\lambda)}m), \quad (7)$$

for  $a, b \in C, v \in M$ .

Similarly, a conformal action of a Lie conformal algebra  $L$  on a module  $M$  meets (6) and the conformal analogue of the Jacobi identity:

$$(a_{(\lambda)}(b_{(\mu)}m)) - (b_{(\mu)}(a_{(\lambda)}m)) = ((a_{(\lambda)}b)_{(\lambda+\mu)}m),$$

for  $a, b \in L, m \in M$ .

**Example 2.2.** Given a 1-generated free  $H$ -module  $M = \mathbb{k}[\partial]u$  and two scalars  $\Delta, \alpha \in \mathbb{k}$ , one may define conformal action of  $\text{Vir}$  on  $M$  as

$$(v_{(\lambda)}u) = (\alpha + \partial + \Delta\lambda)u. \quad (8)$$

Denote the conformal  $\text{Vir}$ -module obtained by  $M(\alpha, \Delta)$ . For  $\Delta \neq 0$ , this is an irreducible  $\text{Vir}$ -module, and every finite irreducible  $\text{Vir}$ -module is isomorphic to an appropriate  $M_{(\alpha, \Delta)}$  [7].

If  $M$  is a module over an (associative or Lie) conformal algebra  $C$ , then  $M$  is also a module over the ordinary (associative or Lie, respectively) algebra  $\mathcal{A}_+(C)$ . Namely, for  $a \in C, n \in \mathbb{Z}_+, u \in M$  the element  $a(n)u$  is the coefficient at  $\lambda^n/n!$  of  $(a_{(\lambda)}u)$ :

$$a_{(\lambda)}u = \sum_{n \geq 0} \frac{\lambda^n}{n!} a(n)u.$$

For every conformal  $C$ -module  $M$  over an associative conformal algebra  $C$ , the space  $M$  is also a  $C^{(-)}$ -module relative to the same conformal action. The converse construction has a restriction due to locality. For example, the module  $M_{(\alpha, \Delta)}$  over the Virasoro (Lie) conformal algebra is also a module over its universal enveloping associative conformal algebra  $U(2)$  if and only if  $\Delta = 0$  or  $\Delta = 1$  but the module  $M_{(\alpha, \Delta)}$  over the Virasoro (Lie) conformal algebra is also a module over its universal enveloping associative conformal algebra  $U(3)$  for all  $\Delta$ .

## 2.6 Hochschild cohomology for associative algebras

Let  $\Lambda$  be an associative algebra with a unit over a field  $\mathbb{k}$ . Let us start with the bar resolution (see [14])  $\mathbf{B}_\bullet = (\mathbf{B}_n, d_n)$  where

$$\mathbf{B}_n := \Lambda \otimes (\Lambda/\mathbb{k})^{\otimes n}$$

The basis of  $\mathbf{B}_n$  as a  $\Lambda$ -left module is presented by  $[a_1|a_2|\dots|a_n]$ , where  $a_i$  are nontrivial (i.e., not equal to 1) reduced normal forms relative to the fixed Gröbner–Shirshov basis.

The differential  $d_n : \mathbf{B}_n \rightarrow \mathbf{B}_{n-1}$  is defined as follows:

$$d_n([a_1|\dots|a_n]) = (a_1 \otimes 1)[a_2|\dots|a_n] + \sum_{i=1}^{n-1} (-1)^i [a_1|\dots|N(a_i a_{i+1})|\dots|a_n].$$

Suppose  $M$  is an “ordinary”  $\Lambda$ -left module over  $\Lambda$ . Denote

$$C_B^n = \text{Hom}_\Lambda(\mathbf{B}_n, M)$$

this is the space of Hochschild cochains.

The Hochschild differential

$$\Delta_B^n : C_B^n \rightarrow C_B^{n+1}$$

is obtained through

$$(\Delta_B^n \varphi)(\mathbf{x}) = \varphi d_{n+1}(\mathbf{x}), \quad \varphi \in C_B^n, \quad \mathbf{x} \in \mathbf{B}_{n+1}.$$

The Hochschild cohomology for associative algebra  $\Lambda$  with values in  $M$  is

$$H^n(\Lambda, M) = \ker \Delta_n / \text{Im } \Delta_{n-1}.$$

## 2.7 Hochschild cohomology for associative conformal algebras

Let  $C$  be an associative conformal algebra and  $M$  is  $\Lambda$ -left conformal module. The basic Hochschild complex  $\tilde{C}^\bullet(C, M)$  for associative conformal algebra  $C$  (see [4]) is cochain spaces  $\tilde{C}^n(C, M)$ ,  $n = 1, 2, \dots$ , each of them is the space of all maps

$$\varphi_{\bar{\lambda}} : C^{\otimes n} \rightarrow M[\bar{\lambda}],$$

where  $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ , satisfying the conformal anti-linearity condition:

$$\varphi_{\bar{\lambda}}(a_1, \dots, \partial a_i, \dots, a_n) = -\lambda_i \varphi_{\bar{\lambda}}(a_1, \dots, a_n), \quad i = 1, \dots, n.$$

The Hochschild differential  $d_n : \tilde{C}^n(C, M) \rightarrow \tilde{C}^{n+1}(C, M)$  on the basic complex is given by

$$(d_n \varphi)_{\bar{\lambda}}(a_1, \dots, a_{n+1}) = a_1 (\lambda_1) \varphi_{\bar{\lambda}_0}(a_2, \dots, a_{n+1}) + \sum_{i=1}^n (-1)^i \varphi_{\bar{\lambda}_i}(a_1, \dots, a_i (\lambda_i) a_{i+1}, \dots, a_{n+1}),$$

for  $\bar{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$ ,  $\bar{\lambda}_0 = (\lambda_2, \dots, \lambda_{n+1})$ ,  $\bar{\lambda}_i = (\lambda_1, \dots, \lambda_i + \lambda_{i+1}, \dots, \lambda_{n+1})$ ,  $i = 1, \dots, n$ . The cohomology of the basic Hochschild complex is called *the basic Hochschild cohomology*  $\tilde{H}^\bullet(C, M)$ .

For every  $n \geq 1$ , the cochain space  $\tilde{C}^n(C, M)$  is a left  $\mathbb{k}[\partial]$ -module:

$$(\partial\varphi)_{\bar{\lambda}}(a_1, \dots, a_n) = (\partial + \sum_{i=1}^n \lambda_i)\varphi_{\bar{\lambda}}(a_1, \dots, a_n).$$

For every  $n \geq 1$ , the map  $d_n$  commutes with  $\partial$ . The quotient complex

$$C^\bullet(C, M) = \tilde{C}^\bullet(C, M) / \partial\tilde{C}^\bullet(C, M)$$

is called the *reduced Hochschild complex* and its cohomology is called the *reduced Hochschild cohomology*  $H^\bullet(C, M)$ .

Consider the “ordinary” Hochschild complex  $C^\bullet(\mathcal{A}_+(C), M)$ . The maps

$$\partial_n^* : C^n(\mathcal{A}_+(C), M) \rightarrow C^n(\mathcal{A}_+(C), M)$$

given by

$$(\partial_n^* f)(\alpha_1, \dots, \alpha_n) = \partial f(\alpha_1, \dots, \alpha_n) - \sum_{i=1}^n f(\alpha_1, \dots, \partial\alpha_i, \dots, \alpha_n),$$

and  $\partial a(n) = -na(n-1)$  for  $a \in C$ ,  $n \geq 0$ , commute with the Hochschild differentials.

In [4] was proved that

$$\begin{aligned} C^\bullet(C, M) &\cong C^\bullet(\mathcal{A}_+(C), M) / \partial_\bullet^* C^\bullet(\mathcal{A}_+(C), M), \\ \tilde{H}^\bullet(C, M) &= H^\bullet(\mathcal{A}_+(C), M). \end{aligned}$$

So we can calculate (reduced) Hochschild cohomology of a conformal algebra  $C$  via the Hochschild complex of its coefficient algebra  $\mathcal{A}_+(C)$  and its quotient. To do that, we construct the Anick resolution for  $\mathcal{A}_+(C)$  by means of the Morse matching method which is explained in detail in [2]. We will use the homotopy maps constructed in this way to transfer the map  $\partial_\bullet^*$  to the dual complex obtained from the Anick resolution, as explained in the following sections.

## 2.8 Anick resolution for associative algebras

Let  $\Lambda$  be an associative algebra with a unit over a field  $\mathbb{k}$ . The Anick resolution (see [3])  $A_\bullet = (A_n, \delta_n)$  is complex, where

$$A_n := \Lambda \otimes \mathbb{k}V^{(n-1)}.$$

The basis of  $A_n$  as a  $\Lambda$ -left module is presented by  $V^{(n-1)}$ . Following Anick [3] the elements of  $V^{(n-1)}$  are defined as: the  $V^{(1)}$  is the set of all leading terms of relations from Gröbner–Shirshov basis of  $\Lambda$ . A word  $v = x_{i_1} \dots x_{i_t}$  is an *n-prechain* if and only if there exist  $a_j, b_j \in \mathbb{Z}$ ,  $1 \leq j \leq n$ , satisfying the following conditions:

- $1 = a_1 < a_2 \leq b_1 < \dots < a_n \leq b_{n-1} < b_n = t$ ;
- $x_{i_{a_j}} \dots x_{i_{b_j}} \in V$  for  $1 \leq j \leq n$ .

An  $n$ -prechain  $x_{i_1} \dots x_{i_t}$  is an  $n$ -chain if only if the integers  $a_j, b_j$  can be chosen in such a way that  $x_{i_1} \dots x_{i_t}$  is not an  $m$ -prechain for neither  $s < b_m, 1 \leq m \leq n$ . The set of all  $n$ -chains is denoted  $V^{(n)}$ .

The Anick differential  $\delta_n$  can be constructed by means of the Morse matching method which is explained in detail in [2] or via the following algorithm: define two  $\Lambda$ -linear operators  $\delta'_n$  and  $\delta''_n$  whose recurrent application leads us to the desired  $\delta_n$ .

1. Let  $[w] \in V^{(n)}$ , calculate the ordinary differential

$$\delta'_{n+1}[w] = w_1[w_2 | \dots | w_n | w_{n+1}] + \sum_{j=1}^n (-1)^j [w_1 | w_2 | \dots | s_j | \dots | w_{n+1}] \in \Lambda \otimes (\Lambda/\mathbb{k})^{\otimes n},$$

where  $s_j$  are the  $V$ -reduced forms of the products  $w_j w_{j+1}$ ,  $j = 1, \dots, n$ ;

2. If  $[v] \in (\Lambda/\mathbb{k})^{\otimes n}$  belongs to  $V^{(n-1)}$ , then

$$\delta''_{n+1}[v] = [v]$$

and the computation is finished. Otherwise, suppose  $[v] = [v_1 | \dots | v_n]$  does not belong to  $V^{(n-1)}$  (here all  $v_k$ 's are  $V$ -reduced) then there exists the largest integer  $i \geq 0$  such that  $v_1 \dots v_i \in V^{(i-1)}$ ,  $v_{i+1}$  may be presented as  $v_{i+1} = v'_{i+1} v''_{i+1}$ , and  $[v_1 | \dots | v'_{i+1}]$  belongs to  $V^{(i)}$ . Then set

$$\delta''_{n+1}[v] = (-1)^i \delta'_{n+1}([v_1 | \dots | v'_{i+1} | v''_{i+1} | \dots | v_n]) + [v].$$

If such an index  $i \geq 0$  does not exist then set  $\delta''_{n+1}([v]) = 0$ .

After finitely many steps, the computation of  $\delta''_{n+1}$  finishes. Therefore,

$$\delta_{n+1}(w) = (\delta''_{n+1})^k \delta'_{n+1}[w],$$

where  $k = k(w) \geq 1$  is a sufficiently large integer (see examples in Theorem 3.4).

We have that Anick resolution is homotopy equivalent to bar resolution and homotopy maps  $f_n : B_n \rightarrow A_n$ ,  $g_n : A_n \rightarrow B_n$  are given (see [2]) such that  $\delta_n = f_{n-1} d_n g_n : A_n \rightarrow A_{n-1}$ .

### 3 Anick resolution for $\mathcal{A}_+(U(3))$

**Definition 3.1** ([16]). The algebra  $\mathcal{A}_+(U(3))$  is generated by the elements  $v(n)$ ,  $n \geq 0$ , relative to the following relations:

$$v(n)v(m) - 3v(n-1)v(m+1) \tag{9}$$

$$+ 3v(n-2)v(m+2) - v(n-3)v(m+3) = 0, \quad n \geq 3, m \geq 0,$$

$$v(n)v(m) - v(m)v(n) = (n-m)v(n+m-1), \quad n > m \geq 0. \tag{10}$$



**Theorem 3.2** ([2]). *The Gröbner–Shirshov basis of  $\mathcal{A}_+(U(3))$  consists of the relations*

$$v(1)v(0) = v(0)v(1) + v(0), \quad (11)$$

$$\begin{aligned} v(n)v(m) &= \frac{nm}{n+m-1}v(1)v(n+m-1) - \frac{(n-1)(m-1)}{n+m-1}v(0)v(n+m) \\ &+ \frac{n(n-1)}{n+m-1}v(n+m-1), \quad n \geq 2. \end{aligned} \quad (12)$$

To find the Anick complex, we need two steps. First, we have to find the set of obstructions for  $\mathcal{A}_+(U(3))$  relative to the given Gröbner–Shirshov basis (the set of leading terms in  $\mathcal{A}_+(U(3))$ ) and the set of  $n$ -chains. Next, build a Morse graph and calculate the path weights for every  $n$ -chain  $w$  and  $(n-1)$ -chain  $w'$  for all  $n \geq 1$ , all graphs were building in [2] or via methods in Section 2.8.

**Corollary 3.3.** *The Anick chains  $\Lambda^{(n-1)}$  are of the following form:*

$$[v(m_1)|v(m_2)|\dots|v(m_{n-1})|v(m_n)],$$

where  $m_1, \dots, m_{n-2} \geq 2$  and either  $m_{n-1} \geq 2$  or  $(m_{n-1}, m_n) = (1, 0)$ .

We will write  $[m_1|\dots|m_n]$  instead of  $[v(m_1)|\dots|v(m_n)]$  for the sake of simplicity.

**Theorem 3.4.** *The Anick differential  $\delta_{n+1} : A_{n+1} \rightarrow A_n$  is given by the following rules:*

$$\begin{aligned} \delta_{n+1}[v(i_1)v(i_2)\dots v(i_{n+1})] &= v(i_1)[v(i_2)\dots v(i_{n+1})] \\ &+ \sum_{j=1}^n (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} v(1)[v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\ &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1)[v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\ &+ \sum_{j=1}^n (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1} [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\ &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^{j+1} i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} [v(i_1)\dots v(i_t - 1)\dots v(i_j + i_{j+1})\dots v(i_{n+1})] \\ &+ \sum_{j=1}^n (-1)^{j+1} \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} v(0)[v(i_1)\dots v(i_j + i_{j+1})\dots v(i_{n+1})], \end{aligned}$$

where  $i_1, i_2, \dots, i_n \geq 2$ ,  $i_{n+1} \geq 0$ , and

$$\delta_{n+1}[v(i_1)v(i_2)\dots v(i_{n-1})v(1)v(0)] = v(i_1)[v(i_2)\dots v(i_{n-1})v(1)v(0)]$$

$$\begin{aligned}
 & + \sum_{j=1}^{n-2} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} v(1) [v(i_1) \dots v(i_j + i_{j+1} - 1) \dots v(i_{n-1}) v(1) v(0)] \\
 & + \sum_{j=1}^{n-2} (-1)^{j+1} \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} v(0) [v(i_1) \dots v(i_j + i_{j+1}) \dots v(i_{n-1}) v(1) v(0)] \\
 & + \sum_{j=1}^{n-2} (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1} [v(i_1) \dots v(i_j + i_{j+1} - 1) \dots v(i_{n-1}) v(1) v(0)] \\
 & + \sum_{j=2}^{n-2} \sum_{t=1}^{j-1} (-1)^{j+1} i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} [v(i_1) \dots v(i_t - 1) \dots v(i_j + i_{j+1}) \dots v(i_{n-1}) v(1) v(0)] \\
 & + \sum_{j=2}^{n-2} \sum_{t=1}^{j-1} (-1)^j (i_t - 1) \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} [v(i_1) \dots v(i_j + i_{j+1} - 1) \dots v(i_{n-1}) v(1) v(0)] \\
 & + \sum_{j=1}^{n-1} (-1)^j i_j [v(i_1) \dots v(i_j - 1) \dots v(i_{n-1}) v(1)] + \sum_{j=1}^{n-1} (-1)^j (i_j - 1) [v(i_1) \dots v(i_{n-1}) v(0)] \\
 & + (-1)^n [v(i_1) v(i_2) \dots v(i_{n-1}) v(0)] + (-1)^n v(0) [v(i_1) v(i_2) \dots v(i_{n-1}) v(1)],
 \end{aligned}$$

where  $i_1, i_2, \dots, i_{n-1} \geq 2$ .

*Proof.* We can evaluate  $\delta_{n+1} : A_{n+1} \rightarrow A_n$  by means of the Morse graph as a similar way which is shown in [2] (see Figure 1-4) considering that in [2] was founded  $\delta_{n+1}$  for trivial module or via methods explained in Section 2.8.

For  $n = 1$  we have the following set of obstructions

$$V^{(1)} = \{v(n)v(m), v(1)v(0); n \geq 2, m \geq 0\}.$$

Let us evaluate  $\delta_2 : A_2 \rightarrow A_1$  by means of the methods explained in Section 2.8. Suppose  $[w] = [v(n)|v(m)]$ . Then calculate

$$\begin{aligned}
 \delta_2' [n|m] &= v(n)[v(m)] - \frac{nm}{n+m-1} [v(1)v(n+m-1)] + \frac{(n-1)(m-1)}{n+m-1} [v(0)v(n+m)] \\
 &\quad - \frac{n(n-1)}{n+m-1} [v(n+m-1)]; \quad n \geq 2, m \geq 0.
 \end{aligned}$$

Apply  $\delta_2''$  to each summand:

$$\begin{aligned}
 \delta_2''([v(1)v(n+m-1)]) &= \delta_2'([v(1)|v(n+m-1)]) + [v(1)v(n+m-1)] \\
 &= v(1)[v(n+m-1)], \quad i = 0, \\
 \delta_2''([v(0)v(n+m)]) &= \delta_2'([v(0)|v(n+m)]) + [v(0)v(n+m)] = v(0)[v(n+m)], \quad i = 0,
 \end{aligned}$$

$$\begin{aligned}\delta_2[n|m] &= v(n)[v(m)] - \frac{nm}{n+m-1}v(1)[v(n+m-1)] \\ &\quad + \frac{(n-1)(m-1)}{n+m-1}v(0)[v(n+m)] \\ &\quad - \frac{n(n-1)}{n+m-1}[v(n+m-1)]; \quad n \geq 2, m \geq 0.\end{aligned}$$

Similar way

$$\delta_2[1|0] = v(1)[v(0)] - v(0)[v(1)] - [v(0)].$$

For  $n = 2$  we have

$$V^{(2)} = \{v(n)v(m)v(p), v(n)v(1)v(0); n, m \geq 2, p \geq 0\}.$$

Suppose  $[w] = [v(n)|v(1)|v(0)]$ . Then calculate

$$\begin{aligned}\delta'_3([v(n)|v(1)|v(0)]) &= v(n)[v(1)|v(0)] - [v(1)v(n)|v(0)] \\ &\quad - (n-1)[v(n)v(0)] + [v(n)|v(0)v(1)] + [v(n)v(0)].\end{aligned}$$

Since  $[v(1)|v(0)], [v(n)v(0)] \in V^{(1)}$ , it is enough to calculate  $(\delta''_3)^k([v(1)v(n)|v(0)])$  and  $(\delta''_3)^k([v(n)|v(0)v(1)])$  for  $k = 1, 2, \dots$ . Here

$$\begin{aligned}\delta''_3([v(n)|v(0)v(1)]) &= -\delta'_3([v(n)|v(0)|v(1)]) + [v(n)|v(0)v(1)] \\ &= [v(0)v(n)|v(1)] + n[v(n-1)v(1)], \quad i = 1.\end{aligned}$$

Note that  $[v(n-1)v(1)] \in V^{(1)}$  and calculate

$$\begin{aligned}\delta''_3([v(0)v(n)|v(1)]) &= \delta'_3([v(0)|v(n)|v(1)]) + [v(0)v(n)|v(1)] = v(0)[v(n)|v(1)], \\ \delta''_3([v(1)v(n)|v(0)]) &= \delta'_3([v(1)|v(n)|v(0)]) + [v(1)v(n)|v(0)] = v(1)[v(n)|v(0)], \\ \delta_3[v(n)v(1)v(0)] &= v(n)[v(1)v(0)] + n[v(n-1)v(1)] - v(1)[v(n)v(0)] \\ &\quad - (n-2)[v(n)v(0)] + v(0)[v(n)v(1)]; \quad n \geq 2.\end{aligned}$$

Similar way for

$$\begin{aligned}\delta_3[v(n)v(m)v(p)] &= v(n)[v(m)v(p)] - \frac{nm}{n+m-1}v(1)[v(n+m-1)v(p)] \\ &\quad + \frac{(n-1)(m-1)}{n+m-1}v(0)[v(n+m)v(p)] - \frac{n(n-1)}{n+m-1}[v(n+m-1)v(p)] \\ &\quad + \frac{mp}{m+p-1}v(1)[v(n)v(m+p-1)] + \frac{(n-1)mp}{m+p-1}[v(n)v(m+p-1)] \\ &\quad - \frac{(m-1)(p-1)}{m+p-1}v(0)[v(n)v(m+p)] \\ &\quad - \frac{n(m-1)(p-1)}{m+p-1}[v(n-1)v(m+p)] \\ &\quad + \frac{m(m-1)}{m+p-1}[v(n)v(m+p-1)]; \quad n, m \geq 2, p \geq 0.\end{aligned}$$

And also for  $\delta_{n+1}$ . Suppose  $[w] = [v(i_1)v(i_2)\dots v(i_{n+1})]$ ;  $i_1, \dots, i_n \geq 2, i_{n+1} \geq 0$ . Then calculate

$$\begin{aligned}
 \delta'_{n+1}[v(i_1)|v(i_2)|\dots|v(i_{n+1})] &= v(i_1)[v(i_2)|\dots|v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} \\
 &\times [v(i_1)|v(i_2)|\dots|v(1)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1} \\
 &\times [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^{j+1} \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 &\times [v(i_1)|\dots|v(0)v(i_j + i_{j+1})|\dots|v(i_{n+1})].
 \end{aligned}$$

Since  $[v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \in V^{(n)}$ , it is enough to calculate

$$(\delta''_{n+1})^k([v(i_1)|v(i_2)|\dots|v(1)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})]), \text{ and}$$

$$(\delta''_{n+1})^k([v(i_1)|\dots|v(0)v(i_j + i_{j+1})|\dots|v(i_{n+1})]),$$

for  $k = 1, 2, \dots$ . Here

$$\begin{aligned}
 &\delta''_{n+1}([v(i_1)|v(i_2)|\dots|v(1)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})]) \\
 &= \delta'_{n+1}[v(i_1)|v(i_2)|\dots|v(1)|v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})] \\
 &+ (-1)^j [v(i_1)|v(i_2)|\dots|v(1)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})] \\
 &+ (-1)^{j-1} (i_{j-1} - 1) [v(i_1)|v(i_2)|\dots|v(i_{j-1})|v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})].
 \end{aligned}$$

By repeating this step,

$$\begin{aligned}
 &(\delta''_{n+1})^{j-1}([v(i_1)|v(i_2)|\dots|v(1)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})]) \\
 &= v(1)[v(i_1)|v(i_2)|\dots|v(i_{j-1})|v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})] \\
 &+ \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})].
 \end{aligned}$$

Similar way

$$\begin{aligned}
 &(\delta''_{n+1})^{j-1}([v(i_1)|v(i_2)|\dots|v(0)v(i_j + i_{j+1} - 1)|\dots|v(i_{n+1})]) \\
 &= v(0)[v(i_1)|v(i_2)|\dots|v(i_{j-1})|v(i_j + i_{j+1})|\dots|v(i_{n+1})] \\
 &+ \sum_{t=1}^{j-1} (-1)^{j+1} i_t [v(i_1)\dots v(i_t - 1)\dots v(i_j + i_{j+1})\dots v(i_{n+1})].
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \delta_{n+1}[v(i_1)v(i_2)\dots v(i_{n+1})] &= v(i_1)[v(i_2)\dots v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} v(1) \\
 &\times [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\
 &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\
 &\times [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j(i_j - 1)}{i_j + i_{j+1} - 1} [v(i_1)v(i_2)\dots v(i_j + i_{j+1} - 1)\dots v(i_{n+1})] \\
 &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^{j+1} i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 &\times [v(i_1)\dots v(i_t - 1)\dots v(i_j + i_{j+1})\dots v(i_{n+1})] \\
 &+ \sum_{j=1}^n (-1)^{j+1} \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} v(0) \\
 &\times [v(i_1)\dots v(i_j + i_{j+1})\dots v(i_{n+1})].
 \end{aligned}$$

By this method one can find  $\delta_{n+1}[v(i_1)v(i_2)\dots v(i_{n-1})v(1)v(0)]$ .  $\square$

### 3.1 Anick resolution for differential algebras

Let  $C$  be an associative conformal algebra. During the whole of the rest of the paper  $\Lambda$  represents for the augmented algebra  $\Lambda = \mathcal{A}_+ \oplus \mathbb{k}1$ , where  $\mathcal{A}_+ = \mathcal{A}_+(C)$ , and the augmentation is given by  $\varepsilon(\mathcal{A}_+) = 0$ . Then  $\mathcal{A}_+ = \Lambda/\mathbb{k}$ , and  $\Lambda$  has a derivation  $\partial$  such that  $\partial(a(n)) = -na(n-1)$ ,  $\partial(1) = 0$ .

For every  $n \geq 1$ , there is a  $\mathbb{k}$ -linear map

$$\partial_n : B_n \rightarrow B_n,$$

$$\partial_n(\beta[\alpha_1 | \dots | \alpha_n]) = \partial(\beta)[\alpha_1 | \dots | \alpha_n] + \sum_{i=1}^n \beta[\alpha_1 | \dots | \partial(\alpha_i) | \dots | \alpha_n],$$

$\alpha_i \in \mathcal{A}_+$ ,  $\beta \in \Lambda$ . This is not a morphism of complexes, but it causes a morphism of dual cochain complexes which can be assigned to the Anick resolution via homotopy. That is to say, suppose  $M$  is a left conformal  $C$ -module. Thus,  $M$  is an ‘‘ordinary’’  $\Lambda$ -module. Denote

$$C_B^n = \text{Hom}_\Lambda(B_n, M) \simeq \text{Hom}_{\mathbb{k}}(A^{\otimes n}, M),$$

this is the space of Hochschild cochains. The Hochschild differential

$$\Delta_B^n : C_B^n \rightarrow C_B^{n+1}$$

is obtained through

$$(\Delta_{\mathbf{B}}^n \varphi)(\mathbf{x}) = \varphi d_{n+1}(\mathbf{x}), \quad \varphi \in C_{\mathbf{B}}^n, \quad \mathbf{x} \in B_{n+1}.$$

Note that the  $\Lambda$ -module  $M$  is endowed with a derivation also denoted by  $\partial$  (the same as in the definition of a conformal module), so that  $\partial(a(n)u) = -na(n-1)u + a(n)\partial(u)$ , for  $a \in C$ ,  $u \in M$ ,  $n \in \mathbb{Z}_+$ .

Then for every  $n \geq 1$  the map

$$D_{\mathbf{B}}^n : C_{\mathbf{B}}^n \rightarrow C_{\mathbf{B}}^n$$

is expressed as

$$(D_{\mathbf{B}}^n \varphi)(\mathbf{x}) = \partial(\varphi(\mathbf{x})) - \varphi(\partial_n(\mathbf{x})), \quad \varphi \in C_{\mathbf{B}}^n, \quad \mathbf{x} \in B_{n+1},$$

and it is a morphism of complexes:

$$D_{\mathbf{B}}^{n+1} \Delta_{\mathbf{B}}^n = \Delta_{\mathbf{B}}^n D_{\mathbf{B}}^n.$$

Let us convert the mapping  $D_{\mathbf{B}}^\bullet$  from  $C_{\mathbf{B}}^\bullet$  to the complex  $C_{\mathbf{A}}^\bullet$  created on the spaces

$$C_{\mathbf{A}}^n = \text{Hom}_{\Lambda}(A_n, M) \simeq \text{Hom}_{\mathbb{k}}(\mathbb{k}V^{(n-1)}, M)$$

with the differential  $\Delta_{\mathbf{A}}^n : C_{\mathbf{A}}^n \rightarrow C_{\mathbf{A}}^{n+1}$  given by

$$\Delta_{\mathbf{A}}^n(\psi) = \psi d_{n+1} = \psi f_n d_{n+1} g_{n+1} = \Delta_{\mathbf{B}}^n(\psi f_n) g_{n+1}.$$

The homotopy equivalence between  $\mathbf{A}_\bullet$  and  $\mathbf{B}_\bullet$  transforms  $D_{\mathbf{B}}^n$  to

$$D_{\mathbf{A}}^n : C_{\mathbf{A}}^n \rightarrow C_{\mathbf{A}}^n$$

such that

$$D_{\mathbf{A}}^n \psi = D_{\mathbf{B}}^n(\psi f_n) g_n.$$

For every  $\mathbf{a} \in A_n$ , we have

$$(D_{\mathbf{A}}^n \psi)(\mathbf{a}) = \partial(\psi f_n(g_n(\mathbf{a}))) - (\psi f_n)(\partial_n(g_n(\mathbf{a}))) = \partial(\psi(\mathbf{a})) - (\psi f_n)(\partial_n(g_n(\mathbf{a}))).$$

**Proposition 3.5** ([1]). *For a conformal algebra  $C$  and a conformal  $C$ -module  $M$ , the reduced Hochschild complex  $C^\bullet(C, M)$  is homotopy equivalent to  $C_{\mathbf{A}}^\bullet/D_{\mathbf{A}}^\bullet C_{\mathbf{A}}^\bullet$ .*

## 4 Application of Anick resolution on $\mathcal{A}_+(U(3))$

**Lemma 4.1** ([1]). *The elements of  $C^n$  are in one-to-one correspondence with scalar sequences  $\alpha_{(i_1, i_2, \dots, i_n)}$ ,  $[i_1 | i_2 | \dots | i_n] \in A_n$ .*

**Theorem 4.2.** *For the conformal module  $M_{(\alpha, \Delta)}$  over  $U(3)$ , where  $\Delta \neq 0$ , we have*

$$\dim_{\mathbb{k}} H^1(U(3), M_{(\alpha, \Delta)}) = \begin{cases} 2, & \Delta = 1, \alpha = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the space  $H^1(U(3), M)$  which is isomorphic to the space of non-coboundary cocycles in  $C^1 = \tilde{C}^1 / D^1 \tilde{C}^1$ . Suppose  $\varphi \in \tilde{C}^1 = \text{Hom}_{\Lambda}(A_1, M)$ . By Lemma 4.1 we may assume  $\varphi[n] = \alpha_n \in \mathbb{k}$  for  $n \geq 0$ . Then the differential  $\Delta^1 \varphi$  takes the following values on the Anick 1-chains

$$\begin{aligned} (\Delta^1 \varphi)[1|0] &= \varphi(\delta_2[1|0]) = v(1)\alpha_0 - v(0)\alpha_1 - \alpha_0 = \Delta\alpha_0 - (\partial + a)\alpha_1 - \alpha_0 \\ &= (\Delta - 1)\alpha_0 - (\partial + a)\alpha_1, \\ (\Delta^1 \varphi)[n|m] &= \varphi(\delta_2[n|m]) = v(n)\alpha_m - \frac{nm}{n+m-1}v(1)\alpha_{n+m-1} \\ &\quad + \frac{(n-1)(m-1)}{n+m-1}v(0)\alpha_{n+m} - \frac{n(n-1)}{n+m-1}\alpha_{n+m-1} \\ &= -\frac{nm}{n+m-1}\Delta\alpha_{n+m-1} + \frac{(n-1)(m-1)}{n+m-1}(\partial + a)\alpha_{n+m} \\ &\quad - \frac{n(n-1)}{n+m-1}\alpha_{n+m-1}. \end{aligned}$$

In order to find the constants that define  $\Delta^1(\varphi + D^1 \tilde{C}^1) \in C^2$  choose  $\psi \in \tilde{C}^2$  such that

$$\psi[1|0] = -\alpha_1, \quad \psi[n|m] = \frac{(n-1)(m-1)}{n+m-1}\alpha_{n+m},$$

Then

$$\begin{aligned} (\Delta^1 \varphi - D^2 \psi)[n|m] &= -\frac{nm}{n+m-1}\Delta\alpha_{n+m-1} + \frac{(n-1)(m-1)}{n+m-1}a\alpha_{n+m} \\ &\quad - \frac{n(n-1)}{n+m-1}\alpha_{n+m-1} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{n+m-1} \\ &\quad - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{n+m-1}; \quad n \geq 2, m \geq 0, \\ (\Delta^1 \varphi - D^2 \psi)[1|0] &= (\Delta - 1)\alpha_0 - a\alpha_1, \end{aligned} \tag{13}$$

for all  $[n|m] \in A_2$ . Therefore,  $\varphi + D^1 \tilde{C}^1$  is a 1-cocycle in  $C^1$  if and only if:

CASE 1:  $\alpha = 0$ . Put  $m = 1$  in (13) to obtain

$$\begin{aligned} (\Delta^1 \varphi - D^2 \psi)[1|0] &= (\Delta - 1)\alpha_0 = 0, \\ (\Delta^1 \varphi - D^2 \psi)[n|1] &= -(\Delta + n - 2)\alpha_n = 0, \quad n \geq 2. \end{aligned}$$

For all other Anick 1-chains  $[n|m]$ , the desired value is proportional to  $\alpha_{n+m-1}$ , so  $\alpha_1$  does not emerge in these expressions.

Therefore, if  $\varphi - D^1\psi$  is a cocycle in  $C^1$  then  $\alpha_n = 0$  for all  $n \geq 2$  except, maybe, for  $n = 2 - \Delta$ . The latter is impossible for  $n \geq 3$  since

$$(\Delta^1\varphi - D^2\psi)[n-1|2] = -\frac{2(n-1)}{n}\Delta\alpha_n - \frac{(n-1)(n-2)}{n}\alpha_n - (n-3)\alpha_n = \frac{2}{n}\alpha_n.$$

Finally, we obtain the description of cocycles in  $C^1$  for various  $\Delta$ :

- $\Delta = 1$ :  $\alpha_0$  and  $\alpha_1$  take arbitrary values,  $\alpha_n = 0$  for  $n \geq 2$ ;
- $\Delta \neq 1$ :  $\alpha_1$  is arbitrary,  $\alpha_0 = \alpha_n = 0$  for  $n \geq 2$ .

Coboundary cocycles in  $\tilde{C}^1$  are given by  $\Delta^0h$ , where  $h \in \text{Hom}_\Lambda(\Lambda, M)$ . Modulo  $D^0\tilde{C}^0$ , we may assume  $h(1) = \beta u$ ,  $\beta \in \mathbb{k}$ . Then  $(\Delta^0h)[n] = v(n)\beta u$ . Choose  $\psi \in \tilde{C}^1$  such that  $\psi[0] = \beta u$  and  $\psi[n] = 0$  for  $n \geq 1$ . Then

$$(\Delta^0h - D^1\psi)[n] = \begin{cases} 0, & n = 0, \\ (\Delta - 1)\beta u, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Hence, the space of coboundaries in  $C^1$  is 1-dimensional for  $\Delta \neq 1$  and zero otherwise.

As a result, for the 1st Hochschild cohomology of  $U(3)$  we have

$$\dim_{\mathbb{k}} H^1(U(3), M(0, \Delta)) = \begin{cases} 2, & \Delta = 1, \\ 0, & \text{otherwise.} \end{cases}$$

CASE 2:  $\alpha \neq 0$ . Put  $m = 0$  in (13) to obtain

$$\begin{aligned} (\Delta^1\varphi - D^2\psi)[1|0] &= (\Delta - 1)\alpha_0 - a\alpha_1, \\ (\Delta^1\varphi - D^2\psi)[n|0] &= -a\alpha_n, \quad n \geq 2. \end{aligned}$$

If  $\delta\varphi - D^1\psi$  is a cocycle in  $C^1$  then for all  $n \geq 2$

$$\alpha_n = 0 \quad \text{and} \quad \alpha_1 = \frac{\Delta - 1}{a}\alpha_0.$$

Coboundary cocycles in  $\tilde{C}^1$  are given by  $\delta_0h$ , where  $h \in \text{Hom}_\Lambda(\Lambda, M)$ . Modulo  $D^0\tilde{C}^0$ , we may assume  $h(0) = cu$ ,  $c = \frac{\alpha_0}{a} \in \mathbb{k}$ . Then  $(\Delta^0h)[0] = v(0)cu = (\partial + a)cu$ . Choose  $\psi \in \tilde{C}^1$  such that  $\psi[0] = cu$  and  $\psi[n] = 0$  for  $n \geq 1$ . Then

$$(\Delta^0h - D^1\psi)[n] = \begin{cases} ac = \alpha_0, & n = 0, \\ 0, & n \geq 1. \end{cases}$$



Hence, the space of coboundaries in  $C^1$  is 1-dimensional for  $a \neq 0$ .

As a result, for the 1st Hochschild cohomology of  $U(3)$  we have

$$\dim_{\mathbb{k}} H^1(U(3), M_{(\alpha, \Delta)}) = \begin{cases} 2, & \Delta = 1, \alpha = 0 \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 4.3.** *The cohomology groups of  $U(3)$  with the values in  $M_{(\alpha, \Delta)}$  where  $a \neq 0$  and  $\Delta \in \mathbb{k}$  are trivial.*

*Proof.* By Lemma 4.1 we may assume  $\varphi([v(n)v(m)]) = \alpha_{(n,m)}$ ,  $\varphi([v(1)v(0)]) = \alpha_{(1,0)} \in \mathbb{k}$  for  $n \geq 2$  and  $m \geq 0$ . The differential  $\Delta^2\varphi$  takes the following values on the Anick 2-chains:

$$\begin{aligned} (\Delta^2\varphi)[v(n)v(m)v(p)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p)} \\ &+ \frac{(n-1)(m-1)}{n+m-1}(\partial + \alpha)\alpha_{(n+m,p)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p)} \\ &+ \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1)} \\ &- \frac{(m-1)(p-1)}{m+p-1}(\partial + \alpha)\alpha_{(n,m+p)} - \frac{n(m-1)(p-1)}{m+p-1}\alpha_{(n-1,m+p)} \\ &+ \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1)}; \quad n, m \geq 2, p \geq 0, \\ (\Delta^2\varphi)[v(n)v(1)v(0)] &= -\Delta\alpha_{(n,0)} - (n-2)\alpha_{(n,0)} + (\partial + \alpha)\alpha_{(n,1)} + n\alpha_{(n-1,1)}; \quad n \geq 2. \end{aligned}$$

Reduce the result by means of  $D^3\psi$ , where  $\psi \in \tilde{C}^3$  is given by

$$\begin{aligned} \psi(n, m, p) &= \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p)} - \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p)}; \quad n, m \geq 2, p \geq 0, \\ \psi(n, 1, 0) &= \alpha_{(n,1)}; \quad n \geq 2. \end{aligned}$$

Then

$$\begin{aligned} (\Delta^2\varphi - D^3\psi)(n, m, p) &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p)} \\ &+ \frac{(n-1)(m-1)}{n+m-1}\alpha\alpha_{(n+m,p)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p)} \\ &+ \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1)} \\ &- \frac{(m-1)(p-1)}{m+p-1}\alpha\alpha_{(n,m+p)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1)} \\ &- \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p)} - \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1)} \end{aligned}$$

$$\begin{aligned}
 & - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,p)} + \frac{m(m-2)(p-1)}{m+p-2} \alpha_{(n,m+p-1)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2} \alpha_{(n,m+p-1)}; \quad n, m \geq 2, p \geq 0, \\
 (\Delta^2 \varphi - D^3 \psi)(n, 1, 0) & = -(\Delta + n - 2) \alpha_{(n,0)} + \alpha \alpha_{(n,1)}; \quad n \geq 2.
 \end{aligned}$$

Hence,  $\varphi + D^2 \tilde{C}^2$  is a 2-cocycle in  $C^2$  if and only if and for  $p = 0$

$$\begin{aligned}
 -\alpha \alpha_{(n,m)} & = -\frac{nm}{n+m-1} \Delta \alpha_{(n+m-1,0)} + \frac{(n-1)(m-1)}{n+m-1} \alpha \alpha_{(n+m,0)} - \frac{n(n-1)}{n+m-1} \alpha_{(n+m-1,0)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2} \alpha_{(n+m-1,0)} - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,0)}; \quad n, m \geq 2, \\
 \alpha \alpha_{(n,1)} & = (\Delta + n - 2) \alpha_{(n,0)}; \quad n \geq 2.
 \end{aligned}$$

Therefore, cocycles in  $C^2$  are determined by  $\alpha_{(n,0)}$  for  $n \geq 2$ .

Choose  $\varphi_1 \in \tilde{C}^1$ ,  $\psi_1 \in \tilde{C}^2$ , such that

$$\varphi_1[n] = \beta_n; \quad n \geq 0, \quad \psi_1[1|0] = \beta_1, \quad \psi_1[n|m] = \frac{(n-1)(m-1)}{n+m-1} \beta_{n+m}; \quad n \geq 2, m \geq 0$$

where

$$\beta_n = \begin{cases} -\frac{\alpha_{(n,0)}}{\alpha}; & n \geq 1. \\ 0; & \text{otherwise.} \end{cases}$$

Then  $\Delta^1 \varphi_1 - D^2 \psi_1$  represents a coboundary in  $C^2$ , and

$$(\Delta^1 \varphi_1 - D^2 \psi_1)[n|0] = -\alpha \beta_n = \alpha_{(n,0)} = \varphi(n, 0); \quad n \geq 1.$$

Hence, every 2-cocycle is a coboundary.

Similar way, evaluate  $\Delta^n \varphi - D^{n+1} \psi$ :

$$\begin{aligned}
 (\Delta^n \varphi - D^{n+1} \psi)[i_1|i_2|\dots|i_{n+1}] & = \sum_{j=1}^n \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})} \\
 & + \sum_{j=1}^n (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n+1})} \\
 & + \sum_{j=1}^n (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})} \\
 & + \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=3}^{n+1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n+1})} \\
 & + \sum_{t=1}^n (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n+1})} \\
 & + \sum_{t=1}^n (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n+1})},
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^n \varphi - D^{n+1} \psi)[i_1 | i_2 | \dots | i_{n-1} | 1 | 0] & = \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + (-1)^{n-1} \Delta \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 & + \sum_{j=1}^{n-2} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-1}, 1, 0)} \\
 & + (-1)^{n-1} a \alpha_{(i_1, i_2, \dots, i_{n-1}, 1)} \\
 & + \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + (-1)^{n-1} \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 & + \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\
 & \times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-1} (-1)^j (i_t - 1) \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 & + \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 & \times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)},
 \end{aligned}$$

where  $[i_1|i_2|\dots|i_n|i_{n+1}] \in V^n$ . We have that  $\varphi - D^n \tilde{C}^n$  is a  $n$ -cocycle in  $C^n$  if and only if

$$\begin{aligned}
 (-1)^n \alpha \alpha_{(i_1, i_2, \dots, i_n)} &= \sum_{j=1}^{n-1} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 0)} \\
 &+ \sum_{j=1}^{n-1} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n, 0)} \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 0)} \\
 &+ \sum_{j=2}^{n-1} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 0)} \\
 &+ \sum_{t=3}^n \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{t-1}, \dots, i_n, 0)} \\
 &+ \sum_{t=1}^{n-1} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n+1})} \\
 &+ \sum_{t=1}^{n-1} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n, 0)}, \\
 \\
 (-1)^n \alpha \alpha_{(i_1, i_2, \dots, i_{n-1}, 1)} &= \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ (-1)^{n-1} \Delta \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-1}, 1, 0)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ (-1)^{n-1} \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &+ \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ \sum_{t=1}^{n-1} (-1)^j (i_t - 1) \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)}.
 \end{aligned}$$

Therefore, cocycles in  $C^n$  are determined by  $\alpha_{(i_1, i_2, \dots, i_{n-1}, 0)}$  where  $i_1, \dots, i_{n-2} \geq 2$ ,  $i_{n-1} \geq 1$ .

Choose  $\varphi_1 \in \tilde{C}^{n-1}$ ,  $\psi_1 \in \tilde{C}^n$ , such that

$$\begin{aligned}
 \varphi_1(i_1, i_2, \dots, i_{n-1}) &= \beta_{(i_1, i_2, \dots, i_{n-1})} \\
 \psi_1(i_1, i_2, \dots, i_n) &= \sum_{j=1}^{n-1} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n)} \\
 \psi_1(i_1, i_2, \dots, i_{n-2}, 1, 0) &= \sum_{j=1}^{n-2} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-2}, 1, 0)} \\
 &\quad + (-1)^{n-1} \beta_{(i_1, i_2, \dots, i_{n-1}, 1)},
 \end{aligned}$$

where

$$\beta_{(i_1, i_2, \dots, i_{n-1})} = \begin{cases} \frac{(-1)^{n-1} \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)}}{\alpha}; & i_1, \dots, i_{n-2} \geq 2, i_{n-1} \geq 1 \\ 0; & \text{otherwise.} \end{cases}$$

Then  $\Delta^{n-1} \varphi_1 - D^n \psi_1$  represents a coboundary in  $C^n$ , and for all  $i_1, \dots, i_{n-2} \geq 2$  and  $i_{n-1} \geq 1$ , we get

$$\begin{aligned}
 (\Delta^{n-1} \varphi_1 - D^n \psi_1)(i_1, i_2, \dots, i_{n-1}, 0) &= (-1)^{n-1} \alpha \alpha_{(i_1, i_2, \dots, i_{n-1})} = \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &= \varphi_{(i_1, i_2, \dots, i_{n-1}, 0)}.
 \end{aligned}$$

Hence, the element  $\varphi - D^n \tilde{C}^n$  is a coboundary in  $C^n$ . □

**Theorem 4.4.** For the conformal module  $M_{(0, \Delta)}$  where  $\Delta \neq 0$  over  $U(3)$ , we have

$$\dim_{\mathbb{k}} H^2(U(3), M_{(0, \Delta)}) = \begin{cases} 1, & \Delta = 1 \\ 0, & \Delta \neq 0, 1. \end{cases}$$

*Proof.* For  $\alpha = 0$ , the differential  $\Delta^2 \varphi$  takes the following values on the Anick 2-chains:

$$\begin{aligned}
 (\Delta^2\varphi)[v(n)v(m)v(p)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p)} \\
 &+ \frac{(n-1)(m-1)}{n+m-1}\partial\alpha_{(n+m,p)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p)} \\
 &+ \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1)} \\
 &- \frac{(m-1)(p-1)}{m+p-1}\partial\alpha_{(n,m+p)} - \frac{n(m-1)(p-1)}{m+p-1}\alpha_{(n-1,m+p)} \\
 &+ \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1)}; \quad n, m \geq 2, p \geq 0, \\
 (\Delta^2\varphi)[v(n)v(1)v(0)] &= -\Delta\alpha_{(n,0)} - (n-2)\alpha_{(n,0)} + \partial\alpha_{(n,1)} + n\alpha_{(n-1,1)}; \quad n \geq 2.
 \end{aligned}$$

Reduce the result by means of  $D^3\psi$ , where  $\psi \in \tilde{C}^3$  is given by

$$\begin{aligned}
 \psi(n, m, p) &= \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p)} - \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p)}; \quad n, m \geq 2, p \geq 0, \\
 \psi(n, 1, 0) &= \alpha_{(n,1)}; \quad n \geq 2.
 \end{aligned}$$

Namely,

$$\begin{aligned}
 (\Delta^2\varphi - D^3\psi)(n, m, p) &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p)} \\
 &+ \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1)} \\
 &+ \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p)} \\
 &- \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p)} + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1)} \\
 &- \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1)} + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1)}; \\
 &n, m \geq 2, p \geq 0, (m, p) \neq (2, 0), \\
 (\Delta^2\varphi - D^3\psi)(n, 2, 0) &= -\frac{2n}{n+1}\Delta\alpha_{(n+1,0)} - \frac{n(n-1)}{n+1}\alpha_{(n+1,0)} - (n-2)\alpha_{(n+1,0)}, \\
 (\Delta^2\varphi - D^3\psi)(n, 1, 0) &= -(\Delta + n - 2)\alpha_{(n,0)}; \quad n \geq 2.
 \end{aligned}$$

Suppose  $\varphi - D^3\tilde{C}^2$  is a cocycle in  $C^2$ . Since

$$(\Delta^2\varphi - D^3\psi)(n, 1, 0) = 0, \quad (\Delta^2\varphi - D^3\psi)(n-1, 2, 0) = 0, \quad (\Delta^2\varphi - D^3\psi)(n, m, 1) = 0,$$

therefore

$$\alpha_{(2,0)} = 0, \quad \alpha_{(n,0)} = 0; \quad n \geq 3,$$

$$\begin{aligned}
 -(\Delta + n + m - 3)\alpha_{(n,m)} &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,1)} \\
 &\quad -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,1)} \\
 &\quad -\frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,1)}; \quad n, m \geq 2.
 \end{aligned} \tag{14}$$

Choose  $\varphi_1 \in \tilde{C}^1, \psi_1 \in \tilde{C}^2$  such that

$$\varphi_1[n] = \beta_n, \quad \psi_1[1|0] = \beta_1, \quad \psi_1[n|m] = \frac{(n-1)(m-1)}{n+m-1}\beta_{n+m},$$

and

$$\begin{aligned}
 (\Delta^1\varphi_1 - D^2\psi_1)[n|m] &= -\frac{nm}{n+m-1}\Delta\beta_{n+m-1} \\
 &\quad -\frac{n(n-1)}{n+m-1}\beta_{n+m-1} - \frac{n(n-2)(m-1)}{n+m-2}\beta_{n+m-1} \\
 &\quad -\frac{m(n-1)(m-2)}{n+m-2}\beta_{n+m-1}; \quad n \geq 2, m \geq 0, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[1|0] &= (\Delta - 1)\beta_0,
 \end{aligned}$$

where

$$\beta_k = \begin{cases} \frac{\alpha_{(1,0)}}{\Delta - 1}, & k = 0, \Delta \neq 1, \\ -\frac{\alpha_{(2,1)}}{\Delta}, & k = 2, \Delta \neq 0, \\ -\frac{\alpha_{(k,1)}}{\Delta + k - 2}, & k \geq 3, k \neq 2 - \Delta, \\ -\frac{k}{2}\alpha_{(k-1,2)}, & k \geq 3, k = 2 - \Delta. \end{cases}$$

Then  $\Delta^1\varphi_1$  is a coboundary in  $\tilde{C}^2$ , and

$$\begin{aligned}
 (\Delta^1\varphi_1 - D^2\psi_1)[2|1] &= -\Delta\beta_2 = \alpha_{(2,1)} = \varphi(2, 1); \quad \Delta \neq 0, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[1|0] &= (\Delta - 1)\beta_0 = \alpha_{(1,0)} = \varphi(1, 0); \quad \Delta \neq 1, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[k|1] &= (-\Delta - k + 2)\beta_k = \alpha_{(k,1)} = \varphi(k, 1); \quad k \neq 2 - \Delta, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[k-1|2] &= -\frac{2(k-1)}{k}\Delta\beta_k - \frac{(k-1)(k-2)}{k}\beta_k - (k-3)\beta_k \\
 &= -\frac{2}{k}\beta_k = \alpha_{(k-1,2)} = \varphi(k-1, 2); \quad k = 2 - \Delta.
 \end{aligned}$$

Finally, we obtain the description of cocycles in  $C^2$  for various  $\Delta$ :

- There are  $n_1, m_1$  such that  $\Delta = 3 - n_1 - m_1 = 2 - k$ , where  $n_1, m_1 \geq 2$ . From (14) we have

$$0 = \frac{n_1 m_1 (m_1 + 1) + m_1 (m_1 - 1) (m_1 - 2)}{(n_1 + m_1 - 1)(n_1 + m_1 - 2)} \alpha_{(n_1 + m_1 - 1, 1)}.$$

Therefore

$$\alpha_{(k, 1)} = 0; \quad k = 2 - \Delta.$$

If  $\varphi - D^3\psi$  is a cocycle in  $C^2$  then  $(\Delta^2\varphi - D^3\psi)(n_1, m_1 - 1, 2) = 0$ ;  $m_1 \geq 3$ , if and only if:

$$\begin{aligned} -\frac{2}{m_1} \alpha_{(n_1, m_1)} &= -\frac{n_1(m_1 - 1)}{n_1 + m_1 - 2} \Delta \alpha_{(n_1 + m_1 - 2, 2)} - \frac{n_1(n_1 - 1)}{n_1 + m_1 - 2} \alpha_{(n_1 + m_1 - 2, 2)} \\ &\quad - \frac{n_1(n_1 - 2)(m_1 - 2)}{n_1 + m_1 - 3} \alpha_{(n_1 + m_1 - 2, 2)} \\ &\quad - \frac{(m_1 - 1)(n_1 - 1)(m_1 - 3)}{n_1 + m_1 - 3} \alpha_{(n_1 + m_1 - 2, 2)}. \end{aligned}$$

Hence, in this case, cocycles in  $C^2$  are either zero or determined by  $\alpha_{(k-1, 2)}$  such that  $\Delta + n_1 + m_1 - 3 = \Delta + k - 2 = 0$ .

- $\Delta = n + m - 3 \neq 0$ , in this case, cocycles in  $C^2$  are either zero or determined by  $\alpha_{(k, 1)}$  (see (14)) such that  $\Delta = 2 - k \neq 0$ .

In both cases, every 2-cocycle is a coboundary or zero except  $\alpha_{(1, 0)}$  when  $\Delta = 1$ .  $\square$

**Theorem 4.5.** *For  $n \geq 3$ , the cohomology groups of  $U(3)$  with the values in  $M_{(0, \Delta)}$  where  $0 \neq \Delta \in \mathbb{k}$  are trivial.*

*Proof.* For  $n = 3$ , in the same way as the previous theory, we may assume

$$\varphi([v(n)v(m)v(p)]) = \alpha_{(n, m, p)}, \varphi([v(n)v(1)v(0)]) = \alpha_{(n, 1, 0)} \in \mathbb{k},$$

for  $n, m \geq 2$  and  $p \geq 0$ .

The differential  $\Delta^3\varphi$  takes the following values on the Anick 3-chains:

$$\begin{aligned} (\Delta^3\varphi)[v(n)v(m)v(p)v(q)] &= -\frac{nm}{n+m-1} \Delta \alpha_{(n+m-1, p, q)} - \frac{p(p-1)}{p+q-1} \alpha_{(n, m, p+q-1)} \\ &\quad + \frac{(n-1)(m-1)}{n+m-1} \partial \alpha_{(n+m, p, q)} - \frac{n(n-1)}{n+m-1} \alpha_{(n+m-1, p, q)} \\ &\quad + \frac{mp}{m+p-1} \Delta \alpha_{(n, m+p-1, q)} + \frac{(n-1)mp}{m+p-1} \alpha_{(n, m+p-1, q)} \\ &\quad - \frac{(m-1)(p-1)}{m+p-1} \partial \alpha_{(n, m+p, q)} + \frac{(p-1)(q-1)}{p+q-1} \partial \alpha_{(n, m, p+q)} \\ &\quad - \frac{pq}{p+q-1} \Delta \alpha_{(n, m, p+q-1)} - \frac{(m-1)pq}{p+q-1} \alpha_{(n, m, p+q-1)} \end{aligned}$$



$$\begin{aligned}
& -\frac{(n-1)pq}{p+q-1}\alpha_{(n,m,p+q-1)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,q)} \\
& + \frac{m(p-1)(q-1)}{p+q-1}\alpha_{(n,m-1,p+q)}; \quad n, m, p \geq 2, q \geq 0,
\end{aligned}$$

$$\begin{aligned}
(\Delta^3\varphi)[v(n)v(m)v(1)v(0)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,1,0)} \\
& + \frac{(n-1)(m-1)}{n+m-1}\partial\alpha_{(n+m,1,0)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1,0)} \\
& + \Delta\alpha_{(n,m,0)} + (n+m-3)\alpha_{(n,m,0)} - m\alpha_{(n,m-1,1)} \\
& - n\alpha_{(n-1,m,1)} - \partial\alpha_{(n,m,1)}; \quad n, m \geq 2.
\end{aligned}$$

Reduce the result by means of  $D^4\psi$ , where  $\psi \in \tilde{C}^4$  is given by

$$\begin{aligned}
\psi[v(n)v(m)v(p)v(q)] &= \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,q)} \\
& - \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,q)} \\
& + \frac{(p-1)(q-1)}{p+q-1}\alpha_{(n,m,p+q)}; \quad n, m, p \geq 2, q \geq 0,
\end{aligned}$$

$$\psi[v(n)v(m)v(1)v(0)] = \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,1,0)} - \alpha_{(n,m,1)}; \quad n, m \geq 2.$$

Namely,

$$\begin{aligned}
(\Delta^3\varphi - D^4\psi)[v(n)v(m)v(p)v(q)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,q)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,q)} \\
& + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,q)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,q)} \\
& - \frac{pq}{p+q-1}\Delta\alpha_{(n,m,p+q-1)} - \frac{(m-1)pq}{p+q-1}\alpha_{(n,m,p+q-1)} \\
& - \frac{(n-1)pq}{p+q-1}\alpha_{(n,m,p+q-1)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,q)} \\
& - \frac{p(p-1)}{p+q-1}\alpha_{(n,m,p+q-1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,q)} \\
& - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p,q)} \\
& + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1,q)}
\end{aligned}$$

$$\begin{aligned}
 & -\frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1,q)} + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1,q)} \\
 & -\frac{p(p-2)(q-1)}{p+q-2}\alpha_{(n,m,p+q-1)} - \frac{q(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,q-1)} \\
 & + \frac{q(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,q-1)} - \frac{q(p-1)(q-2)}{p+q-2}\alpha_{(n,m,p+q-1)},
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^3\varphi - D^4\psi)[v(n)v(m)v(1)v(0)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,1,0)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1,0)} \\
 & + \Delta\alpha_{(n,m,0)} + (n+m-3)\alpha_{(n,m,0)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,1,0)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,1,0)}.
 \end{aligned}$$

Hence,  $\varphi - D^4\tilde{C}^4$  is a 3-cocycle in  $C^4$  for various  $\Delta$ .

CASE 1:  $\Delta + n + m - 3 \neq 0$ ;  $n, m \geq 2$ .

$$\begin{aligned}
 -(\Delta + n + m - 3)\alpha_{(n,m,0)} &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,1,0)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1,0)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,1,0)} - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,1,0)}.
 \end{aligned}$$

CASE 2:  $\Delta + n + m + p - 4 \neq 0$ ;  $n, m, p \geq 2$ .

$$\begin{aligned}
 (\Delta + n + m + p - 4)\alpha_{(n,m,p)} &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,1)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,1)} \\
 & + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,1)} \\
 & + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,1)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p,1)} \\
 & + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1,1)} \\
 & - \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1,1)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1,1)} \\
 & - \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,0)} + \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,0)}.
 \end{aligned}$$

CASE 3:  $\Delta + n_1 + m_1 - 3 = 0$ ;  $n_1, m_1 \geq 2$ .

$$\begin{aligned} \alpha_{(n_1+m_1-1,1,0)} &= 0, \\ \frac{-2m_1+4}{m_1}\alpha_{(n_1,m_1,0)} &= -\frac{n_1(m_1-1)}{n_1+m_1-2}\Delta\alpha_{(n_1+m_1-2,2,0)} - \frac{n_1(n_1-1)}{n_1+m_1-2}\alpha_{(n_1+m_1-2,2,0)} \\ &\quad - \frac{n_1(n_1-2)(m_1-2)}{n_1+m_1-3}\alpha_{(n_1+m_1-2,2,0)} \\ &\quad - \frac{(m_1-1)(n_1-1)(m_1-3)}{n_1+m_1-3}\alpha_{(n_1+m_1-2,2,0)}. \end{aligned}$$

CASE 4:  $\Delta + n_2 + m_2 + p_2 - 4 = 0$ ;  $n_2, m_2, p_2 \geq 2$ .

$$\begin{aligned} \frac{2p_2-4}{p_2}\alpha_{(n_2,m_2,p_2)} &= -\frac{n_2m_2}{n_2+m_2-1}\Delta\alpha_{(n_2+m_2-1,p_2-1,2)} - \frac{n_2(n_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2-1,p_2-1,2)} \\ &\quad + \frac{m_2(p_2-1)}{m_2+p_2-2}\Delta\alpha_{(n_2,m_2+p_2-2,2)} + \frac{(n_2-1)m_2(p_2-1)}{m_2+p_2-2}\alpha_{(n_2,m_2+p_2-2,2)} \\ &\quad + \frac{m_2(m_2-1)}{m_2+p_2-2}\alpha_{(n_2,m_2+p_2-2,2)} - \frac{n_2(n_2-2)(m_2-1)}{n_2+m_2-2}\alpha_{(n_2+m_2-1,p_2-1,2)} \\ &\quad - \frac{m_2(n_2-1)(m_2-2)}{n_2+m_2-2}\alpha_{(n_2+m_2-1,p_2-1,2)} \\ &\quad + \frac{m_2(m_2-2)(p_2-2)}{m_2+p_2-3}\alpha_{(n_2,m_2+p_2-2,2)} \\ &\quad - \frac{(p_2-1)(n_2-1)(m_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2,p_2-2,2)} \\ &\quad + \frac{(p_2-1)(m_2-1)(p_2-3)}{m_2+p_2-3}\alpha_{(n_2,m_2+p_2-2,2)} \\ &\quad - \frac{2(n_2-1)(m_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2,p_2-1,1)} + \frac{2(m_2-1)(p_2-2)}{m_2+p_2-2}\alpha_{(n_2,m_2+p_2-1,1)}. \end{aligned}$$

The Case 4 can be obtained from Case 3 when  $p_2 = 2$  and  $m_1 \geq 3$  where

$$\Delta + n_1 + m_1 - 3 = \Delta + n_1 + m_1 - 1 + 2 - 4 = \Delta + n_2 + m_2 + p_2 - 4 = 0.$$

Therefore

$$\begin{aligned} \alpha_{(n_2,m_2,2)} &= \alpha_{(n_1,m_1-1,2)}, \\ \frac{2}{m_2}\alpha_{(n_2,m_2,1)} &= -\frac{n_2(m_2-1)}{n_2+m_2-2}\Delta\alpha_{(n_2+m_2-2,2,1)} - \frac{n_2(n_2-1)}{n_2+m_2-2}\alpha_{(n_2+m_2-2,2,1)} \\ &\quad - \frac{n_2(n_2-2)(m_2-2)}{n_2+m_2-3}\alpha_{(n_2+m_2-2,2,1)} \\ &\quad - \frac{(m_2-1)(n_2-1)(m_2-3)}{n_2+m_2-3}\alpha_{(n_2+m_2-2,2,1)} \\ &\quad - \frac{(n_2-1)(m_2-2)}{n_2+m_2-2}\alpha_{(n_2+m_2-1,2,0)} + \frac{m_2-1}{m_2+1}\alpha_{(n_2,m_2+1,0)}. \end{aligned}$$

Choose  $\varphi_1 \in \tilde{C}^2$  and  $\psi_1 \in \tilde{C}^3$  such that

$$\varphi_1(n, m) = \beta_{(n,m)} \quad , \quad \psi_1(n, 1, 0) = \beta_{(n,1)}; \quad n \geq 2, m \geq 0.$$

$$\psi_1(n, m, p) = \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m,p)} - \frac{(m-1)(p-1)}{m+p-1} \beta_{(n,m+p)}; \quad n, m \geq 2, p \geq 0.$$

Namely,

$$\begin{aligned} (\Delta^2 \varphi_1 - D^3 \psi_1)(n, m, p) = & -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1,p)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1,p)} \\ & + \frac{mp}{m+p-1} \Delta \beta_{(n,m+p-1)} + \frac{(n-1)mp}{m+p-1} \beta_{(n,m+p-1)} \\ & + \frac{m(m-1)}{m+p-1} \beta_{(n,m+p-1)} - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,p)} \\ & - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,p)} + \frac{m(m-2)(p-1)}{m+p-2} \beta_{(n,m+p-1)} \\ & - \frac{p(n-1)(m-1)}{n+m-1} \beta_{(n+m,p-1)} + \frac{p(m-1)(p-2)}{m+p-2} \beta_{(n,m+p-1)}, \end{aligned}$$

$$\begin{aligned} (\Delta^2 \varphi_1 - D^3 \psi_1)(n, m, 1) = & -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1,1)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1,1)} \\ & + (\Delta + n + m - 3) \beta_{(n,m)} - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,1)} \\ & - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,1)} - \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m,0)}, \end{aligned}$$

$$\begin{aligned} (\Delta^2 \varphi_1 - D^3 \psi_1)(n, m, 0) = & -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1,0)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1,0)} \\ & - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,0)} - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,0)}, \end{aligned}$$

$$(\Delta^2 \varphi_1 - D^3 \psi_1)(n, 1, 0) = -(\Delta + n - 2) \beta_{(n,0)},$$

where

$$\beta_{(k,0)} = \begin{cases} -\frac{\alpha_{(k,1,0)}}{\Delta+k-2}; & k \neq 2 - \Delta \\ 0; & \text{otherwise.} \end{cases}$$

Then  $\Delta^2 \varphi_1$  is a coboundary in  $\tilde{C}^3$ , and

$$(\Delta^3 \varphi_1 - D^4 \psi_1)(k, 1, 0) = \alpha_{(k,1,0)} = \varphi_{(k,1,0)}; \quad k \neq 2 - \Delta.$$

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi_1^{(i)}$  and  $\psi_1^{(i)}$  for  $i = 1, 2, 3$ , as following:

- For

$$\beta_{(k,0)} = \begin{cases} \frac{k}{2}\alpha_{(k,2,0)}; & k = 2 - \Delta, \\ 0; & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} (\Delta^3\varphi_1^{(1)} - D^4\psi_1^{(1)})(k, 2, 0) &= \alpha_{(k,2,0)} = \varphi_{(k,2,0)}; & k = 2 - \Delta, \\ (\Delta^3\varphi_1^{(1)} - D^4\psi_1^{(1)})(k, 1, 0) &= 0; & k \neq 2 - \Delta. \end{aligned}$$

- For

$$\beta_{(k,1)} = \begin{cases} \frac{k}{2}\alpha_{(k,2,1)}; & k = 2 - \Delta, \\ 0; & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} (\Delta^2\varphi_1^{(2)} - D^3\psi_1^{(2)})(k, 2, 1) &= \alpha_{(k,2,1)} = \varphi_{(k,2,1)}, \\ (\Delta^3\varphi_1^{(2)} - D^4\psi_1^{(2)})(k, 1, 0) &= 0, \\ (\Delta^3\varphi_1^{(2)} - D^4\psi_1^{(2)})(k, 2, 0) &= 0. \end{aligned}$$

- Also if we put

$$\beta_{(n,m)} = \begin{cases} \frac{\alpha_{(n,m,1)}}{\Delta+n+m-3}; & \Delta + n + m - 3 \neq 0, n, m \geq 2 \\ -\frac{m}{2}\alpha_{(n,m-1,2)}; & \Delta + n + m - 3 = 0, n, m \geq 2. \\ 0; & \text{otherwise,} \end{cases}$$

we get  $\varphi_1^{(3)}$  and  $\psi_1^{(3)}$  such that

$$\begin{aligned} (\Delta^2\varphi_1^{(3)} - D^3\psi_1^{(3)})(n, m, 1) &= \varphi_{(n,m,1)}, \\ (\Delta^2\varphi_1^{(3)} - D^3\psi_1^{(3)})(n, m - 1, 2) &= \varphi_{(n,m-1,2)}, \\ (\Delta^2\varphi_1^{(3)} - D^3\psi_1^{(3)})(n, m, p) &= 0; \quad \text{otherwise.} \end{aligned}$$

Hence, every 3-cocycle is a coboundary.

The differential  $\Delta^4\varphi$  takes the following values on the Anick 4-chains:

$$\begin{aligned} (\Delta^4\varphi)[v(n)v(m)v(p)v(q)v(r)] &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,q,r)} + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,q,r)} \\ &\quad - \frac{pq}{p+q-1}\Delta\alpha_{(n,m,p+q-1,r)} + \frac{qr}{q+r-1}\Delta\alpha_{(n,m,p,q+r-1)} \\ &\quad + \frac{(n-1)(m-1)}{n+m-1}\partial\alpha_{(n+m,p,q,r)} - \frac{(m-1)(p-1)}{m+p-1}\partial\alpha_{(n,m+p,q,r)} \\ &\quad + \frac{(p-1)(q-1)}{p+q-1}\partial\alpha_{(n,m,p+q,r)} - \frac{(q-1)(r-1)}{q+r-1}\partial\alpha_{(n,m,p,q+r)} \\ &\quad - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,q,r)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,q,r)} \end{aligned}$$

$$\begin{aligned}
 & -\frac{p(p-1)}{p+q-1}\alpha_{(n,m,p+q-1,r)} + \frac{q(q-1)}{q+r-1}\alpha_{(n,m,p,q+r-1)} \\
 & + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m,p-1,q,r)} - \frac{(n-1)pq}{p+q-1}\alpha_{(n,m,p+q-1,r)} \\
 & - \frac{(m-1)pq}{p+q-1}\alpha_{(n,m,p+q-1,r)} + \frac{(n-1)qr}{q+r-1}\alpha_{(n,m,p,q+r-1)} \\
 & + \frac{(m-1)qr}{q+r-1}\alpha_{(n,m,p,q+r-1)} + \frac{(p-1)qr}{q+r-1}\alpha_{(n,m,p,q+r-1)} \\
 & - \frac{n(m-1)(p-1)}{m+p-1}\alpha_{(n-1,m+p,q,r)} + \frac{n(p-1)(q-1)}{p+q-1}\alpha_{(n-1,m,p+q,r)} \\
 & + \frac{m(p-1)(q-1)}{p+q-1}\alpha_{(n,m-1,p+q,r)} - \frac{n(q-1)(r-1)}{q+r-1}\alpha_{(n-1,m,p,q+r)} \\
 & - \frac{m(q-1)(r-1)}{q+r-1}\alpha_{(n,m-1,p,q+r)} - \frac{p(q-1)(r-1)}{q+r-1}\alpha_{(n,m,p-1,q+r)}, \\
 (\Delta^4\varphi)[v(n)v(m)v(p)v(1)v(0)] & = -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,1,0)} + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,1,0)} \\
 & + \frac{(n-1)(m-1)}{n+m-1}\partial\alpha_{(n+m,p,1,0)} - \frac{(m-1)(p-1)}{m+p-1}\partial\alpha_{(n,m+p,1,0)} \\
 & + \partial\alpha_{(n,m,p,1)} - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,1,0)} \\
 & + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,1,0)} + \frac{n(m-1)(p-1)}{m+p-1}\alpha_{(n-1,m+p,1,0)} \\
 & + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p,1,0)} + \Delta\alpha_{(n,m,0)} + (n+m-3)\alpha_{(n,m,0)} \\
 & - n\alpha_{(n-1,m,p,1)} + m\alpha_{(n,m-1,p,1)} - p\alpha_{(n,m,p-1,1)}.
 \end{aligned}$$

Reduce the result by means of  $D^5\psi$ , where  $\psi \in \tilde{C}^5$  is given by

$$\begin{aligned}
 \psi[v(n)v(m)v(p)v(q)v(r)] & = +\frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,q,r)} - \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,q,r)} \\
 & + \frac{(p-1)(q-1)}{p+q-1}\alpha_{(n,m,p+q,r)} - \frac{(q-1)(r-1)}{q+r-1}\alpha_{(n,m,p,q+r)}. \\
 \psi[v(n)v(m)v(p)v(1)v(0)] & = +\frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,1,0)} - \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,1,0)} \\
 & + \alpha_{(n,m,p,1)}.
 \end{aligned}$$

Namely,

$$\begin{aligned}
 (\Delta^4\varphi - D^5\psi)[v(n)v(m)v(p)v(q)v(r)] & = -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,q,r)} \\
 & + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,q,r)}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{pq}{p+q-1} \Delta \alpha_{(n,m,p+q-1,r)} + \frac{qr}{q+r-1} \Delta \alpha_{(n,m,p,q+r-1)} - \frac{n(n-1)}{n+m-1} \alpha_{(n+m-1,p,q,r)} \\
 & + \frac{m(m-1)}{m+p-1} \alpha_{(n,m+p-1,q,r)} - \frac{p(p-1)}{p+q-1} \alpha_{(n,m,p+q-1,r)} + \frac{q(q-1)}{q+r-1} \alpha_{(n,m,p,q+r-1)} \\
 & + \frac{(n-1)mp}{m+p-1} \alpha_{(n,m+p-1,q,r)} - \frac{(n-1)pq}{p+q-1} \alpha_{(n,m,p+q-1,r)} - \frac{(m-1)pq}{p+q-1} \alpha_{(n,m,p+q-1,q,r)} \\
 & + \frac{(n-1)qr}{q+r-1} \alpha_{(n,m,p,q+r-1)} + \frac{(m-1)qr}{q+r-1} \alpha_{(n,m,p,q+r-1)} + \frac{(p-1)qr}{q+r-1} \alpha_{(n,m,p,q+r-1)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2} \alpha_{(n+m-1,p,q,r)} - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,p,q,r)} \\
 & + \frac{m(m-2)(p-1)}{m+p-2} \alpha_{(n,m+p-1,q,r)} - \frac{p(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p-1,q,r)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2} \alpha_{(n,m+p-1,q,r)} - \frac{p(p-2)(q-1)}{p+q-2} \alpha_{(n,m,p+q-1,r)} \\
 & - \frac{q(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p,q-1,r)} + \frac{q(m-1)(p-1)}{m+p-1} \alpha_{(n,m+p,q-1,r)} \\
 & - \frac{q(p-1)(q-2)}{p+q-2} \alpha_{(n,m,p+q-1,r)} + \frac{q(q-2)(r-1)}{q+r-2} \alpha_{(n,m,p,q+r-1)} \\
 & - \frac{r(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p,q,r-1)} + \frac{r(m-1)(p-1)}{m+p-1} \alpha_{(n,m+p,q,r-1)} \\
 & - \frac{r(p-1)(q-1)}{p+q-1} \alpha_{(n,m,p+q,r-1)} + \frac{r(q-1)(r-2)}{q+r-2} \alpha_{(n,m,p,q+r-1)},
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^4 \varphi - D^5 \psi)[v(n)v(m)v(p)v(1)v(0)] = & - \frac{nm}{n+m-1} \Delta \alpha_{(n+m-1,p,1,0)} \\
 & + \frac{mp}{m+p-1} \Delta \alpha_{(n,m+p-1,1,0)} \\
 & - \frac{n(n-1)}{n+m-1} \alpha_{(n+m-1,p,1,0)} \\
 & + \frac{m(m-1)}{m+p-1} \alpha_{(n,m+p-1,1,0)} \\
 & - (\Delta + n + m + p - 4) \alpha_{(n,m,p,0)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2} \alpha_{(n+m-1,p,1,0)} \\
 & + \frac{n(m-1)(p-1)}{m+p-1} \alpha_{(n-1,m+p,1,0)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,p,1,0)}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{n(m-2)(p-1)}{m+p-2} \alpha_{(n,m+p-1,1,0)} - \frac{p(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p-1,1,0)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2} \alpha_{(n,m+p-1,1,0)}.
 \end{aligned}$$

Hence,  $\varphi - D^5 \tilde{C}^5$  is a 4-cocycle in  $C^4$ .

CASE 1:  $\Delta + n + m + p + q - 5 \neq 0$ ;  $n, m, p, q \geq 2$ .

$$\begin{aligned}
 -(\Delta + n + m + p + q - 5) \alpha_{(n,m,p,q)} &= -\frac{nm}{n+m-1} \Delta \alpha_{(n+m-1,p,q,1)} \\
 & + \frac{mp}{m+p-1} \Delta \alpha_{(n,m+p-1,q,1)} - \frac{pq}{p+q-1} \Delta \alpha_{(n,m,p+q-1,1)} \\
 & - \frac{n(n-1)}{n+m-1} \alpha_{(n+m-1,p,q,1)} + \frac{m(m-1)}{m+p-1} \alpha_{(n,m+p-1,q,1)} \\
 & - \frac{p(p-1)}{p+q-1} \alpha_{(n,m,p+q-1,1)} + \frac{(n-1)mp}{m+p-1} \alpha_{(n,m+p-1,q,1)} \\
 & - \frac{(n-1)pq}{p+q-1} \alpha_{(n,m,p+q-1,q,1)} - \frac{(m-1)pq}{p+q-1} \alpha_{(n,m,p+q-1,q,1)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2} \alpha_{(n+m-1,p,q,1)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,p,q,1)} \\
 & + \frac{m(m-2)(p-1)}{m+p-2} \alpha_{(n,m+p-1,q,1)} \\
 & - \frac{p(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p-1,q,1)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2} \alpha_{(n,m+p-1,q,1)} \\
 & - \frac{p(p-2)(q-1)}{p+q-2} \alpha_{(n,m,p+q-1,1)} \\
 & - \frac{q(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p,q-1,1)} \\
 & + \frac{q(m-1)(p-1)}{m+p-1} \alpha_{(n,m+p,q-1,1)} \\
 & - \frac{q(p-1)(q-2)}{p+q-2} \alpha_{(n,m,p+q-1,1)} \\
 & - \frac{(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p,q,0)} \\
 & + \frac{(m-1)(p-1)}{m+p-1} \alpha_{(n,m+p,q,0)} \\
 & - \frac{(p-1)(q-1)}{p+q-1} \alpha_{(n,m,p+q,0)}.
 \end{aligned}$$



CASE 2:  $\Delta + n + m + p - 4 \neq 0$ ;  $n, m, p \geq 2$ .

$$\begin{aligned}
 (\Delta + n + m + p - 4)\alpha_{(n,m,p,0)} &= -\frac{nm}{n+m-1}\Delta\alpha_{(n+m-1,p,1,0)} + \frac{mp}{m+p-1}\Delta\alpha_{(n,m+p-1,1,0)} \\
 &\quad - \frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,1,0)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,1,0)} \\
 &\quad - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,1,0)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,1,0)} \\
 &\quad - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p,1,0)} \\
 &\quad + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1,1,0)} \\
 &\quad - \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1,1,0)} \\
 &\quad + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1,1,0)}.
 \end{aligned}$$

CASE 3:  $\Delta + n_2 + m_2 + p_2 - 4 = 0$ ;  $n_2, m_2, p_2 \geq 2$ .

$$\begin{aligned}
 \frac{2p_2-4}{p_2}\alpha_{(n_2,m_2,p_2,0)} &= -\frac{n_2m_2}{n_2+m_2-1}\Delta\alpha_{(n_2+m_2-1,p_2-1,2,0)} + \frac{m_2(p_2-1)}{m_2+p_2-2}\Delta\alpha_{(n_2,m_2+p_2-2,2,0)} \\
 &\quad - \frac{n_2(n_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2-1,p_2-1,2,0)} + \frac{m_2(m_2-1)}{m_2+p_2-2}\alpha_{(n_2,m_2+p_2-2,2,0)} \\
 &\quad + \frac{m_2(n_2-1)(p_2-1)}{m_2+p_2-2}\alpha_{(n_2,m_2+p_2-2,2,0)} \\
 &\quad - \frac{n_2(n_2-2)(m_2-1)}{n_2+m_2-2}\alpha_{(n_2+m_2-1,p_2-1,2,0)} \\
 &\quad - \frac{m_2(n_2-1)(m_2-2)}{n_2+m_2-2}\alpha_{(n_2+m_2-1,p_2-1,2,0)} \\
 &\quad + \frac{m_2(m_2-2)(p_2-2)}{m_2+p_2-3}\alpha_{(n_2,m_2+p_2-2,2,0)} \\
 &\quad - \frac{(p_2-1)(n_2-1)(m_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2,p_2-3,2,0)} \\
 &\quad + \frac{(p_2-1)(m_2-1)(p_2-3)}{m_2+p_2-3}\alpha_{(n_2,m_2+p_2-2,2,0)} \\
 &\quad - \frac{2(n_2-1)(m_2-1)}{n_2+m_2-1}\alpha_{(n_2+m_2,p_2-1,1,0)} \\
 &\quad + \frac{2(m_2-1)(p_2-2)}{m_2+p_2-3}\alpha_{(n_2,m_2+p_2-1,1,0)}.
 \end{aligned}$$

We can obtain from Case 4 a new special case when  $p_2 = 2$ . Let  $n_1 = n_2$  and let  $m_1 = m_2 + 1 \geq 3$  then

$$\Delta + n_1 + m_1 - 3 = \Delta + n_1 + m_1 - 1 + 2 - 4 = \Delta + n_2 + m_2 + p_2 - 4 = 0.$$

Therefore

$$\begin{aligned}
 \alpha_{(n_2, m_2, 2, 0)} &= \alpha_{(n_1, m_1 - 1, 2, 0)} \\
 \frac{2}{m_2} \alpha_{(n_2, m_2, 1, 0)} &= -\frac{n_2(m_2 - 1)}{n_2 + m_2 - 2} \Delta \alpha_{(n_2 + m_2 - 2, 2, 1, 0)} - \frac{n_2(n_2 - 1)}{n_2 + m_2 - 2} \alpha_{(n_2 + m_2 - 2, 2, 1, 0)} \\
 &\quad - \frac{n_2(n_2 - 2)(m_2 - 2)}{n_2 + m_2 - 3} \alpha_{(n_2 + m_2 - 2, 2, 1, 0)} \\
 &\quad - \frac{(m_2 - 1)(n_2 - 2)(m_2 - 3)}{n_2 + m_2 - 3} \alpha_{(n_2 + m_2 - 2, 2, 1, 0)}, \\
 \\
 \frac{-2}{p_2} \alpha_{(n_2, m_2, p_2, 1)} &= -\frac{n_2 m_2}{n_2 + m_2 - 1} \Delta \alpha_{(n_2 + m_2 - 1, p_2 - 1, 2, 1)} + \frac{m_2(p_2 - 1)}{m_2 + p_2 - 2} \Delta \alpha_{(n_2, m_2 + p_2 - 2, 2, 1)} \\
 &\quad - \frac{n_2(n_2 - 1)}{n_2 + m_2 - 1} \alpha_{(n_2 + m_2 - 1, p_2 - 1, 2, 1)} + \frac{m_2(m_2 - 1)}{m_2 + p_2 - 2} \alpha_{(n_2, m_2 + p_2 - 2, 2, 1)} \\
 &\quad + \frac{(n_2 - 1)m_2(p_2 - 1)}{m_2 + p_2 - 2} \alpha_{(n_2, m_2 + p_2 - 2, 2, 1)} \\
 &\quad - \frac{n_2(n_2 - 2)(m_2 - 1)}{n_2 + m_2 - 2} \alpha_{(n_2 + m_2 - 1, p_2 - 1, 2, 1)} \\
 &\quad - \frac{m_2(n_2 - 1)(m_2 - 2)}{n_2 + m_2 - 2} \alpha_{(n_2 + m_2 - 1, p_2 - 1, 2, 1)} \\
 &\quad + \frac{m_2(m_2 - 2)(p_2 - 2)}{m_2 + p_2 - 3} \alpha_{(n_2, m_2 + p_2 - 2, 2, 1)} \\
 &\quad - \frac{(p_2 - 1)(n_2 - 1)(m_2 - 1)}{n_2 + m_2 - 1} \alpha_{(n_2 + m_2, p_2 - 2, 2, 1)} \\
 &\quad + \frac{(p_2 - 1)(m_2 - 1)(p_2 - 3)}{m_2 + p_2 - 3} \alpha_{(n_2, m_2 + p_2 - 2, 2, 1)} \\
 &\quad - \frac{(n_2 - 1)(m_2 - 1)}{n_2 + m_2 - 1} \alpha_{(n_2 + m_2, p_2 - 1, 2, 0)} \\
 &\quad + \frac{(m_2 - 1)(p_2 - 2)}{m_2 + p_2 - 2} \alpha_{(n_2, m_2 + p_2 - 1, 2, 0)} \\
 &\quad - \frac{(p_2 - 2)}{p_2} \alpha_{(n_2, m_2, p_2 + 1, 0)}.
 \end{aligned}$$

CASE 4:  $\Delta + n_3 + m_3 + p_3 + q_3 - 5 = 0$ ;  $n_3, m_3, p_3 \geq 2, q_3 \geq 3$ .

$$\begin{aligned}
 \frac{-2q_3 + 4}{q_3} \alpha_{(n_3, m_3, p_3, q_3)} &= -\frac{n_3 m_3}{n_3 + m_3 - 1} \Delta \alpha_{(n_3 + m_3 - 1, p_3, q_3 - 1, 2)} \\
 &\quad + \frac{m_3 p_3}{m_3 + p_3 - 1} \Delta \alpha_{(n_3, m_3 + p_3 - 1, q_3 - 1, 2)} - \frac{p_3(q_3 - 1)}{p_3 + q_3 - 2} \Delta \alpha_{(n_3, m_3, p_3 + q_3 - 2, 2)} \\
 &\quad - \frac{n_3(n_3 - 1)}{n_3 + m_3 - 1} \alpha_{(n_3 + m_3 - 1, p_3, q_3 - 1, 2)} + \frac{m_3(m_3 - 1)}{m_3 + p_3 - 1} \alpha_{(n_3, m_3 + p_3 - 1, q_3 - 1, 2)}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{p_3(p_3-1)}{p_3+q_3-2} \alpha_{(n_3, m_3, p_3+q_3-2, 2)} + \frac{(n_3-1)m_3p_3}{m_3+p_3-1} \alpha_{(n_3, m_3+p_3-1, q_3-1, 2)} \\
 & - \frac{(n_3-1)p_3(q_3-1)}{p_3+q_3-2} \alpha_{(n_3, m_3, p_3+q_3-2, 2)} - \frac{(m_3-1)p_3(q_3-1)}{p_3+q_3-2} \alpha_{(n_3, m_3, p_3+q_3-2, 2)} \\
 & - \frac{n_3(n_3-2)(m_3-1)}{n_3+m_3-2} \alpha_{(n_3+m_3-1, p_3, q_3-1, 2)} - \frac{m_3(n_3-1)(m_3-2)}{n_3+m_3-2} \alpha_{(n_3+m_3-1, p_3, q_3-1, 2)} \\
 & + \frac{m_3(m_3-2)(p_3-1)}{m_3+p_3-2} \alpha_{(n_3, m_3+p_3-1, q_3-1, 2)} - \frac{p_3(n_3-1)(m_3-1)}{n_3+m_3-1} \alpha_{(n_3+m_3, p_3-1, q_3-1, 2)} \\
 & + \frac{p_3(m_3-1)(p_3-2)}{m_3+p_3-2} \alpha_{(n_3, m_3+p_3-1, q_3-1, 2)} - \frac{p_3(p_3-2)(q_3-2)}{p_3+q_3-3} \alpha_{(n_3, m_3, p_3+q_3-2, 2)} \\
 & - \frac{(q_3-1)(n_3-1)(m_3-1)}{n_3+m_3-1} \alpha_{(n_3+m_3, p_3, q_3-2, 2)} + \frac{(q_3-1)(m_3-1)(p_3-1)}{m_3+p_3-1} \alpha_{(n_3, m_3+p_3, q_3-2, 2)} \\
 & - \frac{(q_3-1)(p_3-1)(q_3-3)}{m_3+p_3-3} \alpha_{(n_3, m_3, p_3+q_3-2, 2)} - \frac{2(n_3-1)(m_3-1)}{n_3+m_3-1} \alpha_{(n_3+m_3, p_3, q_3-1, 1)} \\
 & + \frac{2(m_3-1)(p_3-1)}{m_3+p_3-1} \alpha_{(n_3, m_3+p_3, q_3-1, 1)} - \frac{2(p_3-1)(q_3-2)}{p_3+q_3-2} \alpha_{(n_3, m_3, p_3+q_3-1, 1)}.
 \end{aligned}$$

Choose  $\varphi_1 \in \tilde{C}^3$  and  $\psi_1 \in \tilde{C}^4$  such that

$$\begin{aligned}
 \varphi_1[v(n)v(m)v(p)] &= \beta_{(n, m, p)}; \quad n, m \geq 2, p \geq 0, \\
 \varphi_1[v(n)v(1)v(0)] &= \beta_{(n, 1, 0)}; \quad n \geq 2, \\
 \psi_1[v(n)v(m)v(p)v(q)] &= \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m, p, q)} \\
 & - \frac{(m-1)(p-1)}{m+p-1} \beta_{(n, m+p, q)} \\
 & + \frac{(p-1)(q-1)}{p+q-1} \beta_{(n, m, p+q)}; \quad n, m, p \geq 2, q \geq 0, \\
 \psi_1[v(n)v(m)v(1)v(0)] &= \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m, 1, 0)} - \beta_{(n, m, 1)}; \quad n, m \geq 2.
 \end{aligned}$$

Namely,

$$\begin{aligned}
 (\Delta^3 \varphi_1 - D^4 \psi_1)[v(n)v(m)v(p)v(q)] &= -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1, p, q)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1, p, q)} \\
 & + \frac{mp}{m+p-1} \Delta \beta_{(n, m+p-1, q)} + \frac{(n-1)mp}{m+p-1} \beta_{(n, m+p-1, q)} \\
 & - \frac{pq}{p+q-1} \Delta \beta_{(n, m, p+q-1)} - \frac{(m-1)pq}{p+q-1} \beta_{(n, m, p+q-1)} \\
 & - \frac{(n-1)pq}{p+q-1} \beta_{(n, m, p+q-1)} + \frac{m(m-1)}{m+p-1} \beta_{(n, m+p-1, q)}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{p(p-1)}{p+q-1} \beta_{(n,m,p+q-1)} - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,p,q)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,p,q)} + \frac{m(m-2)(p-1)}{m+p-2} \beta_{(n,m+p-1,q)} \\
 & - \frac{p(n-1)(m-1)}{n+m-1} \beta_{(n+m,p-1,q)} + \frac{p(m-1)(p-2)}{m+p-2} \beta_{(n,m+p-1,q)} \\
 & - \frac{p(p-2)(q-1)}{p+q-2} \beta_{(n,m,p+q-1)} - \frac{q(n-1)(m-1)}{n+m-1} \beta_{(n+m,p,q-1)} \\
 & + \frac{q(m-1)(p-1)}{m+p-1} \beta_{(n,m+p,q-1)} - \frac{q(p-1)(q-2)}{p+q-2} \beta_{(n,m,p+q-1)},
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^3 \varphi_1 - D^4 \psi_1)[v(n)v(m)v(1)v(0)] &= -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1,1,0)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1,1,0)} \\
 & - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,1,0)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,1,0)} \\
 & + (\Delta + n + m - 3) \beta_{(n,m,0)},
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^3 \varphi_1 - D^4 \psi_1)[v(n)v(m)v(p)v(0)] &= -\frac{nm}{n+m-1} \Delta \beta_{(n+m-1,p,0)} - \frac{n(n-1)}{n+m-1} \beta_{(n+m-1,p,0)} \\
 & + \frac{mp}{m+p-1} \Delta \beta_{(n,m+p-1,0)} + \frac{(n-1)mp}{m+p-1} \beta_{(n,m+p-1,0)} \\
 & + \frac{m(m-1)}{m+p-1} \beta_{(n,m+p-1,0)} - \frac{n(n-2)(m-1)}{n+m-2} \beta_{(n+m-1,p,0)} \\
 & - \frac{m(n-1)(m-2)}{n+m-2} \beta_{(n+m-1,p,0)} \\
 & + \frac{m(m-2)(p-1)}{m+p-2} \beta_{(n,m+p-1,0)} \\
 & - \frac{p(n-1)(m-1)}{n+m-1} \beta_{(n+m,p-1,0)} \\
 & + \frac{p(m-1)(p-2)}{m+p-2} \beta_{(n,m+p-1,0)},
 \end{aligned}$$

where

- for

$$\beta_{(n,m,0)} = \begin{cases} \frac{\alpha_{(n,m,1,0)}}{\Delta+n+m-3}; & \Delta + n + m - 3 \neq 0, n, m \geq 2, \\ 0; & \text{otherwise,} \end{cases}$$

then  $\Delta^3\varphi_1$  is a coboundary in  $\tilde{C}^4$ , and

$$(\Delta^3\varphi_1 - D^4\psi_1)[v(n)v(m)v(1)v(0)] = \varphi(n, m, 1, 0); \quad \Delta + n + m - 3 \neq 0.$$

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi_1^{(i)}$  and  $\psi_1^{(i)}$  for  $i = 1, 2, 3, 4$  as following:

- For

$$\beta_{(n,1,0)} = \begin{cases} \frac{n}{2}\alpha_{(n-1,2,1,0)}; & \Delta + n - 1 = 0, n \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} (\Delta^3\varphi_1^{(1)} - D^4\psi_1^{(1)})[v(n-1)v(2)v(1)v(0)] &= \varphi(n-1, 2, 1, 0); \quad \Delta + n - 1 = 0. \\ (\Delta^3\varphi_1^{(1)} - D^4\psi_1^{(1)})[v(n)v(m)v(p)v(q)] &= 0; \quad \text{otherwise.} \end{aligned}$$

- Also, if we put

$$\beta_{(n,m,0)} = \begin{cases} \frac{m}{2m-4}\alpha_{(n,m-1,2,0)}; & \Delta + n + m - 3 = 0, n \geq 2, m \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we get new  $\varphi_1^{(2)}$  and  $\psi_1^{(2)}$  such that

$$\begin{aligned} (\Delta^3\varphi_1^{(2)} - D^4\psi_1^{(2)})[v(n)v(m-1)v(2)v(0)] &= \varphi(n, m-1, 2, 0); \quad \Delta + n + m - 3 = 0, \\ (\Delta^3\varphi_1^{(2)} - D^4\psi_1^{(2)})[v(n)v(m)v(p)v(q)] &= 0; \quad \text{otherwise.} \end{aligned}$$

- For

$$\beta_{(n,m,1)} = \begin{cases} -\frac{m}{2}\alpha_{(n,m-1,2,1)}; & \Delta + n + m - 2 = 0, n \geq 2, m \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} (\Delta^3\varphi_1^{(3)} - D^4\psi_1^{(3)})[v(n)v(m-1)v(2)v(1)] &= \varphi(n, m-1, 2, 1); \quad \Delta + n + m - 2 = 0, \\ (\Delta^3\varphi_1^{(3)} - D^4\psi_1^{(3)})[v(n)v(m)v(p)v(q)] &= 0; \quad \text{otherwise.} \end{aligned}$$

- For

$$\beta_{(n,m,p)} = \begin{cases} -\frac{\alpha_{(n,m,p,0)}}{\Delta+n+m+p-4}; & \Delta + n + m + p - 4 \neq 0, n, m, p \geq 2, \\ \frac{p}{-2p+4}\alpha_{(n,m,p-1,2)}; & \Delta + n + m + p - 4 = 0, n, m \geq 2, p \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} (\Delta^3\varphi_1^{(4)} - D^4\psi_1^{(4)})[v(n)v(m)v(p)v(1)] &= \varphi(n, m, p, 1); \quad \Delta + n + m + p - 4 \neq 0, \\ (\Delta^3\varphi_1^{(4)} - D^4\psi_1^{(4)})[v(n)v(m)v(p-1)v(2)] &= \varphi(n, m, p-1, 2); \\ &\quad \text{where } \Delta + n + m + p - 4 = 0, \\ (\Delta^3\varphi_1^{(4)} - D^4\psi_1^{(4)})[v(n)v(m)v(p)v(q)] &= 0; \quad \text{otherwise.} \end{aligned}$$

Hence, every 4-cocycle is a coboundary.

In a similar way,

$$\begin{aligned}
 (\Delta^n \varphi - D^{n+1} \psi)[i_1 | i_2 | \dots | i_{n+1}] &= \sum_{j=1}^n \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})} \\
 &+ \sum_{j=1}^n (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})} \\
 &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n+1})} \\
 &+ \sum_{t=3}^{n+1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n+1})} \\
 &+ \sum_{t=1}^n (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n+1})} \\
 &+ \sum_{t=1}^n (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n+1})}
 \end{aligned}$$

$$\begin{aligned}
 (\Delta^n \varphi - D^{n+1} \psi)[i_1 | i_2 | \dots | i_{n-1} | 1 | 0] &= \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ (-1)^{n-1} \Delta \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ (-1)^{n-1} \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &+ \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\
 &\times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ \sum_{t=1}^{n-1} (-1)^{n-1} (i_t - 1) \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} \\
 &+ \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 &\times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1}, 1, 0)} \\
 &+ \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)}
 \end{aligned}$$

$$+ \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)}.$$

Hence,  $\varphi - D^{n+1} \tilde{C}^{n+1}$  is a  $n$ -cocycle in  $C^n$  for various  $\Delta$ :

CASE 1:  $\Delta + i_1 + \dots + i_n - n - 1 \neq 0$ ;  $i_1, \dots, i_n \geq 2$ .

$$\begin{aligned} (-1)^{n+1} (\Delta + i_1 + \dots + i_n - n - 1) \alpha_{(i_1, i_2, \dots, i_n)} &= \sum_{j=1}^{n-1} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 1)} \\ &+ \sum_{j=1}^{n-1} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 1)} \\ &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\ &\quad \times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 1)} \\ &+ \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\ &\quad \times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_n, 1)} \\ &+ \sum_{t=1}^n (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \\ &\quad \times \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n, 1)} \\ &+ \sum_{t=1}^n (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \\ &\quad \times \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n, 1)}. \end{aligned}$$

CASE 2:  $\Delta + i_1 + \dots + i_{n-1} - n \neq 0$ ;  $i_1, \dots, i_{n-1} \geq 2$ .

$$\begin{aligned} (-1)^n (\Delta + i_1 + \dots + i_{n-1} - n) \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} &= \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\ &+ \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \\ &\quad \times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\ &+ \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\ &\quad \times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)}
 \end{aligned}$$

CASE 3:  $\Delta + i_1 + \dots + i_{n-1} - n = 0$ ;  $i_1, \dots, i_{n-1} \geq 2$ .

$$\begin{aligned}
 \frac{2(-1)^n}{i_{n-2}} \alpha_{(i_1, i_2, \dots, i_{n-2}, 1, 0)} & = \sum_{j=1}^{n-4} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + \frac{(-1)^{n-3} i_{n-3} (i_{n-2} - 1)}{i_{n-3} + i_{n-2} - 2} \Delta \alpha_{(i_1, \dots, i_{n-4}, i_{n-3} + i_{n-2} - 2, 2, 1, 0)} \\
 & + \sum_{j=1}^{n-4} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + (-1)^{n-3} \frac{i_{n-3} (i_{n-3} - 1)}{i_{n-3} + i_{n-2} - 2} \alpha_{(i_1, \dots, i_{n-3} + i_{n-2} - 2, 2, 1, 0)} \\
 & + \sum_{t=1}^{n-5} \sum_{j=2}^{t+1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + \sum_{t=1}^{n-4} (-1)^{n-3} \frac{i_{n-3} (i_{n-2} - 1)}{i_{n-3} + i_{n-2} - 2} (i_t - 1) \alpha_{(i_1, \dots, i_{n-3} + i_{n-2} - 2, 2, 1, 0)} \\
 & + \sum_{t=3}^{n-4} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-2}, 2, 1, 0)} \\
 & + (-1)^{n-3} \frac{2(i_{n-3} - 1)(i_{n-2} - 2)}{i_{n-3} + i_{n-2} - 2} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + \sum_{t=1}^{n-4} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + (-1)^{n-3} i_{n-3} \frac{(i_{n-3} - 2)(i_{n-2} - 2)}{i_{n-3} + i_{n-2} - 3} \alpha_{(i_1, \dots, i_{n-3} + i_{n-2} - 2, 2, 1, 0)} \\
 & + \sum_{t=1}^{n-4} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-2} - 1, 2, 1, 0)} \\
 & + (-1)^{n-3} \frac{(i_{n-2} - 1)(i_{n-3} - 1)(i_{n-2} - 3)}{i_{n-3} + i_{n-2} - 3} \alpha_{(i_1, \dots, i_{n-3} + i_{n-2} - 2, 2, 1, 0)},
 \end{aligned}$$



$$\begin{aligned}
 (-1)^n \frac{2i_{n-1} - 4}{i_{n-1}} \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} &= \sum_{j=1}^{n-3} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ \frac{(-1)^{n-2} i_{n-2} (i_{n-1} - 1)}{i_{n-2} + i_{n-1} - 2} \Delta \alpha_{(i_1, \dots, i_{n-3}, i_{n-2} + i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ (-1)^{n-2} \frac{i_{n-2} (i_{n-2} - 1)}{i_{n-2} + i_{n-1} - 2} \alpha_{(i_1, \dots, i_{n-3}, i_{n-2} + i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{t=1}^{n-4} \sum_{j=2}^{t+1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ \sum_{t=1}^{n-3} (-1)^{n-2} \frac{i_{n-2} (i_{n-1} - 1)}{i_{n-2} + i_{n-1} - 2} (i_t - 1) \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j i_j \frac{(i_j - 2)(i_{j+1} - 1)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ (-1)^{n-2} i_{n-2} \frac{(i_{n-2} - 2)(i_{n-1} - 2)}{i_{n-2} + i_{n-1} - 3} \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j i_{j+1} \frac{(i_j - 1)(i_{j+1} - 2)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ (-1)^{n-2} \frac{(i_{n-1} - 1)(i_{n-2} - 1)(i_{n-1} - 3)}{i_{n-2} + i_{n-1} - 3} \\
 &\times \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{j=1}^{n-4} \sum_{t=j+2}^{n-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 &\times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{(i_{n-1} - 1)(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 &\times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-1} - 2, 2, 0)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{2(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1} - 1, 1, 0)} \\
 &+ (-1)^{n-2} \frac{2(i_{n-2} - 1)(i_{n-1} - 2)}{i_{n-2} + i_{n-1} - 3} \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 1, 1, 0)}
 \end{aligned}$$

$$\begin{aligned}
 -\frac{2(-1)^n}{i_{n-1}}\alpha_{(i_1, i_2, \dots, i_{n-1}, 1)} &= \sum_{j=1}^{n-3} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ \frac{(-1)^{n-2} i_{n-2} (i_{n-1} - 1)}{i_{n-2} + i_{n-1} - 2} \Delta \alpha_{(i_1, \dots, i_{n-3}, i_{n-2} + i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ (-1)^{n-2} \frac{i_{n-2} (i_{n-2} - 1)}{i_{n-2} + i_{n-1} - 2} \alpha_{(i_1, \dots, i_{n-3}, i_{n-2} + i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{t=1}^{n-4} \sum_{j=2}^{t+1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ \sum_{t=1}^{n-3} (-1)^{n-2} \frac{i_{n-2} (i_{n-1} - 1)}{i_{n-2} + i_{n-1} - 2} (i_t - 1) \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j i_j \frac{(i_j - 2)(i_{j+1} - 1)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ (-1)^{n-2} i_{n-2} \frac{(i_{n-2} - 2)(i_{n-1} - 2)}{i_{n-2} + i_{n-1} - 3} \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j i_{j+1} \frac{(i_j - 1)(i_{j+1} - 2)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ (-1)^{n-2} \frac{(i_{n-1} - 1)(i_{n-2} - 1)(i_{n-1} - 3)}{i_{n-2} + i_{n-1} - 3} \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{j=1}^{n-4} \sum_{t=j+2}^{n-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1} - 1, 2, 1)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{(i_{n-1} - 1)(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-1} - 2, 2, 1)} \\
 &+ \sum_{j=1}^{n-3} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1} - 1, 2, 0)} \\
 &+ (-1)^{n-2} \frac{(i_{n-2} - 1)(i_{n-1} - 2)}{i_{n-2} + i_{n-1} - 3} \alpha_{(i_1, \dots, i_{n-2} + i_{n-1} - 1, 2, 0)} \\
 &+ (-1)^{n-1} \frac{i_{n-1} - 2}{i_{n-1}} \alpha_{(i_1, \dots, i_{n-1} + 1, 0)}.
 \end{aligned}$$

$$\begin{aligned}
 (-1)^{n+1} \frac{2i_n - 4}{i_n} \alpha_{(i_1, i_2, \dots, i_n)} &= \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n - 1, 2)} \\
 &+ \frac{(-1)^{n-1} i_{n-1} (i_n - 1)}{i_{n-1} + i_n - 2} \Delta \alpha_{(i_1, \dots, i_{n-1} + i_n - 2, 2)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n - 1, 2)} \\
 &+ (-1)^{n-1} \frac{i_{n-1} (i_{n-1} - 1)}{i_{n-1} + i_n - 2} \alpha_{(i_1, \dots, i_{n-1} + i_n - 2, 2)} \\
 &+ \sum_{t=1}^{n-3} \sum_{j=2}^{t+1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n - 1, 2)} \\
 &+ \sum_{t=1}^{n-2} (-1)^{n-1} \frac{i_{n-1} (i_n - 1)}{i_{n-1} + i_n - 2} (i_t - 1) \alpha_{(i_1, \dots, i_{n-1} + i_n - 2, 2)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j i_j \frac{(i_j - 2)(i_{j+1} - 1)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n - 1, 2)} \\
 &+ (-1)^{n-1} \frac{i_n (i_{n-1} - 2)(i_n - 2)}{i_{n-1} + i_n - 3} \alpha_{(i_1, \dots, i_{n-1} + i_n - 2, 2)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j i_{j+1} \frac{(i_j - 1)(i_{j+1} - 2)}{i_j + i_{j+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n - 1, 2)} \\
 &+ (-1)^{n-1} \frac{(i_n - 1)(i_{n-1} - 1)(i_n - 3)}{i_{n-1} + i_n - 3} \alpha_{(i_1, \dots, i_{n-1} + i_n - 2, 2)} \\
 &+ \sum_{j=1}^{n-3} \sum_{t=j+1}^{n-1} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_n - 1, 2)} \\
 &+ \sum_{j=1}^{n-2} (-1)^j \frac{(i_n - 1)(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n - 1, 2)} \\
 &+ \sum_{j=1}^{n-1} (-1)^j \frac{2(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n - 1, 1)} \\
 &+ (-1)^{n-1} \frac{2(i_{n-1} - 1)(i_n - 2)}{i_{n-1} + i_n - 2} \alpha_{(i_1, \dots, i_{n-1} + i_n - 1, 1)}.
 \end{aligned}$$

Let  $\varphi_1 \in \tilde{C}^{n-1}$ ,  $\psi_1 \in \tilde{C}^n$ , such that

$$\begin{aligned}
 \varphi_1(i_1, i_2, \dots, i_{n-1}) &= \beta_{(i_1, i_2, \dots, i_{n-1})}, \\
 \psi_1(i_1, i_2, \dots, i_n) &= \sum_{j=1}^{n-1} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n)},
 \end{aligned}$$

$$\begin{aligned} \psi_1(i_1, i_2, \dots, i_{n-2}, 1, 0) &= \sum_{j=1}^{n-2} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-2}, 1, 0)} \\ &\quad + (-1)^{n-1} \beta_{(i_1, i_2, \dots, i_{n-1}, 1)}. \end{aligned}$$

Namely,

$$\begin{aligned} (\Delta^{n-1} \varphi_1 - D^n \psi_1)[i_1 | i_2 | \dots | i_n] &= \sum_{j=1}^{n-1} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n)} \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n)} \\ &\quad + \sum_{j=2}^{n-1} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n)} \\ &\quad + \sum_{t=3}^n \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_n)} \\ &\quad + \sum_{t=1}^{n-1} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \beta_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n)} \\ &\quad + \sum_{t=1}^{n-1} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \beta_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n)}, \end{aligned}$$

$$\begin{aligned} (\Delta^{n-1} \varphi_1 - D^n \psi_1)[i_1 | i_2 | \dots | i_{n-2} | 1 | 0] &= \sum_{j=1}^{n-2} \frac{(-1)^j i_j i_{j+1}}{i_j + i_{j+1} - 1} \Delta \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2}, 1, 0)} \\ &\quad + \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2}, 1, 0)} \\ &\quad + \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\ &\quad \times \beta_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-2}, 1, 0)} \\ &\quad + (-1)^{n-1} (\Delta + i_1 + \dots + i_{n-2} - n + 1) \beta_{(i_1, i_2, \dots, i_{n-2}, 0)} \\ &\quad + \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\ &\quad \times \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-2}, 1, 0)} \\ &\quad + \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \beta_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-2}, 1, 0)} \end{aligned}$$

$$+ \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \beta_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-2}, 1, 0)},$$

where

- for

$$\beta_{(i_1, \dots, i_{n-2}, 0)} = \begin{cases} \frac{\alpha_{(i_1, \dots, i_{n-2}, 1, 0)}}{\Delta + i_1 + \dots + i_{n-2} - n + 1}; & \Delta + i_1 + \dots + i_{n-2} - n + 1 \neq 0, i_1, \dots, i_{n-2} \geq 2, \\ 0; & \text{otherwise,} \end{cases}$$

then  $\Delta^{n-1}\varphi_1$  is a coboundary in  $\tilde{C}^n$ , and

$$(\Delta^{n-1}\varphi_1 - D^n\psi_1)[v(i_1) \dots v(i_{n-2})v(1)v(0)] = \varphi(i_1, \dots, i_{n-2}, 1, 0);$$

where  $\Delta + i_1 + \dots + i_{n-2} - n + 1 \neq 0$ .

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi_1^{(i)}$  and  $\psi_1^{(i)}$  for  $i = 1, 2, 3, 4$  as following:

- For

$$\beta_{(i_1, \dots, i_{n-3}, 1, 0)} = \begin{cases} \frac{(-1)^n i_1}{2} \alpha_{(i_1 - 1, \dots, i_{n-3}, 2, 1, 0)}; & \Delta + i_1 + \dots + i_{n-3} - n + 3 = 0, i_1 \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we have

$$(\Delta^{n-1}\varphi_1^{(1)} - D^n\psi_1^{(1)})[v(i_1 - 1) \dots v(i_{n-3})v(2)v(1)v(0)] = \varphi(i_1 - 1, \dots, i_{n-3}, 2, 1, 0);$$

where  $\Delta + i_1 + \dots + i_{n-3} - n + 3 = 0$ ,

$$(\Delta^{n-1}\varphi_1^{(1)} - D^n\psi_1^{(1)})[v(i_1) \dots v(i_n)] = 0; \quad \text{otherwise.}$$

- Also if we put

$$\beta_{(i_1, \dots, i_{n-2}, 0)} = \begin{cases} \frac{(-1)^n i_{n-2}}{2i_{n-2} - 4} \alpha_{(i_1, \dots, i_{n-2} - 1, 2, 0)}; & \Delta + i_1 + \dots + i_{n-2} - n + 1 = 0, i_{n-2} \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we get new  $\varphi_1^{(2)}$  and  $\psi_1^{(2)}$  such that

$$(\Delta^{n-1}\varphi_1^{(2)} - D^n\psi_1^{(2)})[v(i_1) \dots v(i_{n-2} - 1)v(2)v(0)] = \varphi(i_1, \dots, i_{n-2} - 1, 2, 0);$$

where  $\Delta + i_1 + \dots + i_{n-2} - n + 1 = 0$ ,

$$(\Delta^{n-1}\varphi_1^{(2)} - D^n\psi_1^{(2)})[v(i_1) \dots v(i_n)] = 0; \quad \text{otherwise.}$$

- For

$$\beta_{(i_1, \dots, i_{n-2}, 1)} = \begin{cases} \frac{(-1)^{n-1} i_{n-2}}{2} \alpha_{(i_1, \dots, i_{n-2} - 1, 2, 1)}; & \Delta + i_1 + \dots + i_{n-2} - n + 2 = 0 \text{ and} \\ & i_{n-2} \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} (\Delta^{n-1}\varphi_1^{(3)} - D^n\psi_1^{(3)})[v(i_1) \dots v(i_{n-2}-1)v(2)v(1)] &= \varphi(i_1, \dots, i_{n-2}-1, 2, 1); \\ \text{where } \Delta + i_1 + \dots + i_{n-2} - n + 2 &= 0, \\ (\Delta^{n-1}\varphi_1^{(3)} - D^n\psi_1^{(3)})[v(i_1) \dots v(i_n)] &= 0; \quad \text{otherwise.} \end{aligned}$$

• Also if

$$\beta_{(i_1, \dots, i_{n-1})} = \begin{cases} \frac{(-1)^{n-1}\alpha_{(i_1, \dots, i_{n-1}, 0)}}{\Delta + i_1 + \dots + i_{n-1} - n}; & \Delta + i_1 + \dots + i_{n-1} - n \neq 0, \\ \frac{(-1)^{n-1}i_{n-1}}{2i_{n-1}-4}\alpha_{(i_1, \dots, i_{n-2}, i_{n-1}-1, 2)}; & \Delta + i_1 + \dots + i_{n-1} - n = 0, i_{n-1} \geq 3, \\ 0; & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} (\Delta^{n-1}\varphi_1^{(4)} - D^n\psi_1^{(4)})[v(i_1) \dots v(i_{n-1})v(1)] &= \varphi(i_1, \dots, i_{n-1}, 1); \\ \text{where } \Delta + i_1 + \dots + i_{n-1} - n &\neq 0, \\ (\Delta^{n-1}\varphi_1^{(4)} - D^n\psi_1^{(4)})[v(i_1) \dots v(i_{n-1}-1)v(2)] &= \varphi(i_1, \dots, i_{n-1}-1, 2); \\ \text{where } \Delta + i_1 + \dots + i_{n-1} - n &= 0 \\ (\Delta^{n-1}\varphi_1^{(4)} - D^n\psi_1^{(4)})[v(i_1) \dots v(i_n)] &= 0; \quad \text{otherwise.} \end{aligned}$$

Hence, every  $n$ -cocycle is a coboundary.  $\square$

**Theorem 4.6.** *For a conformal module  $M_{(0,0)}$  over the associative conformal algebra  $U(2)$  we have*

$$\dim_{\mathbb{k}} H^n(U(3), M_{(0,0)}) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ 1, & n = 3, \\ 0, & n \geq 4. \end{cases}$$

*Proof.* We have proven in Theorem 4.2 that

$$\begin{aligned} (\Delta^1\varphi - D^2\psi)[1|0] &= -\alpha_0, \\ (\Delta^1\varphi - D^2\psi)[n|1] &= -(n-2)\alpha_n, \quad n \geq 2. \end{aligned}$$

Therefore, if  $\Delta^1\varphi - D^1\psi$  is a cocycle in  $C^1$  then  $\alpha_n = 0$  for all  $n \geq 0$  except, maybe, for  $n = 1, 2$ .

Coboundary cocycles in  $\tilde{C}^1$  are given by  $\Delta^0h$ , where  $h \in \text{Hom}_{\Lambda}(\Lambda, M)$ . Modulo  $D^0\tilde{C}^0$ , we may assume  $h(1) = \beta u$ ,  $\beta = -\alpha_1 \in \mathbb{k}$ . Then  $(\Delta^0h)[n] = v(n)\beta u$ . Choose  $\psi \in \tilde{C}^1$  such that  $\psi[0] = \beta u$  and  $\psi[n] = 0$  for  $n \geq 1$ . Then

$$(\Delta^0h - D^1\psi)[n] = \begin{cases} 0, & n = 0, \\ -\beta u, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

and cocycles are determined by  $\alpha_2$ .

We have proven in Theorem 4.4 that:

$$\begin{aligned}
 (\Delta^2\varphi - D^3\psi)(n, m, p) &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1)} \\
 &+ \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p)} \\
 &- \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p)} + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1)} \\
 &- \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1)} + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1)}, \\
 (\Delta^2\varphi - D^3\psi)(n, 1, 0) &= -(n-2)\alpha_{(n,0)}.
 \end{aligned}$$

Hence,  $\varphi - D^3\tilde{C}^3$  is a 2-cocycle in  $C^2$  if and only if

$$\begin{aligned}
 \alpha_{(n,0)} &= 0; \quad n \geq 1; \quad n \neq 2, \\
 -(n+m-3)\alpha_{(n,m)} &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,1)} \\
 &- \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,1)}.
 \end{aligned}$$

Therefore, cocycles in  $C^2$  are determined by  $\alpha_{(n+m-1,1)}$ ,  $\alpha_{(2,1)}$  and  $\alpha_{(2,0)}$  for all  $n, m \geq 2$ .  
Choose

$$\varphi_1 \in \tilde{C}^1, \quad \psi_1 \in \tilde{C}^2 = \text{Hom}_\Lambda(A_2, M),$$

such that

$$\begin{aligned}
 \varphi_1[n] &= \beta_n, \quad \psi_1[1|0] = \beta_1, \quad \psi_1[n|m] = \frac{(n-1)(m-1)}{n+m-1}\beta_{n+m}, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[n|m] &= -\frac{n(n-1)}{n+m-1}\beta_{n+m-1} - \frac{n(n-2)(m-1)}{n+m-2}\beta_{n+m-1} \\
 &- \frac{m(n-1)(m-2)}{n+m-2}\beta_{n+m-1} \\
 (\Delta^1\varphi_1 - D^2\psi_1)[1|0] &= -\beta_0,
 \end{aligned}$$

where

$$\beta_n = -\frac{\alpha_{(n,1)}}{n-2}; \quad n \geq 3.$$

Then  $\Delta^1\varphi_1$  is coboundary in  $\tilde{C}^2$  and

$$\begin{aligned}
 (\Delta^1\varphi_1 - D^2\psi_1)[n|1] &= (-n+2)\beta_n = \alpha_{(n,1)} = \varphi(n, 1); \quad n \geq 3, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[2|1] &= 0, \\
 (\Delta^1\varphi_1 - D^2\psi_1)[2|0] &= 0.
 \end{aligned}$$

Hence,  $\varphi - \Delta^1\varphi_1 \in D^2\tilde{C}^2$ , so every cocycle in  $C^2$  is a coboundary or zero except  $\alpha_{(2,0)}$  and  $\alpha_{(2,1)}$ .

We have proven in Theorem 4.5 that:

$$\begin{aligned} -(n+m-3)\alpha_{(n,m,0)} &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,1,0)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,1,0)} \\ &\quad - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,1,0)}, \\ (n+m+p-4)\alpha_{(n,m,p)} &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,1)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,1)} \\ &\quad + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,1)} \\ &\quad - \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p,1)} + \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1,1)} \\ &\quad - \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1,1)} + \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1,1)} \\ &\quad - \frac{(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,0)} + \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,0)}. \end{aligned}$$

Choose

$$\varphi_1 \in \tilde{C}^2, \quad \psi_1 \in \tilde{C}^3 = \text{Hom}_\Lambda(A_3, M)$$

such that

$$\begin{aligned} (\Delta^2\varphi_1 - D^3\psi_1)(n, m, 1) &= -\frac{n(n-1)}{n+m-1}\beta_{(n+m-1,1)} + (n+m-3)\beta_{(n,m)} \\ &\quad - \frac{n(n-2)(m-1)}{n+m-2}\beta_{(n+m-1,1)} - \frac{m(n-1)(m-2)}{n+m-2}\beta_{(n+m-1,1)} \\ &\quad - \frac{(n-1)(m-1)}{n+m-1}\beta_{(n+m,0)}, \\ (\Delta^2\varphi_1 - D^3\psi_1)(n, 1, 0) &= -(n-2)\beta_{(n,0)}, \end{aligned}$$

where

$$\beta_{(n,0)} = \begin{cases} -\frac{\alpha_{(n,1,0)}}{n-2}; & n \geq 3, \\ 0; & \text{otherwise.} \end{cases}$$

Then

$$(\Delta^2\varphi_1 - D^3\psi_1)(n, 1, 0) = \alpha_{(n,1,0)} = \varphi_{(n,1,0)}; \quad n \neq 2.$$

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi'_1$  and  $\psi'_1$ :

$$\beta_{(n,m)} = \begin{cases} \frac{\alpha_{(n,m,1)}}{n+m-3}; & n, m \geq 2, \\ 0; & \text{otherwise,} \end{cases}$$



$$\begin{aligned}(\Delta^2\varphi'_1 - D^3\psi'_1)(n, 1, 0) &= 0; \quad n \neq 2, \\(\Delta^2\varphi'_1 - D^3\psi'_1)(n, m, 1) &= \alpha_{(n,m,1)} = \varphi_{(n,m,1)}.\end{aligned}$$

Hence,  $\varphi - \Delta^2\varphi_1 \in D^3\tilde{C}^3$ , so every cocycle in  $C^3$  is a coboundary or zero except  $\alpha_{(2,1,0)}$ . We have proven in Theorem 4.5 that: for all  $n, m, p, q \geq 2$

$$\begin{aligned}-(n+m+p+q-5)\alpha_{(n,m,p,q)} &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,q,1)} - \frac{(p-1)(q-1)}{p+q-1}\alpha_{(n,m,p+q,0)} \\&+ \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,q,1)} - \frac{p(p-1)}{p+q-1}\alpha_{(n,m,p+q-1,1)} \\&+ \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,q,1)} - \frac{(n-1)pq}{p+q-1}\alpha_{(n,m,p+q-1,q,1)} \\&- \frac{(m-1)pq}{p+q-1}\alpha_{(n,m,p+q-1,q,1)} - \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,q,1)} \\&- \frac{m(n-1)(m-2)}{n+m-2}\alpha_{(n+m-1,p,q,1)} \\&+ \frac{m(m-2)(p-1)}{m+p-2}\alpha_{(n,m+p-1,q,1)} \\&- \frac{p(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p-1,q,1)} \\&+ \frac{p(m-1)(p-2)}{m+p-2}\alpha_{(n,m+p-1,q,1)} \\&- \frac{p(p-2)(q-1)}{p+q-2}\alpha_{(n,m,p+q-1,1)} \\&- \frac{q(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,q-1,1)} \\&+ \frac{q(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,q-1,1)} \\&- \frac{q(p-1)(q-2)}{p+q-2}\alpha_{(n,m,p+q-1,1)} \\&- \frac{r(n-1)(m-1)}{n+m-1}\alpha_{(n+m,p,q,0)} \\&+ \frac{(m-1)(p-1)}{m+p-1}\alpha_{(n,m+p,q,0)},\end{aligned}$$

$$\begin{aligned}(n+m+p-4)\alpha_{(n,m,p,0)} &= -\frac{n(n-1)}{n+m-1}\alpha_{(n+m-1,p,1,0)} + \frac{m(m-1)}{m+p-1}\alpha_{(n,m+p-1,1,0)} \\&- \frac{n(n-2)(m-1)}{n+m-2}\alpha_{(n+m-1,p,1,0)} + \frac{(n-1)mp}{m+p-1}\alpha_{(n,m+p-1,1,0)}\end{aligned}$$

$$\begin{aligned}
 & - \frac{m(n-1)(m-2)}{n+m-2} \alpha_{(n+m-1,p,1,0)} + \frac{m(m-2)(p-1)}{m+p-2} \alpha_{(n,m+p-1,1,0)} \\
 & - \frac{p(n-1)(m-1)}{n+m-1} \alpha_{(n+m,p-1,1,0)} + \frac{p(m-1)(p-2)}{m+p-2} \alpha_{(n,m+p-1,1,0)}.
 \end{aligned}$$

Let  $\varphi_1 \in \tilde{C}^3$  and  $\psi_1 \in \tilde{C}^4$  such that

$$\begin{aligned}
 \varphi_1[v(n)v(m)v(p)] &= \beta_{(n,m,p)}; \quad n \geq 2, m \geq 1, p \geq 0, \\
 \psi_1[v(n)v(m)v(p)v(q)] &= \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m,p,q)} \\
 & - \frac{(m-1)(p-1)}{m+p-1} \beta_{(n,m+p,q)} \\
 & + \frac{(p-1)(q-1)}{p+q-1} \beta_{(n,m,p+q)}; \quad n, m, p \geq 2, q \geq 0, \\
 \psi_1[v(n)v(m)v(1)v(0)] &= \frac{(n-1)(m-1)}{n+m-1} \beta_{(n+m,1,0)} - \beta_{(n,m,1)}; \quad n, m \geq 2.
 \end{aligned}$$

where

$$\beta_{(n,m,0)} = \begin{cases} \frac{\alpha_{(n,m,1,0)}}{n+m-3}, & n, m \geq 2, \\ 0; & \text{otherwise.} \end{cases}$$

Then

$$(\Delta^3 \varphi_1 - D^4 \psi_1)(n, m, 1, 0) = \alpha_{(n,m,1,0)} = \varphi_{(n,m,1,0)}; \quad n, m \geq 2.$$

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi'_1$  and  $\psi'_1$ :

$$\beta_{(n,m,p)} = \begin{cases} -\frac{\alpha_{(n,m,p,1)}}{n+m+p-4}; & n, m, p \geq 2, \\ 0; & \text{otherwise,} \end{cases}$$

$$(\Delta^3 \varphi'_1 - D^4 \psi'_1)(n, m, 1, 0) = 0; \quad n, m \geq 2,$$

$$(\Delta^3 \varphi'_1 - D^4 \psi'_1)(n, m, p, 1) = \alpha_{(n,m,p,1)} = \varphi_{(n,m,p,1)}.$$

Hence  $\varphi - \Delta^3 \varphi_1 \in D^4 \tilde{C}^4$ , so every cocycle in  $C^4$  is a coboundary.

We have proven in Theorem 4.5 that: for all  $i_1, \dots, i_n \geq 2, i_{n+1} \geq 0$ ,

$$\begin{aligned}
 (-1)^{n+1} (i_1 + \dots + i_n - n - 1) \alpha_{(i_1, i_2, \dots, i_n)} &= \sum_{j=1}^{n-1} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 1)} \\
 &+ \sum_{j=2}^n \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\
 &\times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_n, 1)}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=3}^{n+1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_n, 1)} \\
 & + \sum_{t=1}^n (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n, 1)} \\
 & + \sum_{t=1}^n (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_n, 1)},
 \end{aligned}$$

$$\begin{aligned}
 (-1)^n (i_1 + \dots + i_{n-1} - n) \alpha_{(i_1, i_2, \dots, i_{n-1}, 0)} & = \sum_{j=1}^{n-2} (-1)^j \frac{i_j (i_j - 1)}{i_j + i_{j+1} - 1} \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{j=1}^{n-2} \sum_{t=1}^{j-1} (-1)^j \frac{i_j i_{j+1}}{i_j + i_{j+1} - 1} (i_t - 1) \\
 & \times \alpha_{(i_1, \dots, i_j + i_{j+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=3}^{n-1} \sum_{j=1}^{t-2} (-1)^j i_t \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \\
 & \times \alpha_{(i_1, \dots, i_j + i_{j+1}, \dots, i_t - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_t \frac{(i_t - 2)(i_{t+1} - 1)}{i_t + i_{t+1} - 2} \\
 & \times \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)} \\
 & + \sum_{t=1}^{n-2} (-1)^t i_{t+1} \frac{(i_t - 1)(i_{t+1} - 2)}{i_t + i_{t+1} - 2} \\
 & \times \alpha_{(i_1, \dots, i_t + i_{t+1} - 1, \dots, i_{n-1}, 1, 0)}.
 \end{aligned}$$

Let  $\varphi_1 \in \tilde{C}^{n-1}$ ,  $\psi_1 \in \tilde{C}^n$ , such that

$$\varphi_1(i_1, i_2, \dots, i_{n-1}) = \beta_{(i_1, i_2, \dots, i_{n-1})},$$

$$\begin{aligned}
 \psi_1(i_1, i_2, \dots, i_n) & = \sum_{j=1}^{n-1} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_n)} \\
 \psi_1(i_1, i_2, \dots, i_{n-2}, 1, 0) & = \sum_{j=1}^{n-2} (-1)^j \frac{(i_j - 1)(i_{j+1} - 1)}{i_j + i_{j+1} - 1} \beta_{(i_1, \dots, i_j + i_{j+1}, \dots, i_{n-2}, 1, 0)} \\
 & + (-1)^{n-1} \beta_{(i_1, i_2, \dots, i_{n-1}, 1)},
 \end{aligned}$$

where

$$\beta_{(i_1, i_2, \dots, i_{n-2}, 0)} = \begin{cases} \frac{(-1)^n \alpha_{(i_1, i_2, \dots, i_{n-2}, 1, 0)}}{i_1 + \dots + i_{n-2} - n + 1}; & i_1, \dots, i_{n-2} \geq 2, \\ 0; & \text{otherwise.} \end{cases}$$

Then

$$(\Delta^{n-1}\varphi_1 - D^n\psi_1)(i_1, i_2, \dots, i_{n-2}, 1, 0) = \varphi(i_1, i_2, \dots, i_{n-2}, 1, 0); \quad i_1, \dots, i_{n-2} \geq 2.$$

Let us repeat the construction with new values of  $\beta$ 's to get  $\varphi'_1$  and  $\psi'_1$ :

$$\beta_{(i_1, \dots, i_{n-1})} = \begin{cases} -\frac{(-1)^n \alpha_{(i_1, \dots, i_{n-1}, 1)}}{i_1 + \dots + i_{n-1} - n}; & i_1, \dots, i_{n-1} \geq 2, \\ 0; & \text{otherwise.} \end{cases}$$

$$\begin{aligned} (\Delta^{n-1}\varphi'_1 - D^n\psi'_1)(i_1, i_2, \dots, i_{n-2}, 1, 0) &= 0; \quad i_1, \dots, i_{n-2} \geq 2, \\ (\Delta^{n-1}\varphi'_1 - D^n\psi'_1)(i_1, i_2, \dots, i_{n-2}, i_{n-1}, 1) &= \varphi(i_1, i_2, \dots, i_{n-2}, i_{n-1}, 1); \quad i_1, \dots, i_{n-1} \geq 2. \end{aligned}$$

Hence  $\varphi - \Delta^{n-1}\varphi_1 \in D^n\tilde{C}^n$ , so every cocycle in  $C^n$  is a coboundary.  $\square$

**Corollary 4.7.** *Let  $M$  be a finite module over  $U(3)$ . Then  $H^k(U(3), M) = 0$  for all  $k \geq 4$ .*

*Proof.* Let  $M$  be a finite conformal module over  $U(3)$ . Then, in particular,  $M$  is a finite module over the Virasoro Lie conformal algebra  $\text{Vir}$ . Hence there exists a chain of  $\text{Vir}$ -submodules (see, e.g., [10, Lemma 3.3])

$$0 = M_{-1} \subset M_0 \subset \dots \subset M_n = M,$$

where  $M_i/M_{i-1}$ ,  $i = 0, \dots, n$ , is either isomorphic to a  $\text{Vir}$ -module  $M_{(\alpha, \Delta)}$ , or trivial torsion-free module  $\mathbb{k}[\partial]u$  with  $(v \text{ }_{(\lambda)} u) = 0$ , or coincides with its torsion (hence, trivial). Note that a  $\text{Vir}$ -submodule of an  $U(3)$ -module  $M$  is itself a  $U(3)$ -module, therefore, all  $M_i$  are  $U(3)$ -modules and so are  $M_i/M_{i-1}$ . Hence, all irreducible quotients are of type  $M_{(\alpha, \Delta)}$ .

The case of torsion module was considered in [2]. There is no difference between the scalar  $U(3)$ -module  $\mathbb{k}$  and the trivial torsion-free module  $M = \mathbb{k}[\partial]u$ . One may also apply the technique of Theorem 4.5 to the case of the trivial module  $M$  as above to prove that  $H^k(U(3), M) = 0$  for all  $k \geq 4$ . Finally, both Theorem 4.3 and Theorem 4.5 imply that

$$H^k(U(3), M_i/M_{i-1}) = 0, \quad i = 0, \dots, n,$$

for all  $k \geq 4$ . The short exact sequence of modules

$$0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_i/M_{i-1} \rightarrow 0$$

leads to the long exact sequence of cohomology groups

$$\begin{aligned} \dots \rightarrow H^k(U(3), M_{i-1}) &\rightarrow H^k(U(3), M_i) \rightarrow H^k(U(3), M_i/M_{i-1}) \\ &\rightarrow H^{k+1}(U(3), M_{i-1}) \rightarrow H^{k+1}(U(3), M_i) \rightarrow H^{k+1}(U(3), M_i/M_{i-1}) \\ &\rightarrow \dots \end{aligned}$$

for every  $i = 1, \dots, n$ . Since

$$H^k(U(3), M_0) = 0, \quad H^k(U(3), M_1/M_0) = 0$$

for all  $k \geq 4$ , we obtain  $H^k(U(3), M_1) = 0$ . Proceed by induction on  $i = 1, \dots, n$  to obtain  $H^k(U(3), M_n) = 0$ , for all  $k \geq 4$ .  $\square$

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