Communications in Mathematics **33** (2025), no. 1, Paper no. 7
DOI: https://doi.org/10.46298/cm.14455

On the shape of the connected components of the complement of two-dimensional Brownian random interlacements

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Abstract. We study the limiting shape of the connected components of the vacant set of two-dimensional Brownian random interlacements: we prove that the connected component around x is close in distribution to a rescaled *Brownian amoeba* in the regime when the distance from $x \in \mathbb{C}$ to the closest trajectory is small (which, in particular, includes the cases $x \to \infty$ with fixed intensity parameter α, and $\alpha \to \infty$ with fixed x). We also obtain a new family of martingales built on the conditioned Brownian motion, which may be of independent interest.

Contents

MSC 2020: 60K35; 60J65
Keywords: conditioned trajectories, Doob's h-transform, Brownian amoeba.
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1 Introduction

In this paper, we study the model of Brownian random interlacements in two dimensions. The discrete random interlacements in higher dimensions are "canonical Poissonian soups" of doubly infinite simple random walk's trajectories (or "loops passing through infinity"); they were introduced by Sznitman in [15] and then studied quite extensively (see, e.g. [5]). That original construction cannot be used in low dimensions where a simple random walk's trajectory is recurrent so that even a single trajectory will a.s. fill the whole space; one can, however, construct a similar object in a natural way using the *conditioned* (on avoiding the origin) trajectories. This was done in [4] in two dimensions and in [1] in one dimension. In all dimensions, random interlacements are related to (simple) random walks on tori: the traces left by the random walk up to a certain time on a box (with a size much smaller than that of the torus) can be well approximated by the random interlacements restricted to that box (in dimensions 1 and 2 one also needs to condition the center of the box to be not visited by the walk).

It is also natural to consider a continuous space/time counterpart of the above-mentioned models: namely, the "interlacement soup" made of Brownian trajectories (which, in the case of lower dimensions, also needs some conditioning). This was done in [16] for the original random interlacement model and in [3] for the two-dimensional case.

One of the main objects of interest in the context of random interlacements is *the vacant set*, which is the complement of the union of the trajectories that make up the interlacements. This article is part of a program focused on studying the geometric properties of the vacant set of the two-dimensional random interlacements (discrete and continuous), as well as the vacant set of a single-conditioned trajectory. Note that in the two-dimensional case, the vacant set of Brownian interlacements only has bounded connected components; the paper [3] contains some results mainly related to the (linear) size of these components. In particular, it was shown there (see also [2]) that the vacant set contains an infinite number of nonoverlapping disks of constant radius whenever $\alpha \leq 1$, where α is the intensity parameter of the model. In addition, the radius of the largest disk around x, which is fully contained in the vacant set is $\exp(-Z\alpha \ln^2 |x|)$, where Z is a random variable with some (explicit) distribution. In Theorem 2.4 below, we extend that last result by studying not only the distance to the closest trajectory but also to the second closest one, and so on, but the main focus of this paper is to study the geometric shapes of the connected components of the vacant set. We mention that the connected components of the complement of *one* The Brownian trajectory was studied in a number of papers, notably (in chronological order) [6, 8, 12, 17]; in particular, the limiting shape was the main subject of [17]. It is interesting to note that, despite the fact that we are dealing with many trajectories here, in some regimes it is still typically only one trajectory that defines the shape of a given connected component, cf. Theorem 2.6 below. Then, in Theorem 2.7 we also consider a situation when the shape of the connected component is typically determined by many trajectories: in the limit $\alpha \to \infty$, we prove some geometric properties of the "central cell" (that is, the connected component which contains the unit disk centered at the origin, which the trajectories are conditioned not to touch).

Another contribution of this paper we have to mention is the following: in Section 3.1, we present a family of functions which, when applied to the conditioned Brownian motion, result in (local) martingales (see Proposition 3.4 below). That can be seen as a continuous counterpart of Proposition 2.4 of [13] or Proposition 4.10 of [14], where the corresponding families for the simple conditioned two-dimensional random walk were discussed. It is generally very useful to be able to construct such martingales since it allows, e.g., estimating hitting probabilities of various sets via the optional stopping theorem; therefore, we hope that Proposition 3.4 will find its further applications.

2 Formal definitions and results

In the following, we will identify $\mathbb{R}^2$ and $\mathbb{C}$ via $x = (x_1, x_2) = x_1 + ix_2$, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^2$ as well as the modulus in $\mathbb{C}$. Also, let $\mathsf{B}(x,r) = \{y : |x-y| \leq r\}$ be the closed disk of radius r centered in x, and abbreviate $\mathsf{B}(r) := \mathsf{B}(0,r)$.

Let W be the standard two-dimensional Brownian motion. The main ingredient of Brownian random interlacement is the Brownian motion is conditioned on not hitting the unit disk $\mathsf{B}(1)$, which will be denoted by $\widehat{W}$; formally, it is the Doob's h-transform of W with respect to $h(x) = \ln|x|$. The process $\widehat{W}$ can be formally defined via its transition kernel $\hat{p}$: for $|x| > 1, |y| \geq 1$,

$$\hat{p}(t,x,y) = p_0(t,x,y)\frac{\ln|y|}{\ln|x|}. \tag{1}$$

where p_0 denotes the transition subprobability density of W killed on hitting the unit disk $\mathsf{B}(1)$. It is possible to show (see [3]) that the diffusion $\widehat{W}$ obeys the stochastic differential equation

$$d\widehat{W}_t = \frac{\widehat{W}_t}{|\widehat{W}_t|^2 \ln|\widehat{W}_t|}dt + dW_t. \tag{2}$$

Sometimes, it can be useful to work with an alternative definition of the diffusion $\widehat{W}$ using polar coordinates $\widehat{W}_t = (\mathcal{R}_t \cos\Theta_t, \mathcal{R}_t \sin\Theta_t)$. With $W^{(1,2)}$ two independent standard one-dimensional Brownian motions, let us consider the stochastic differential equations

$$d\mathcal{R}_t = \Big(\frac{1}{\mathcal{R}_t \ln \mathcal{R}_t} + \frac{1}{2\mathcal{R}_t}\Big)dt + dW_t^{(1)}, \tag{3}$$

$$d\Theta_t = \frac{1}{\mathcal{R}_t}dW_t^{(2)} \tag{4}$$

(note that the diffusion Θ takes values in the whole $\mathbb{R}$, so we are considering a Brownian motion on the Riemann surface); it is an elementary exercise in stochastic calculus to show that (2) is equivalent to (3)–(4). We also note that, as shown in [3], even though the radial drift component in the above stochastic differential equations is not well defined $\partial\mathsf{B}(1)$, it is still possible to define the process $\widehat{W}$ starting at $\partial\mathsf{B}(1)$ and staying outside the unit disk for all $t > 0$.

Let us define

$$\tau(x,r) = \inf\big\{t > 0 : W_t \in \partial\mathsf{B}(x,r)\big\},$$
$$\widehat{\tau}(x,r) = \inf\big\{t > 0 : \widehat{W}_t \in \partial\mathsf{B}(x,r)\big\}$$

to be the hitting times of the boundary of the disk $\mathsf{B}(x,r)$ with respect to the (two-dimensional) Brownian motion and its conditioned version; we abbreviate $\tau(r) := \tau(0,r)$ and $\widehat{\tau}(r) := \widehat{\tau}(0,r)$.

Recall the definition of *Wiener moustache* (Definition 2.4 of [3]):

Definition 2.1. *Let U be a random variable with a uniform distribution in $[0,2\pi]$, and let $(\mathcal{R}^{(1,2)},\Theta^{(1,2)})$ be two independent copies of the processes defined by (3)–(4), with a common initial point $(1,U)$. Then, the* Wiener moustache *η is defined as the union of ranges of the two trajectories, that is,*

$$\eta = \big\{re^{i\theta} : \text{ there exist } k \in \{1,2\}, t \geq 0 \text{ such that } \mathcal{R}^{(k)}_t = r, \Theta^{(k)}_t = \theta\big\}.$$

We also need to recall the definition of capacity for subsets of $\mathbb{R}^2$. Let A be a compact subset of $\mathbb{R}^2$ such that $\mathsf{B}(1) \subset A$. Denote by hm_A the *harmonic measure* (from infinity) on A, that is, the entrance law in A for the Brownian motion starting from infinity (cf. e.g. Theorem 3.46 of [11]). We define the (Brownian) capacity of A as

$$\mathrm{cap}(A) = \frac{2}{\pi}\int_A \ln|y|\, d\,\mathrm{hm}_A(y). \tag{5}$$

Also, for *any* compact subset A of $\mathbb{R}^2$, we define $\widehat{\mathrm{cap}}(A) := \mathrm{cap}\big(\mathsf{B}(1) \cup A\big)$.

Now, we recall the definition of two-dimensional Brownian random interlacements [3]:

Definition 2.2. *Let $\alpha > 0$ and consider a Poisson point process $(\rho^\alpha_k, k \in \mathbb{Z})$ on $\mathbb{R}_+$ with intensity $r(\rho) = \frac{2\alpha}{\rho}, \rho \in \mathbb{R}_+$. Let $(\eta_k, k \in \mathbb{Z})$ be a sequence of i.i.d. Wiener moustaches. Fix $b \geq 0$. Then, the model of Brownian Random Interlacements (BRI) on level α truncated at b is defined in the following way:*

$$\mathrm{BRI}(\alpha; b) = \bigcup_{k:\rho^\alpha_k \geq b} \rho^\alpha_k \eta_k\,. \tag{6}$$

We also define the vacant set *$\mathcal{V}^{\alpha;b}$: it is the set of points of the plane that do not belong to trajectories of* $\mathrm{BRI}(\alpha; b)$

$$\mathcal{V}^{\alpha;b} = \mathbb{R}^2 \setminus \mathrm{BRI}(\alpha; b).$$

Let us abbreviate $\mathrm{BRI}(\alpha) := \mathrm{BRI}(\alpha; 1)$ and $\mathcal{V}^\alpha := \mathcal{V}^{\alpha;1}$. It is a characteristic property of Brownian random interlacements that (recall Proposition 2.11 of [3])

$$\mathbb{P}\big[A \cap \mathrm{BRI}(\alpha) = \emptyset\big] = \exp\big(-\pi\alpha\,\widehat{\mathrm{cap}}(A)\big). \tag{7}$$

An important observation is that the above Poisson process is the image of a homogeneous Poisson process of rate 1 in $\mathbb{R}$ under the map $x \mapsto e^{x/2\alpha}$; this follows from the mapping theorem for Poisson processes (see e.g. Section 2.3 of [7]). Because of that, we may write

$$\rho_k^\alpha = \exp\Big(\frac{Y_1 + \dots + Y_k}{2\alpha}\Big), \tag{8}$$

where $Y_1, \dots, Y_k$ are i.i.d. Exponential(1) random variables. Also, as mentioned in Remark 2.7 of [3], one can actually construct BRI(α) for all $\alpha > 0$ simultaneously, in such a way that BRI(α_1) *dominates* BRI(α_2) for $\alpha_1 > \alpha_2$: for this, one can consider a Poisson process of rate 1 in $\mathbb{R}_+^2$ (with coordinates (ρ, u)) and then take those points that lie below the curve $u = \frac{2\alpha}{\rho}$ when constructing BRI(α).

It is clear that the above construction is not invariant with respect to translations of $\mathbb{R}^2$. Let us also mention an equivalent construction which, in some sense, recovers the translation invariance property (the random interlacement is obtained as an image of an object which is "more translationally invariant"). For that, let us first observe that the following fact holds (recall that the Bessel process of dimension 3, also denoted here as Bes(3), is the norm of the three-dimensional standard Brownian motion):

Proposition 2.3. *Let $\widehat{W}$ be the conditioned Brownian motion started somewhere outside* $\mathsf{B}(1)$ *(or on its boundary). Then, there exists a pair of independent processes (Z, B), where Z is Bes(3) and B is a Brownian motion such that*

$$\widehat{W}_t = \exp(Z_{G_t} + iB_{G_t}), \text{ where } G_t = \int_0^t \frac{ds}{|\widehat{W}_s|^2}. \tag{9}$$

Proof. This follows from the skew-product representation of the Brownian motion, cf. e.g. Theorem 7.19 of [9], together with the well-known fact that Bes(3) is the (one-dimensional) Brownian motion conditioned on never hitting the origin (cf. [10, 18]). □

Now, let $\mathbb{S}^1_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle of radius 1; then, one can naturally define the exponential map from $\mathbb{R} \times \mathbb{S}^1_{2\pi}$ to $\mathbb{C}$ by $\exp(r, \theta) = \exp(r + i\theta)$. For $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1_{2\pi}$, we call a Bessel moustache (attached to that point) a pair of independent trajectories $(r + Z_t^{(1,2)}, \theta + B_t^{(1,2)}, t \geq 0)$, where $Z^{(k)}$ is a Bes(3) process (taking values in $\mathbb{R}_+$ and starting at 0) and $B^{(k)}$ is an independent Brownian motion on $\mathbb{S}^1_{2\pi}$ starting at 0.

Then, due to (8) and Proposition 2.3, we can define BRI(α) in the following way[1] (see Figure 1):

- take a Poisson point process of rate $2\alpha \times \frac{1}{2\pi}$ on $\mathbb{R}_+ \times \mathbb{S}^1_{2\pi}$ (i.e., take a one-dimensional Poisson process with rate 2α and rotate the points randomly on $\mathbb{S}^1_{2\pi}$);
- attach Bessel moustaches to these points independently;
- transfer that picture to $\mathbb{C}$ using the exponential map.

[1] In fact, we will obtain the same model if the interlacement trajectories are viewed as subsets of $\mathbb{C}$; note the time change in (9).

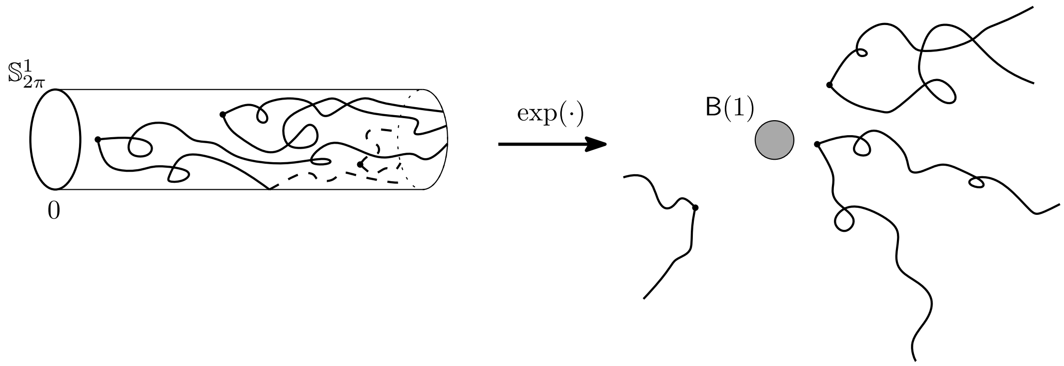

Figure 1: On the equivalent definition of BRI(α).

We also remark that in a similar way, one can obtain the process BRI($\alpha; b$) by taking a Poisson process on $[\ln b, +\infty) \times \mathbb{S}^1_{2\pi}$ in the above construction.

Now, we are almost ready to state our results, but we still need to recall a couple of notations from [3]: for $x \notin \mathsf{B}(1)$,

$$\ell_x = \int_{\partial\mathsf{B}(1)} \ln|x-z|\, H(x,dz) = \frac{|x|^2-1}{2\pi}\int_{\partial\mathsf{B}(1)} \frac{\ln|x-z|}{|x-z|^2}\,dz,$$

where $H(x,\cdot)$ is the entrance measure (with respect to the Brownian motion) to $\mathsf{B}(1)$ starting from x (see (24) below). As argued in (3.9)–(3.10) of [3],

$$\ell_x = \big(1+O(|x|^{-1})\big)\ln|x| \text{ as } |x|\to\infty \text{ and } \ell_x = \ln(|x|-1)+O(1) \text{ as } |x|\downarrow 1. \tag{10}$$

Let us also define (for the specific purpose of being inside O's, given that $\ell_\cdot$ changes sign and can be equal to 0) $\tilde{\ell}_x := |\ell_x| \vee 1$.

We recall another notation from [3]: for $x\in\mathbb{R}^2$, $\Phi_x(\alpha)$ denotes the distance from x to the closest trajectory of BRI(α). We now extend this by defining $\Phi_x^{(0)}(\alpha) := 0$, $\Phi_x^{(1)}(\alpha) := \Phi_x(\alpha)$, $\Phi_x^{(2)}(\alpha)$ to be the distance to the second closest trajectory, $\Phi_x^{(3)}(\alpha)$ to be the distance to the third closest trajectory, and so on; note that a.s. it holds that $0 < \Phi_x^{(1)}(\alpha) < \Phi_x^{(2)}(\alpha) < \Phi_x^{(3)}(\alpha) < \dots$. Let us denote also

$$\widetilde{Y}_x^{(1)}(\alpha) = \frac{2\alpha\ln^2|x|}{\ln(\Phi_x^{(1)}(\alpha)^{-1})}, \quad \widetilde{Y}_x^{(j)}(\alpha) = \frac{2\alpha\ln^2|x|}{\ln(\Phi_x^{(j)}(\alpha)^{-1})} - \frac{2\alpha\ln^2|x|}{\ln(\Phi_x^{(j-1)}(\alpha)^{-1})}, \quad j\geq 2.$$

The following result states that $\widetilde{Y}_x^{(m)}(\alpha)$ are approximately Exponential(1) for $m\geq 1$, and also that $\widetilde{Y}_x^{(j+1)}(\alpha)$ is approximately independent of $\widetilde{Y}_x^{(1)}(\alpha),\dots,\widetilde{Y}_x^{(j)}(\alpha)$:

Theorem 2.4. *Assume that $x\notin\mathsf{B}(1)$, and let $b_1,\dots,b_j>0$. Then*

$$\mathbb{P}\big[\widetilde{Y}_x^{(1)}(\alpha) > s\big] = e^{-s}\big(1 + s\times O\big(\Psi_1^{(s)}+\Psi_2^{(s)}+\Psi_3^{(s)}\big)\big), \tag{11}$$

and, for $j \geq 1$,

$$\begin{aligned}&\mathbb{P}\big[\widetilde{Y}_x^{(j+1)}(\alpha) > s \mid \widetilde{Y}_x^{(1)}(\alpha) = b_1, \ldots, \widetilde{Y}_x^{(j)}(\alpha) = b_j\big]\\ &= e^{-s}\big(1 + s(b+s) \times O\big(\Psi_1^{(b+s)} + \Psi_2^{(b)} + \Psi_2^{(b+s)} + \Psi_3^{(b)} + \Psi_3^{(b+s)}\big)\big),\end{aligned} \tag{12}$$

where $b := b_1 + \cdots + b_j$*, and*

$$\begin{aligned}\Psi_1^{(h)} &= \frac{\tilde{\ell}_x h}{\alpha \ln^2 |x|},\\ \Psi_2^{(h)} &= \frac{1 + \alpha h^{-1} \ln^2 |x| + \ln |x|}{\exp(2h^{-1}\alpha \ln^2 |x|)|x| \ln |x|},\\ \Psi_3^{(h)} &= \frac{1 + \Psi_1^{(h)}}{\exp(2h^{-1}\alpha \ln^2 |x|)(|x| - 1)}.\end{aligned}$$

The above result can be interpreted informally in the following way:

$$\Big(\frac{2\alpha \ln^2 |x|}{\ln(\Phi_x^{(1)}(\alpha)^{-1})}, \frac{2\alpha \ln^2 |x|}{\ln(\Phi_x^{(2)}(\alpha)^{-1})}, \frac{2\alpha \ln^2 |x|}{\ln(\Phi_x^{(3)}(\alpha)^{-1})}, \ldots\Big)$$

is approximately a Poisson process of rate 1 in $\mathbb{R}_+$, as long as the error term in (12) is small. (Note that the quantity inside $O(\ldots)$ in (12) does not depend on j.) Let us also observe that the error term in (12) is $O(\alpha^{-1})$ when x is fixed and $\alpha \to \infty$, and is $O(\frac{1}{\ln |x|})$ when α is fixed and $|x| \to \infty$.

We need to define another important object, which is derived from the Wiener moustache.

Definition 2.5. *Let* η *be a Wiener moustache. The* Brownian amoeba $\mathfrak{A}$ *is the connected component of the origin in the complement of* $\eta \subset \mathbb{R}^2$*, see Figure 2.*

Observe that the Brownian amoeba a.s. contains $\mathsf{B}(1)$ (except for the point where the Wiener moustache touches the unit disk). To the best of our knowledge, this object first appeared in [17] (it is the one with the distribution $\mathcal{L}_1$ there) as the limiting shape of the connected components of the complement of *one* Brownian trajectory. A remarkable property of the Brownian amoeba is that if we move the origin to a uniformly randomly chosen (with respect to the area) point in $\mathfrak{A}$ and rescale it properly (so that it touches the boundary of the unit disk centered at the new origin), then the resulting object has the same law; this is Proposition 22 of [17].

Next, we formulate the main result of this paper, which says that in certain regimes, the connected components of the vacant set converge in distribution to the Brownian amoeba. Analogously to Definition 2.5, for $x \in \mathbb{R}^2$, define

$$\mathfrak{C}_x(\alpha) = \text{the connected component of } x \text{ in } \mathbb{R}^2 \setminus \mathrm{BRI}(\alpha),$$

with the convention that $\mathfrak{C}_x(\alpha) = \emptyset$ if $x \in \mathrm{BRI}(\alpha)$. In the next result, we show that $\mathfrak{C}_0(\alpha)/\Phi_0(\alpha)$ converges to $\mathfrak{A}$ in total variation distance as $\alpha \to 0$, and also $\mathfrak{C}_x(\alpha)/\Phi_x(\alpha)$

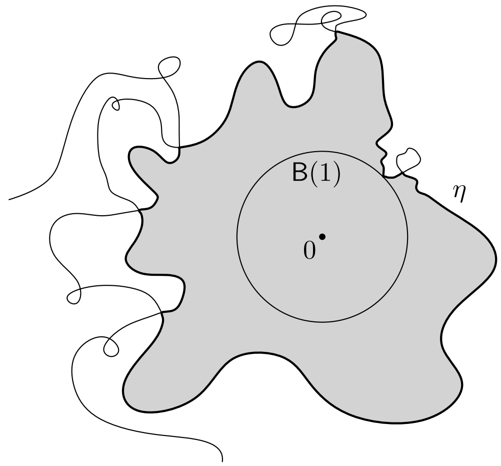

Figure 2: The Brownian amoeba $\mathfrak{A}$

converges to $\mathfrak{A}$ in total variation distance under certain conditions (which we discuss in more detail below). In the following, $dist_{TV}(X,Y)$ denotes the total variation distance between the laws of random objects X and Y (recall that this total variation distance is defined as $\inf \mathbb{P}[X \neq Y]$, where the infimum is taken over all couplings of X and Y).

Theorem 2.6. *(i) We have for a positive constant* c_1

$$dist_{TV}\Big(\frac{\mathfrak{C}_0(\alpha)}{\Phi_0(\alpha)}, \mathfrak{A}\Big) \leq c_1\alpha. \tag{13}$$

(ii) Assume that $|x| > 1$ *and* $\alpha \ln^2 |x| \geq 2$. *Then, for some,* $c_2 > 0$ *it holds that*

$$dist_{TV}\Big(\frac{\mathfrak{C}_x(\alpha) - x}{\Phi_x(\alpha)}, \mathfrak{A}\Big) \leq c_2 \frac{\tilde{\ell}_x \ln^3(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}. \tag{14}$$

We are not sure how sharp is the estimate (14). In any case, note that for the term in the right-hand side of (14) to be small, we always need $\alpha \ln^2 |x|$ to be large, but that might not be enough for convergence to $\mathfrak{A}$ (since there is also the factor $\tilde{\ell}_x$, which grows to infinity as $|x| \to \infty$ or $|x| \downarrow 1$). To give a few examples, we observe that the error term in (14) is

- $O\big(\frac{\ln^3 \alpha}{\alpha}\big)$ when $x \notin \mathsf{B}(1)$ is fixed and $\alpha \to \infty$;
- $O\big(\frac{(\ln\ln|x|)^3}{\ln|x|}\big)$ when $\alpha > 0$ is fixed and $|x| \to \infty$;
- $O\big(\frac{|\ln v| \ln^3(\alpha v^2)}{\alpha v^2}\big)$ when $v := |x| - 1 \downarrow 0$ (in this case, we need α to be "a bit larger" than $(1/v)^2$, i.e., $\alpha = (1/v)^{2+\varepsilon}$ would work).

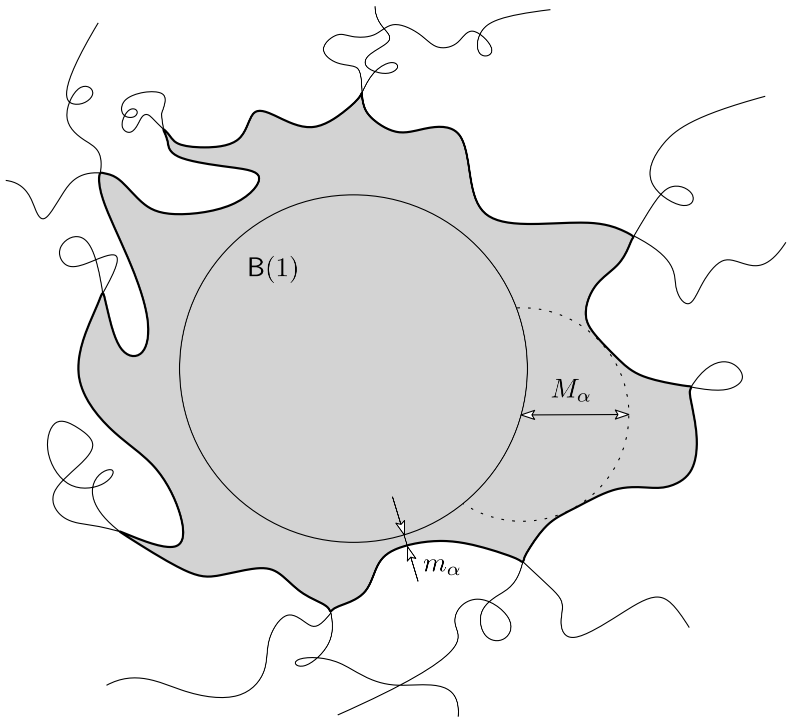

Figure 3: The central cell (in this case, formed by five bi-infinite trajectories).

In any case, it is interesting to observe the similarity between what is seen from the origin in the $\alpha \to 0$ regime and what is seen from $x \notin \mathsf{B}(1)$ the "high intensity" regime: only one trajectory (the one that forms the amoeba) really "matters"; others are much more distant.

We also discuss the "central cell" $\mathfrak{C}_0(\alpha)$ in the regime when $\alpha \to \infty$. In this situation, the cell is likely to be formed by *many* trajectories. Let us denote by

$$m_\alpha = \min_{x \in \partial \mathsf{B}(1)} \Phi_x(\alpha), \tag{15}$$

$$M_\alpha = \max_{x \in \partial \mathsf{B}(1)} \Phi_x(\alpha) \tag{16}$$

the minimal and maximal distances from a boundary point of $\mathsf{B}(1)$ to the boundary of the cell $\mathfrak{C}_0(\alpha)$, see Figure 3. Now, (8) immediately implies that $2\alpha \ln(1 + m_\alpha)$ is an Exp(1) random variable (so, informally, m_α is of order α^{-1} with a random factor in front); also (the second part of) Theorem 2.20 of [3] implies that, for *fixed* $x \in \partial \mathsf{B}(1)$, $\alpha\big(\Phi_x(\alpha)\big)^2$ is *approximately* an Exp(1) random variable (so, informally, the distance from a generic boundary point of $\mathsf{B}(1)$ to the boundary of the cell is of order $\alpha^{-1/2}$; again, with a random factor in front). We now obtain that M_α is concentrated around $\sqrt{\frac{\ln \alpha}{2\alpha}}$ (without any random factors), and, moreover, even the a.s. convergence takes place (here we, of course, assume that, as mentioned earlier, the random interlacement process is constructed for all

$\alpha \geq 0$ simultaneously, in such a way that BRI(α_1) is dominated by BRI(α_2) for $\alpha_1 < \alpha_2$, meaning that M_α is nonincreasing):

Theorem 2.7. *It holds that*

$$\Big(\frac{\ln \alpha}{2\alpha}\Big)^{-1/2} M_\alpha \to 1 \ \textit{a.s., as } \alpha \to \infty. \tag{17}$$

Let us also remark that it would be interesting to investigate the shape of the (suitably rescaled) "interface" (that is, the boundary of $\mathfrak{C}_0(\alpha)$) seen in some window around a typical boundary point (say, 1) of $\mathsf{B}(1)$ in the regime $\alpha \to \infty$.

The rest of the paper will be organized in the following way: in Section 3, we recall some facts and prove a few technical lemmas about the conditioned Brownian motion, notably, in Section 3.1 we introduce and discuss a new martingale for the conditioned Brownian motion. Then, in Section 4 we give the proofs of Theorems 2.4, 2.6, and 2.7.

3 Some auxiliary facts

First, we recall a basic fact about hitting circles centered at the origin by the conditioned Brownian motion. Since $\frac{1}{\ln|\widehat{W}_t|}$ is a local martingale, the optional stopping theorem implies that for any $1 < a < |x| < b < \infty$

$$\mathbb{P}_x[\widehat{\tau}(b) < \widehat{\tau}(a)] = \frac{(\ln a)^{-1} - (\ln|x|)^{-1}}{(\ln a)^{-1} - (\ln b)^{-1}} = \frac{\ln(|x|/a) \times \ln b}{\ln(b/a) \times \ln|x|}. \tag{18}$$

Sending b to infinity in (18) we also obtain that for $1 \leq a \leq |x|$

$$\mathbb{P}_x[\widehat{\tau}(a) = \infty] = 1 - \frac{\ln a}{\ln|x|}. \tag{19}$$

Also, in the following, we will need to refer to the fact that

$$\ln|x| = \ln|y| + \ln\big(1 + \tfrac{|x|-|y|}{|y|}\big) = \ln|y| + O\big(\tfrac{|x-y|}{|y|}\big). \tag{20}$$

We denote by $\widehat{W}^{x,r}$ the Brownian motion conditioned on not hitting $\mathsf{B}(x,r)$; it is straightforward to check that it can be obtained from the Brownian motion conditioned by "canonical" conditioned Brownian motion $\widehat{W}$ via the linear space-time transformation,

$$\widehat{W}^{x,r}_t = x + \widehat{W}_{r^2 t}.$$

Then, we need a fact about the conditional entrance measure of a Brownian motion (we need to state and prove it here because the second statement of Lemma 3.3 of [3] is incorrect, a logarithmic factor is missing there):

Lemma 3.1. *Let $0 < 2r < |x| < R$, and let $\nu_x^{r,R}$ be the conditional entrance measure of the Brownian motion started at x to $\mathsf{B}(r)$, given that $\tau(r) < \tau(R)$. Then, we have (abbreviating $s := |x|$)*

$$\Big|\frac{d\nu_x^{r,R}}{d\,\mathrm{hm}_{\mathsf{B}(r)}} - 1\Big| = O\Big(\frac{r \ln \frac{R}{r}}{s \ln(1+\frac{R}{s})}\Big) \tag{21}$$

(here, $\mathrm{hm}_{\mathsf{B}(r)}$ is the harmonic measure on $\mathsf{B}(r)$, which is also uniform on $\partial\mathsf{B}(r)$).

Proof. Without restricting generality, one can assume that $R-s \geq s/2$ (otherwise, one can just condition the first entry point to $\partial\mathsf{B}\big(\frac{2}{3}s\big)$). Denote also by $\hat{\nu}$ the conditional entrance measure of the Brownian motion started at x to $\partial\mathsf{B}(R)$, given that $\tau(R) < \tau(r)$. Consider $M \subset \partial\mathsf{B}(r)$ and note first that, for y with $|y| \geq 2r$

$$\mathbb{P}_y\big[W_{\tau(r)} \in M\big] = \mathrm{hm}_{\mathsf{B}(r)}(M)\big(1 + O\big(\tfrac{r}{|y|}\big)\big); \tag{22}$$

this is a well-known fact that can be obtained from the explicit formula for the (unconditional) entrance measure to a disk from outside; see, e.g. Theorem 3.44 of [11] or (24) below. Then, we write

$$\begin{aligned}
&\mathbb{P}_x\big[W_{\tau(r)} \in M, \tau(r) < \tau(R)\big]\\
&= \mathbb{P}_x\big[W_{\tau(r)} \in M\big] - \mathbb{P}_x\big[W_{\tau(r)} \in M, \tau(R) < \tau(r)\big]
\end{aligned}$$

(by (22) and conditioning on the entrance point to $\partial\mathsf{B}(R)$)

$$= \mathrm{hm}_{\mathsf{B}(r)}(M)\big(1 + O\big(\tfrac{r}{s}\big)\big) - \mathbb{P}_x[\tau(R) < \tau(r)] \int_{\partial\mathsf{B}(R)} \mathbb{P}_z\big[W_{\tau(r)} \in M\big]\, d\hat{\nu}(z)$$

(again by (22))

$$\begin{aligned}
&= \mathrm{hm}_{\mathsf{B}(r)}(M)\big(1 + O\big(\tfrac{r}{s}\big)\big) - \mathbb{P}_x[\tau(R) < \tau(r)]\, \mathrm{hm}_{\mathsf{B}(r)}(M)\big(1 + O\big(\tfrac{r}{R}\big)\big)\\
&= \mathrm{hm}_{\mathsf{B}(r)}(M)\big(\mathbb{P}_x[\tau(r) < \tau(R)] + O\big(\tfrac{r}{s}\big)\big).
\end{aligned}$$

We then divide the above by $\mathbb{P}_x[\tau(r) < \tau(R)] = \frac{\ln \frac{R}{s}}{\ln \frac{R}{r}}$ to arrive to (21) (recall the remark at the beginning of the proof). □

Next, we recall a fact about the capacity of a union of two disks:

Lemma 3.2. *Assume that $r < |y| - 1$. We have*

$$\widehat{\mathrm{cap}}\,\big(\mathsf{B}(y,r)\big) = \mathrm{cap}\,\big(\mathsf{B}(1) \cup \mathsf{B}(y,r)\big) = \frac{2}{\pi} \cdot \frac{\ln^2|y| + O\big(\frac{r(1+|\ln r|+\ln|y|)\ln|y|}{|y|}\big)}{\ln r^{-1} + \ell_y + \ln|y| + O\big(\frac{r}{|y|-1}\ln\frac{|y|-1}{r}\big)}. \tag{23}$$

Proof. It is a reformulation of Lemma 3.11 (iii) of [3]. □

In the following subsections, we collect some "more advanced" auxiliary results about the conditioned Brownian motion.

3.1 Conditioned Brownian motion: martingales and hitting probabilities

Now, we will need some facts about hitting probabilities for conditioned Brownian motions; however, to be able to estimate these probabilities, we first develop a method which is "cleaner" than the one used, e.g. in Lemma 3.7 of [3].

Lemma 3.3. *Assume that, for some open set $\Lambda \subset \mathbb{R}^2$ such that $\Lambda \cap \mathsf{B}(1) = \emptyset$, a function $g : \Lambda \mapsto \mathbb{R}$ is harmonic. Then $g(\widehat{W})/\ln|\widehat{W}|$ is a local martingale.*

Proof. Let $x \in \Lambda$ and consider any bounded closed subset G of Λ that contains x in its interior. Let τ be the hitting time of $\Lambda \setminus G$ and let $t > 0$. Now, recall (1) and write (note that none of the trajectories up to time $t \wedge \tau$ can touch the unit disk and therefore be killed)

$$\mathbb{E}_x \frac{g(\widehat{W}_{t\wedge\tau})}{\ln|\widehat{W}_{t\wedge\tau}|} = \frac{1}{\ln|x|}\mathbb{E}_x g(W_{t\wedge\tau}) = \frac{g(x)}{\ln|x|}$$

since $g(W)$ is a local martingale, so (since $\widehat{W}$ is also Markovian) we obtain that $\frac{g(\widehat{W})}{\ln|\widehat{W}|}$ is a local martingale. □

Next, we know (see e.g. Theorem 3.44 of [11]) that for $x \notin \mathsf{B}(1)$ and $z \in \partial\mathsf{B}(1)$

$$H(x,z) = \frac{|x|^2 - 1}{2\pi|z-x|^2} \tag{24}$$

is the Poisson kernel on $\mathbb{R}^2 \setminus \mathsf{B}(1)$, i.e., the density of the entrance measure to $\mathsf{B}(1)$ when starting at x. Define for $x, y \notin \mathsf{B}(1)$, $x \neq y$,

$$\begin{aligned} L(x,y) &= \frac{1}{\ln|x|}\Big(\ln|x| - \ln|x-y| + \int_{\partial\mathsf{B}(1)} \ln|z-y| H(x,z)\,dz\Big) \qquad (25)\\ &= 1 + \frac{1}{\ln|x|}\Big(-\ln|x-y| + \frac{|x|^2-1}{2\pi}\int_{\partial\mathsf{B}(1)} \frac{\ln|z-y|}{|z-x|^2}\,dz\Big), \qquad (26) \end{aligned}$$

Writing $x = x_0(1+\delta)$ with $x_0 \in \partial\mathsf{B}(1)$ and $\delta > 0$, this definition can be rewritten as

$$\begin{aligned} L(x,y) &= 1 + \frac{1}{\ln|x|}\Big(-\ln\frac{|x-y|}{|x_0-y|} + \frac{|x|^2-1}{2\pi}\int_{\partial\mathsf{B}(1)} \frac{\ln\frac{|z-y|}{|x_0-y|}}{|z-x|^2}\,dz\Big)\\ &= 1 + \frac{1}{\ln|x|}\Big(-\ln\frac{|x-y|}{|x_0-y|} + \frac{|x|^2-1}{2\pi}\int_0^{\pi} \frac{\ln\frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x|^2}\,d\varphi\Big), \qquad (27) \end{aligned}$$

where $z^{(\varphi)} = x_0 e^{i\varphi}$. To cover the case $\delta = 0$, define for $x_0 \in \partial\mathsf{B}(1), y \notin \mathsf{B}(1)$ (denoting by $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^2$),

$$L(x_0,y) = \frac{(y-x_0,y)}{|x_0-y|^2} + \frac{1}{\pi}\int_0^{\pi} \frac{\ln\frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x_0|^2}\,d\varphi. \tag{28}$$

Then, partly as a consequence of Lemma 3.3, we have

Proposition 3.4. *For any fixed $y \notin \mathsf{B}(1)$, it holds that*

(i) the function $\big(\mathbb{R}^2 \setminus (\mathsf{B}(1) \cup \{y\})\big) \cup \partial\mathsf{B}(1) \to \mathbb{R}, x \mapsto L(x,y)$ is continuous;

(ii) the process $L(\widehat{W}_t, y)$ is a local martingale;

(iii) $L(x,y) \to 0$ as $x \to \infty$;

(iv) for any fixed $r > 0$, $L(x,y)$ is uniformly bounded in $x \in \mathbb{R}^2 \setminus (\mathsf{B}(1) \cup \mathsf{B}(y,r))$;

(v) $L(x,y) > 0$ for all $x \in (\mathbb{R}^2 \setminus \mathsf{B}(1)) \cup \partial\mathsf{B}(1)$, $x \neq y$.

Proof. We fix $y \notin \mathsf{B}(1)$ once and for all. We begin with the proof of (i). The continuity outside of $\mathsf{B}(1)$ is immediate. Let us prove that for all $x_0 \in \partial\mathsf{B}(1)$,

$$\lim_{x\to x_0} L(x,y) = \frac{(y-x_0,y)}{|x_0-y|^2} + \frac{1}{\pi}\int_0^{\pi} \frac{\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x_0|^2}\, d\varphi. \tag{29}$$

To prove that, we assume that $x = (1+\delta)x_0$ and then take the limit $\delta \downarrow 0$ with the help of (27); we will prove that convergence uniformly in x_0, together with the continuity in x_0 of $L(x_0,y)$, and (29) will follow by elementary arguments. We proceed term by term. First, we compute that for fixed $x_0 \in \partial\mathsf{B}(1)$, we have

$$\ln|(1+\delta)x_0| = \ln(1+\delta) = \delta + O(\delta^2) \tag{30}$$

and

$$\begin{aligned}\ln\frac{|(1+\delta)x_0-y|}{|x_0-y|} &= \frac{1}{2}\ln\frac{|x_0-y+\delta x_0|^2}{|x_0-y|^2}\\ &= \frac{1}{2}\ln\Big(1+2\delta\cdot\frac{(x_0-y,x_0)}{|x_0-y|^2}+\delta^2\cdot\frac{1}{|x_0-y|^2}\Big) \qquad (31)\\ &= \delta\cdot\frac{(x_0-y,x_0)}{|x_0-y|^2}+O(\delta^2). \qquad (32)\end{aligned}$$

We derive that, for any $x_0 \in \partial\mathsf{B}(1)$, when $x = x_0(1+\delta)$, we have

$$1 - \lim_{\delta\to 0}\frac{1}{\ln|x|}\ln\frac{|x-y|}{|x_0-y|} = 1 - \frac{(x_0-y,x_0)}{|x_0-y|^2} = \frac{(y-x_0,y)}{|x_0-y|^2}. \tag{33}$$

We argue that this convergence is in fact uniform in x_0. We observe that the coefficients $\frac{(x_0-y,x_0)}{|x_0-y|^2}$ and $\frac{1}{|x_0-y|^2}$ appearing in (31) are bounded, respectively by $\frac{|y|+1}{(|y|-1)^2}$ and $\frac{1}{(|y|-1)^2}$. Using the fact that $|\ln(1+u)-u| \le 2u^2$ for $u \in [-\frac{1}{2},\frac{1}{2}]$, we see that the $O(\delta^2)$ in (32) is uniform in x_0, and thus, so is the convergence in (33).

We also compute $\lim_{\delta\to 0}\frac{|x|^2-1}{\ln|x|} = 2$ and see that this limit is uniform in x_0 (in fact, $\frac{|x|^2-1}{\ln|x|}$ is independent of x_0).

Finally, let us argue using the dominated convergence theorem that, still with $x = (1+\delta)x_0$,

$$\int_0^{\pi} \frac{\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x|^2}\, d\varphi \xrightarrow[\delta\to 0]{} \int_0^{\pi} \frac{\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x_0|^2}\, d\varphi, \tag{34}$$

uniformly in x_0. For the moment, let us reason with fixed x_0. It is clear that the integrand in the left-hand side of (34) converges pointwise towards the integrand in the right-hand side of (34). Observing that $|x_0 - z^{(\varphi)}| \le |x - z^{(\varphi)}|$, we see that the integrand in the right-hand side of (34) dominates the integrand in the left-hand side of (34). Let us check that the integrand in the right-hand side of (34) is integrable; the only difficulty is around 0. On the one hand, as $\varphi \to 0$:

$$\begin{aligned}
\frac{|z^{(\varphi)}-y|^2}{|x_0-y|^2} &= \frac{|x_0 - y + x_0(e^{i\varphi}-1)|^2}{|x_0-y|^2}\\
&= 1 + 2\frac{(x_0-y, x_0(e^{i\varphi}-1))}{|x_0-y|^2} + \frac{|e^{i\varphi}-1|^2}{|x_0-y|^2}\\
&= 1 + 2\varphi\frac{(x_0-y, x_0 i)}{|x_0-y|^2} + 2\frac{(x_0-y, x_0(e^{i\varphi}-1-i\varphi))}{|x_0-y|^2} + \frac{|e^{i\varphi}-1|^2}{|x_0-y|^2} \qquad (35)\\
&= 1 + 2\varphi\frac{(x_0-y, x_0 i)}{|x_0-y|^2} + O(\varphi^2), \qquad (36)
\end{aligned}$$

so

$$\begin{aligned}
\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2} &= \frac{1}{2}\ln\left(\frac{|z^{(\varphi)}-y|^2}{|x_0-y|^2} \times \frac{|z^{(-\varphi)}-y|^2}{|x_0-y|^2}\right)\\
&= \frac{1}{2}\ln(1+O(\varphi^2)) = O(\varphi^2). \qquad (37)
\end{aligned}$$

On the other hand, $|z^{(\varphi)} - x_0|^2 = |e^{i\varphi}-1|^2 = \varphi^2 + O(\varphi^3)$. We obtain that the integrand on the right-hand side of (34) is bounded around 0, thus, it is integrable. The dominated convergence theorem yields the convergence in (34), for fixed x_0.

To conclude, we need to show that this convergence is uniform in x_0. We are simply going to refine the above arguments by showing that our estimates are uniform. Let us denote

$$\begin{aligned}
f(\varphi,\delta) &:= \sup_{x_0\in\partial\mathsf{B}(1)} \left| \frac{\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x_0|^2} - \frac{\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}}{|z^{(\varphi)}-x|^2} \right|\\
&= \left(1 - \frac{|e^{i\varphi}-1|^2}{|e^{i\varphi}-(1+\delta)|^2}\right) \times g(\varphi), \qquad (38)
\end{aligned}$$

with

$$g(\varphi) := \frac{\sup_{x_0\in\partial\mathsf{B}(1)} \left|\ln \frac{|z^{(\varphi)}-y||z^{(-\varphi)}-y|}{|x_0-y|^2}\right|}{|e^{i\varphi}-1|^2}. \tag{39}$$

Let us show that $g(\varphi)$ is bounded: the numerator on the right-hand side of (39) is clearly bounded, so we only have to check that g does not diverge at 0. This comes from (35), (36) and (37): observe that the coefficients in (35) are bounded in x_0. Using the fact that $|\ln(1+u)-u| \le 2u^2$ for $u \in [-\frac{1}{2}, \frac{1}{2}]$, it is straightforward to establish that the $O(\varphi^2)$ in (37) is uniform in x_0, hence g is bounded around 0.

In view of (38), we see that $f(\cdot,\delta)$ is dominated by g and converges pointwise towards 0. The dominated convergence theorem then yields $\int_0^\pi f(\varphi,\delta)d\varphi \underset{\delta\to 0}{\longrightarrow} 0$. The uniform convergence in (34) follows by exchanging the integration with the supremum and the absolute value.

From the boundedness of g, we also derive that the right-hand side of (34), hence also $L(x_0, y)$, is continuous in x_0, as desired. This concludes the proof of item (i).

Next, item (ii) follows from Lemma 3.3 (recall that the Poisson kernel is harmonic with respect to its first variable). Items (iii) and (iv) are straightforward to obtain.

Finally, item (v) follows from the optional stopping theorem: a direct computation implies that $L(z,y) > 1$ for all $z \in \partial \mathsf{B}(y,r)$ with small enough r. Then, the optional stopping theorem together with (ii) and (iii) implies that

$$L(x,y) = \mathbb{P}_x[\widehat{\tau}(y,r) < \infty]\mathbb{E}_x\big(L(\widehat{W}_{\widehat{\tau}(y,r)}, y) \mid \widehat{\tau}(y,r) < \infty\big) > 0.$$

This concludes the proof of Proposition 3.4. □

In the next result, we are interested in the regime when the radius of the disk (to be hit) is small, and the starting point is also close to the disk.

Lemma 3.5. *Consider some fixed $\delta_0 > 0$. For every $x, y \notin B(1)$ such that $|x-y| < \frac{1}{2}$ and $|x-y| \le (|y|-1)^{1+\delta_0}$, we have, uniformly for all $r \in (0, |x-y|)$:*

$$\mathbb{P}_x\big[\widehat{\tau}(y,r) < \infty\big] = \frac{\ln|x-y|^{-1} + \ell_y + \ln|y|}{\ln r^{-1} + \ell_y + \ln|y|}\Big(1 + O\big(\tfrac{|x-y|\tilde{\ell}_y}{(|y|-1)(\ln|x-y|^{-1}+\ln|y|)}\big)\Big), \tag{40}$$

$$\mathbb{P}_x\big[\widehat{\tau}(y,r) = \infty\big] = \frac{\ln\frac{|x-y|}{r}}{\ln r^{-1} + \ell_y + \ln|y|} + O\big(\tfrac{|x-y|\tilde{\ell}_y}{(|y|-1)(\ln|x-y|^{-1}+\ln|y|)}\big). \tag{41}$$

Proof. Instead of relying on Lemma 3.7 (iii) of [3], we will use the optional stopping theorem with the martingale provided by Proposition 3.4 and the stopping time $\widehat{\tau}(y,r)$.

We first establish some estimates. Let us show that according to our assumptions, $\frac{|x-y|}{|y|-1}$ is bounded by 1 and $\frac{|y|-1}{|x|-1}$ is bounded by a constant depending on δ_0. For the first quantity, observe that when $|y| \le 2$, we have $|x-y| \le (|y|-1)^{1+\delta_0} \le |y|-1$, while $|y| > 2$ we simply have $\frac{|x-y|}{|y|-1} < \frac{1}{2}$. For the second quantity, write when $|y| \le 1.7$,

$$|x|-1 \ge |y|-1-|x-y| \ge |y|-1-(|y|-1)^{1+\delta_0} \ge (|y|-1)(1-0.7^{\delta_0}),$$

and note that $|y| > 1.7$ we have $|x|-1 \ge |y|-|x-y|-1 > |y|-3/2$ which yields $\frac{|y|-1}{|x|-1} \le \frac{|y|-1}{|y|-\frac{3}{2}} \le \frac{7}{2}$.

From these facts, we now deduce that, under the conditions of the lemma,

$$H(x,z) = H(y,z)\big(1 + O\big(\tfrac{|x-y|}{|y|-1}\big)\big) \tag{42}$$

uniformly in $z \in \partial \mathsf{B}(1)$. Indeed, using (24), we write

$$\begin{aligned}\frac{H(x,z)}{H(y,z)} &= \frac{|x|^2-1}{|y|^2-1}\cdot\frac{|y-z|^2}{|x-z|^2}\\ &= \Big(1+\frac{|x-y|^2+2(y,x-y)}{|y|^2-1}\Big)\Big(1+\frac{|x-y|^2+2(x-y,x-z)}{|x-z|^2}\Big)\\ &=: (1+T_1)(1+T_2).\end{aligned}$$

Note that, since $|y|\geq 1$, we have $|y|^2-1 \geq |y|(|y|-1)$, and recall that $\frac{|x-y|}{|y|-1}$ is bounded; this shows that T_1 is $O\big(\frac{|x-y|}{|y|-1}\big)$. Then, writing $\frac{|x-y|}{|x-z|} \leq \frac{|x-y|}{|x|-1} = \frac{|x-y|}{|y|-1}\frac{|y|-1}{|x|-1}$, recalling that both fractions are bounded, we also find that T_2 is $O\big(\frac{|x-y|}{|y|-1}\big)$. By the boundedness of $\frac{|x-y|}{|y|-1}$, this implies (42).

We now consider $r \in (0, |x-y|)$. Notice that our assumptions imply $r < |y|-1$, so that $\mathsf{B}(y,r)\cap\mathsf{B}(1)=\emptyset$. Indeed, if $|y|\leq 2$, then $r < |x-y| \leq (|y|-1)^{1+\delta_0} \leq |y|-1$, while if $|y|>2$ then $|y|-1>1>r$.

Recalling (10) and using (20) (to substitute $\ln|x|$ or $\ln|u|$ by $\ln|y|$), we notice that (42) permits us to obtain

$$L(x,y) = \frac{\ln|x-y|^{-1} + \ell_y + \ln|y| + O\big(\frac{|x-y|\tilde{\ell}_y}{|y|-1}\big)}{\ln|y| + O\big(\frac{|x-y|}{|y|}\big)}$$

and, for any $u \in \partial\mathsf{B}(y,r)$

$$L(u,y) = \frac{\ln r^{-1} + \ell_y + \ln|y| + O\big(\frac{r\tilde{\ell}_y}{|y|-1}\big)}{\ln|y| + O\big(\frac{r}{|y|}\big)}.$$

Now, by the assumption that $|x-y|\leq(|y|-1)^{1+\delta_0}$, and lower bounding ℓ_y by $\ln(|y|-1)$, we have

$$\frac{1}{1+\delta_0}\ln|x-y|^{-1} + \ell_y \geq 0$$

hence $\ln|x-y|^{-1} + \ell_y + \ln|y| \geq \frac{\delta_0}{1+\delta_0}(\ln|x-y|^{-1} + \ln|y|)$.

We are now ready to conclude. We obtain from the optional stopping theorem (recall Proposition 3.4 (iii) and (iv))

$$\begin{aligned}\mathbb{P}_x\big[\widehat{\tau}(y,r)<\infty\big] &= \frac{\big(\ln|x-y|^{-1}+\ell_y+\ln|y|+O\big(\frac{|x-y|\tilde{\ell}_y}{|y|-1}\big)\big)\big(\ln|y|+O\big(\frac{r}{|y|}\big)\big)}{\big(\ln r^{-1}+\ell_y+\ln|y|+O\big(\frac{r\tilde{\ell}_y}{|y|-1}\big)\big)\big(\ln|y|+O\big(\frac{|x-y|}{|y|}\big)\big)}\\ &= \frac{\ln|x-y|^{-1}+\ell_y+\ln|y|}{\ln r^{-1}+\ell_y+\ln|y|}\Big(1+O\big(\tfrac{|x-y|\tilde{\ell}_y}{(|y|-1)(\ln|x-y|^{-1}+\ln|y|)}\big)\Big),\end{aligned}$$

thus proving (40). Then, (41) is a direct consequence of (40). □

3.2 Traces of different conditioned Brownian motions

Next, the goal is to be able to compare traces left by the Brownian motion conditioned on not hitting a specific (typically small) disk and the usual conditioned (on not hitting $\mathsf{B}(1)$) Brownian motion additionally conditioned on not hitting that disk. Recall that the Brownian motion conditioned on not hitting $B(y,r)$ is denoted by $\widehat{W}^{y,r}$.

Lemma 3.6. *Assume that $0<r<r'$ and let y be such that $\mathsf{B}(y,r')\cap\mathsf{B}(1)=\emptyset$ (meaning that $s:=|y|-r'-1$ is strictly positive). Let us denote also by $\tau^*(z,r')$ the hitting time of $\partial\mathsf{B}(z,r')$ by $\widehat{W}^{y,r}$. Then, for any $x\in\big(\mathsf{B}(y,r')\setminus\mathsf{B}(y,r)\big)\cup\partial\mathsf{B}(y,r)$*

$$\Big|\frac{d\mathbb{P}_x\big[\widehat{W}_{[0,\widehat{\tau}(y,r')]}\in\cdot\ \mid \widehat{\tau}(y,r')<\widehat{\tau}(y,r)\big]}{d\mathbb{P}_x\big[\widehat{W}^{y,r}_{[0,\tau^*(y,r')]}\in\cdot\,\big]}-1\Big|=O\big(\tfrac{r'}{s\ln s}\big). \tag{43}$$

Proof. First, we note that, analogously to Section 2.1 of [3], it is possible to define the diffusion $\widehat{W}$ conditioned on not touching $\mathsf{B}(y,r)$ anymore even for a starting point on $\partial\mathsf{B}(y,r)$. Since the estimates we obtain below (in the case $x\in\mathsf{B}(y,r')\setminus\mathsf{B}(y,r)$) will be uniform in x, it is enough to prove the result for $x\in\mathsf{B}(y,r')\setminus\mathsf{B}(y,r)$.

Let Γ be a set of (finite) trajectories, having the following property: a trajectory belonging to this set has to start at x, it cannot touch $\mathsf{B}(y,r)$, and it ends on its first visit to $\partial\mathsf{B}(y,r')$. Then, we have (by an obvious adaptation of Lemma 2.1 of [3])

$$\begin{aligned}\mathbb{P}_x\big[\widehat{W}^{y,r}_{[0,\tau^*(y,r')]}\in\Gamma\,\big]&=\mathbb{P}_x\big[W_{[0,\tau(y,r')]}\in\Gamma\ \mid\tau(y,r')<\tau(y,r)\big]\\&=\frac{\mathbb{P}_x\big[W_{[0,\tau(y,r')]}\in\Gamma\,\big]}{\mathbb{P}_x\big[\tau(y,r')<\tau(y,r)\big]},\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_x\big[\widehat{W}_{[0,\widehat{\tau}(y,r')]}\in\Gamma\ \mid\widehat{\tau}(y,r')<\widehat{\tau}(y,r)\big]&=\frac{\mathbb{P}_x\big[\widehat{W}_{[0,\widehat{\tau}(y,r')]}\in\Gamma\,\big]}{\mathbb{P}_x\big[\widehat{\tau}(y,r')<\widehat{\tau}(y,r)\big]}\\&\text{(by Lemma 3.6 of [3])}\\&=\frac{\mathbb{P}_x\big[W_{[0,\tau(y,r')]}\in\Gamma\,\big]}{\mathbb{P}_x\big[\tau(y,r')<\tau(y,r)\big]}\big(1+O\big(\tfrac{r'}{s\ln s}\big)\big),\end{aligned}$$

which implies (43). □

The above result compares the traces left in $\mathsf{B}(y,r')$ by $\widehat{W}^{y,r}$ and (conditioned) $\widehat{W}$ before going out of $\mathsf{B}(y,r')$ for the first time; it is important to observe that it allows us to couple these traces with probability close to 1 (when the right-hand side of (43) is small). With $r':=r\ln^\theta r^{-1}$, we also need to compare the full traces left in $\mathsf{B}(y,r')$ by these two processes:

Lemma 3.7. *Fix $\theta \geq 2$ and $\delta_0 > 0$. Let $r < 1$ and $y \notin \mathsf{B}(1)$ be such that*

$$r \ln^{2\theta} r^{-1} < \min\left(\tfrac{1}{2}, |y| - 1, (|y| - 1)^{1+\delta_0}\right). \tag{44}$$

Then, for any $x \in \big(\mathsf{B}(y, \ln^\theta r^{-1}) \setminus \mathsf{B}(y, r)\big) \cup \partial\mathsf{B}(y, r)$, we have

$$\left|\frac{d\mathbb{P}_x\big[\widehat{W}_{[0,\infty]} \cap \mathsf{B}(y, r\ln^\theta r^{-1}) \in \cdot \mid \widehat{\tau}(y, r) = \infty\big]}{d\mathbb{P}_x\big[\widehat{W}^{y,r}_{[0,\infty]} \cap \mathsf{B}(y, r\ln^\theta r^{-1}) \in \cdot\,\big]} - 1\right| = O\big(\tfrac{\ln\ln r^{-1}}{\ln r^{-1} + \ln|y|} + \tfrac{r\tilde{\ell}_y \ln^{2\theta} r^{-1}}{|y|-1}\big). \tag{45}$$

Proof. Once again, as in the proof of Lemma 3.6, it is enough to obtain the proof for the case $x \in \mathsf{B}(y, \ln^\theta r^{-1}) \setminus \mathsf{B}(y, r)$. Abbreviate $r_1 := r\ln^\theta r^{-1}$, $r_2 := r\ln^{2\theta} r^{-1}$. The idea of the proof is the following: first, by Lemma 3.6, the traces left by the two processes on $\mathsf{B}(y, r_2)$ (and therefore on $\mathsf{B}(y, r_1)$) up to their first hitting time of $\partial\mathsf{B}(y, r_2)$ have almost the same law. Then, both processes may return a few times from $\partial\mathsf{B}(y, r_2)$ to $\partial\mathsf{B}(y, r_1)$, and we argue that the number of such returns has almost the same law (in fact, it is precisely $\mathrm{Geom}_0(1/2)$ for $\widehat{W}^{y,r}$); we also argue that the entrance points of these returns have almost the same law. Then, each pair of additional excursions from $\partial\mathsf{B}(y, r_1)$ to $\partial\mathsf{B}(y, r_2)$ again can be coupled with high probability by Lemma 3.6.

Now, we fill in the details. First, note that (e.g., by (2.17) of [3])

$$\text{for any } x \in \partial\mathsf{B}(y, r_2), \qquad \mathbb{P}_x[\tau^*(y, r_1) < \infty] = \frac{\ln(r_1/r)}{\ln(r_2/r)} = \frac{1}{2}. \tag{46}$$

For the (additionally) conditioned process $\widehat{W}$, we have, for $x \in \partial\mathsf{B}(y, r_2)$

$$\mathbb{P}_x[\widehat{\tau}(y, r_1) < \infty \mid \widehat{\tau}(y, r) = \infty]$$

(ν being the conditional entrance measure to $\partial\mathsf{B}(y, r_1)$ from x)

$$= \frac{\mathbb{P}_x[\widehat{\tau}(y, r_1) < \infty]\mathbb{P}_\nu[\widehat{\tau}(y, r) = \infty]}{\mathbb{P}_x[\widehat{\tau}(y, r) = \infty]}$$

(by Lemma 3.5)

$$= \frac{\frac{\ln r_2^{-1} + \ell_y + \ln|y|}{\ln r_1^{-1} + \ell_y + \ln|y|}\Big(1 + O\big(\frac{r_2\tilde{\ell}_y}{(|y|-1)(\ln r_2^{-1} + \ln|y|)}\big)\Big)\Big(\frac{\ln\frac{r_1}{r}}{\ln r^{-1} + \ell_y + \ln|y|} + O\big(\frac{r_1\tilde{\ell}_y}{(|y|-1)(\ln r_1^{-1} + \ln|y|)}\big)\Big)}{\frac{\ln\frac{r_2}{r}}{\ln r^{-1} + \ell_y + \ln|y|} + O\big(\frac{r_2\tilde{\ell}_y}{(|y|-1)(\ln r_2^{-1} + \ln|y|)}\big)}$$

$$= \frac{\frac{\ln r_2^{-1} + \ell_y + \ln|y|}{\ln r_1^{-1} + \ell_y + \ln|y|} \cdot \frac{\ln\frac{r_1}{r}}{\ln r^{-1} + \ell_y + \ln|y|}}{\frac{\ln\frac{r_2}{r}}{\ln r^{-1} + \ell_y + \ln|y|}}\Big(1 + O\big(\tfrac{r\tilde{\ell}_y \ln^{2\theta} r^{-1}}{|y|-1}\big)\Big)$$

(recall that $\frac{\ln(r_1/r)}{\ln(r_2/r)} = \frac{1}{2}$)

$$= \frac{1}{2}\Big(1 + O\big(\tfrac{\ln\ln r^{-1}}{\ln r^{-1} + \ln|y|} + \tfrac{r\tilde{\ell}_y \ln^{2\theta} r^{-1}}{|y|-1}\big)\Big). \tag{47}$$

So, with (46) and (47), we are now able to couple the excursion counts of the two processes with probability close to 1.

Next, we need to be able to couple the entrance points to $\partial\mathsf{B}(y,r_1)$ with high probability. Let M be a (measurable) subset of $\partial\mathsf{B}(y,r_1)$. First, let us note that Lemma 3.1 implies that

$$\mathbb{P}_u[\widehat{W}^{y,r}_{\tau^*(y,r_1)} \in M \mid \tau^*(y,r_1)<\infty] = \frac{|M|}{2\pi r_1}\big(1+O\big(\tfrac{1}{\ln^\theta r^{-1}}\big)\big) \tag{48}$$

uniformly in $u\in\partial\mathsf{B}(y,r_2)$. Let us obtain an analogue of (48) for $\widehat{W}$ conditioned on $\{\widehat{\tau}(y,r)=\infty\}$. Lemma 3.5 implies that

$$\mathbb{P}_u[\widehat{\tau}(y,r)>\widehat{\tau}(R)] = \frac{2\ln\ln r^{-1}}{\ln r^{-1}+\ell_y+\ln|y|}\Big(1+O\big(\tfrac{r\tilde{\ell}_y\ln^\theta r^{-1}}{|y|-1}\big)\Big)+o(1) \tag{49}$$

as $R\to\infty$, uniformly in $u\in\partial\mathsf{B}(y,r_1)$. Assume without loss of generality that $r_1<r_2/2$. For R such that $\mathsf{B}(y,r_2)\subset\mathsf{B}(R)$, abbreviate $G_R=\big\{\tau(y,r_1)<\tau(R)<\tau(1)\big\}$. Define the (possibly infinite) random variable

$$T_R = \begin{cases} \inf\big\{t\geq 0 : W_t\in\mathsf{B}(y,r_2/2), W_{[t,\tau(y,r_1)]}\cap\partial\mathsf{B}(y,r_2)=\emptyset\big\} & \text{on } G_R,\\ \infty & \text{on } G_R^{\complement}\end{cases}$$

to be the time when the last (before hitting $\partial\mathsf{B}(y,r_1)$) excursion between $\partial\mathsf{B}(y,r_2/2)$ and $\partial\mathsf{B}(y,r_1)\cup\partial\mathsf{B}(y,r_2)$ starts. Note that T_R is not a stopping time; and the law of the excursion that begins at time T_R is the law of a Brownian excursion conditioned to reach $\partial\mathsf{B}(y,r_1)$ before going to $\partial\mathsf{B}(y,r_2)$. We denote that excursion by $\widetilde{W}$ (with its initial time reset to 0) and denote by σ the time it hits $\partial\mathsf{B}(y,r_1)$. Let ν_R be the joint law of the pair (T_R, W_{T_R}) under $\mathbb{P}_x$. Abbreviate also $\mathcal{H}=\mathbb{R}_+\times\partial\mathsf{B}(y,r_2/2)$ and observe that $\int_{\mathcal{H}} d\nu_R(t,y)=\mathbb{P}_x[G_R]$.

We can then write for $x'\in\partial\mathsf{B}(y,r_2)$

$$\begin{aligned}&\mathbb{P}_{x'}\big[\widehat{W}_{\widehat{\tau}(y,r_1)}\in M, \widehat{\tau}(y,r_1)<\infty, \widehat{\tau}(y,r)=\infty\big]\\ &= \lim_{R\to\infty}\mathbb{P}_{x'}\big[\widehat{W}_{\widehat{\tau}(y,r_1)}\in M, \widehat{\tau}(y,r_1)<\widehat{\tau}(R)<\widehat{\tau}(y,r)\big]\end{aligned}$$

(by Lemma 2.1 of [3])

$$\begin{aligned}&= \lim_{R\to\infty}\mathbb{P}_{x'}\big[W_{\tau(y,r_1)}\in M, \tau(y,r_1)<\tau(R)<\tau(y,r)\mid\tau(R)<\tau(1)\big]\\ &= \lim_{R\to\infty}\frac{\ln R}{\ln|x'|}\mathbb{P}_{x'}\big[W_{\tau(y,r_1)}\in M, \tau(y,r_1)<\tau(R)<\tau(y,r)\wedge\tau(1)\big]\\ &= \lim_{R\to\infty}\frac{\ln R}{\ln|x'|}\int_{\mathcal{H}} d\nu_R(t,z)\int_M \mathbb{P}_z[\widetilde{W}_\sigma\in du]\mathbb{P}_u[\tau(R)<\tau(y,r)\wedge\tau(1)]\end{aligned}$$

(again by Lemma 2.1 of [3])

$$= \lim_{R\to\infty}\frac{1}{\ln|x'|}\int_{\mathcal{H}} d\nu_R(t,z)\int_M \mathbb{P}_z[\widetilde{W}_\sigma\in du]\mathbb{P}_u[\widehat{\tau}(y,r)>\widehat{\tau}(R)]\ln|u|. \tag{50}$$

On the other hand, in the same way, one can obtain

$$\begin{aligned}&\mathbb{P}_{x'}\big[\widehat{\tau}(y,r_1)<\infty,\widehat{\tau}(y,r)=\infty\big]\\&=\lim_{R\to\infty}\frac{1}{\ln|x'|}\int_{\mathcal{H}}d\nu_R(t,z)\int_{\partial\mathsf{B}(y,r_1)}\mathbb{P}_z[\widetilde{W}_\sigma\in du]\mathbb{P}_u[\widehat{\tau}(y,r)>\widehat{\tau}(R)]\ln|u|.\end{aligned}\tag{51}$$

Then, (20) implies that $\frac{\ln|u|}{\ln|x'|}=\ln|y|+O(r_2/|y|)$ uniformly in $u\in\partial\mathsf{B}(y,r_1)$, and Lemma 3.1 implies that

$$\mathbb{P}_z[\widetilde{W}_\sigma\in M]=\int_M\mathbb{P}_z[\widetilde{W}_\sigma\in du]=\frac{|M|}{2\pi r_1}\big(1+O\big(\tfrac{1}{\ln^\theta r^{-1}}\big)\big)$$

for any $z\in\partial\mathsf{B}(y,r_2/2)$. Using also (49) together with (50)–(51), we then obtain that

$$\mathbb{P}_{x'}\big[\widehat{W}_{\widehat{\tau}(y,r_1)}\in M\mid\widehat{\tau}(y,r_1)<\infty,\widehat{\tau}(y,r)=\infty\big]=\frac{|M|}{2\pi r_1}\big(1+O\big(\tfrac{1}{\ln^\theta r^{-1}}+\tfrac{r\ln^\theta r^{-1}\tilde{\ell}_y}{|y|-1}\big)\big),\tag{52}$$

which is the desired counterpart of (48) and permits us to couple the starting points of the corresponding excursions with probability at least $1-O\big(\frac{1}{\ln^\theta r^{-1}}+\frac{r\ln^\theta r^{-1}\tilde{\ell}_y}{|y|-1}\big)$.

Using the observation we made after (47), it is now straightforward to conclude the proof of Lemma 3.7 (as outlined at the beginning of the proof). □

3.3 Controlling the size of the Brownian amoeba

Next, we need a result that would permit us to control the size of the Brownian amoeba. For any $r\geq 1$, define the event

$$U_r=\big\{\mathsf{B}(1)\text{ is not connected to }\infty\text{ in }\mathbb{C}\setminus\widehat{W}_{[\widehat{\tau}(r),\widehat{\tau}(2r)]}\big\}.\tag{53}$$

Lemma 3.8. *There exists $\gamma_0>0$ such that for all x with $|x|\geq 1$ we have*

$$\mathbb{P}_x[U_{|x|}]\geq\gamma_0.\tag{54}$$

Proof. Let $r:=|x|$. It is convenient to use Proposition 2.3: it is enough to consider the trace of the process $\exp(Z_t+iB_t)$ (where Z is Bes(3) and B is a standard Brownian motion independent of Z) before Z hits $\ln(2r)$. Denote $\tau^Z(s)=\min\{t\geq 0:e^{Z_t}=s\}$. Let

$$\tilde{\tau}=\min\big\{t>\tau^Z(\tfrac{5}{3}r):e^{Z_t}=\tfrac{3}{2}r\big\}.$$

Define the event (cf. Figure 4; note that what is written in the first line guarantees that the first crossing of $\{z\in\mathbb{C}:\frac{4}{3}r\leq|z|\leq\frac{5}{3}r\}$ is not "too wide")

$$\begin{aligned}U'_r=\Big\{&\tilde{\tau}<\tau^Z(2r),|B_s-B_{\tau^Z(\frac{4}{3}r)}|\leq\tfrac{1}{3}\pi\text{ for all }s\in[\tau^Z(\tfrac{4}{3}r),\tilde{\tau}],\\&|B_{\tilde{\tau}+1}-B_{\tilde{\tau}}|\geq\tfrac{8}{3}\pi,\tfrac{4}{3}r<e^{Z_s}<\tfrac{5}{3}r\text{ for all }s\in[\tilde{\tau},\tilde{\tau}+1]\Big\},\end{aligned}$$

and observe that $\mathbb{P}_x[U_{|x|}]\geq\mathbb{P}_x[U'_{|x|}]$ (given that $|\widehat{W}_0|=r$, on U'_r the origin is indeed disconnected from the infinity). By independence of Z and B it can be easily seen that,

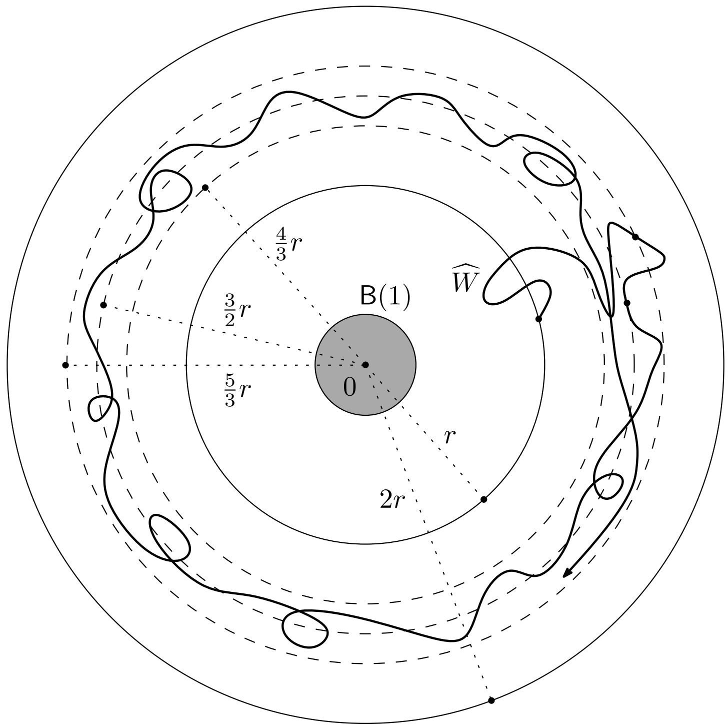

Figure 4: On the definition of the event U'_r

for any y such that $|y| = \frac{4}{3}r$

$$\mathbb{P}_y\big[\tilde{\tau} < \tau^Z(2r), |B_s - B_0| \leq \tfrac{1}{3}\pi \text{ for all } s \leq \tilde{\tau}\big] \geq \gamma_1,$$

and for any y such that $|y| = \frac{3}{2}r$

$$\mathbb{P}_y\big[|B_1 - B_0| \geq \tfrac{8}{3}\pi, \tfrac{4}{3}r < e^{Z_s} < \tfrac{5}{3}r \text{ for all } s \in [0,1]\big] \geq \gamma'_1,$$

for some $\gamma_1, \gamma'_1 > 0$. We conclude the proof using the (strong) Markov property. □

For $A \subset \mathbb{C}$, define $\mathrm{rad}(A) = \sup_{x\in A}|x|$. The above result immediately implies the following fact for the Brownian amoeba $\mathfrak{A}$: $\mathbb{P}[\mathrm{rad}(\mathfrak{A}) > 2^k] \leq (1-\gamma_0)^k$ for any positive integer k. From this, it is straightforward to obtain the following result. (it is similar to Lemma 3 of [17]):

Corollary 3.9. *There is a positive constant γ_2 such that, for all $u > 1$*

$$\mathbb{P}[\mathrm{rad}(\mathfrak{A}) > u] \leq 2u^{-\gamma_2}. \tag{55}$$

4 Proofs of the main results

Proof of Theorem 2.4. For $0 < a < b < \infty$, let $\Lambda_{x,\alpha}(a,b)$ be the set of BRI's trajectories that intersect $\mathsf{B}(x,b)$ but do not intersect $\mathsf{B}(x,a)$. By construction, it holds that $\Lambda_{x,\alpha}(a,b)$

and $\Lambda_{x,\alpha}(c,d)$ are independent whenever $(a,b]\cap(c,d]=\emptyset$, and the number of trajectories in $\Lambda_{x,\alpha}(a,b)$ has Poisson distribution with parameter $\pi\alpha\big(\widehat{\mathrm{cap}}(\mathsf{B}(x,b))-\widehat{\mathrm{cap}}(\mathsf{B}(x,a))\big)$.

For $b>0$, abbreviate $r_b := \exp\big(-\frac{2\alpha\ln^2|x|}{b}\big)$. We have

$$\mathbb{P}[\widetilde{Y}_x^{(1)}(\alpha)>s] = \exp\big(-\pi\alpha\,\widehat{\mathrm{cap}}(\mathsf{B}(x,r_s))\big)$$

and, for $j\geq 1$ (and with $b=b_1+\cdots+b_j$) we can write

$$\begin{aligned}&\mathbb{P}\big[\widetilde{Y}_x^{(j+1)}(\alpha)>s \mid \widetilde{Y}_x^{(1)}(\alpha)=b_1,\ldots,\widetilde{Y}_x^{(j)}(\alpha)=b_j\big]\\&=\mathbb{P}\big[\Lambda_{x,\alpha}(r_b,r_{b+s})=\emptyset\big]\\&=\exp\big(-\pi\alpha\big(\widehat{\mathrm{cap}}(\mathsf{B}(x,r_{b+s}))-\widehat{\mathrm{cap}}(\mathsf{B}(x,r_b))\big)\big).\end{aligned}$$

Next, we use Lemma 3.2 to obtain that

$$\pi\alpha\,\widehat{\mathrm{cap}}(\mathsf{B}(x,r_h)) = h\big(1+O\big(\Psi_1^{(h)}+\Psi_2^{(h)}+\Psi_3^{(h)}\big)\big). \tag{56}$$

Using this in the above calculations together with the fact that (for bounded Ω)

$$e^{-v(1+O(\Omega))} = e^{-v}(1+vO(\Omega)),$$

it is straightforward to conclude the proof of Theorem 2.4. ☐

Proof of Theorem 2.6. For the proof of part (i), denote $\mathfrak{A}_1$ the connected component of the origin formed by the closest trajectory (the one at distance ρ_1^α); it is clear that $\mathfrak{A}_1 \overset{\text{law}}{=} \rho_1^\alpha\mathfrak{A}$ (with $\mathfrak{A}$ being a Brownian amoeba independent of $(\rho_k^\alpha, k\geq 1)$). In the following, we obtain an upper bound for the probability that another trajectory would "touch" the cell formed by the first trajectory. We have (recall (8)) $\rho_2^\alpha/\rho_1^\alpha = \exp(Y_2/(2\alpha))$, where Y_2 is an Exponential(1) random variable, independent of ρ_1^α. Therefore, we can write

$$\begin{aligned}\mathbb{P}[\mathfrak{C}_0^\alpha=\mathfrak{A}_1] &\geq \mathbb{P}\Big[\mathrm{rad}(\mathfrak{A}_1)<\rho_2^\alpha\Big]\\&=\mathbb{P}\Big[\rho_1^\alpha\mathrm{rad}(\mathfrak{A})<\rho_2^\alpha\Big]\\&=\mathbb{P}\Big[\mathrm{rad}(\mathfrak{A})<\exp\Big(\frac{Y_2}{2\alpha}\Big)\Big]\\&=\mathbb{E}\Big(\mathbb{P}\Big[\mathrm{rad}(\mathfrak{A})<\exp\Big(\frac{Y_2}{2\alpha}\Big)\;\Big|\;Y_2\Big]\Big)\end{aligned}$$

(by Corollary 3.9)

$$\begin{aligned}&\geq 1-2\mathbb{E}\exp\Big(-\frac{\gamma_2Y_2}{2\alpha}\Big)\\&=1-2\frac{1}{1+\frac{\gamma_2}{2\alpha}}\\&=1-O(\alpha)\qquad\text{as }\alpha\to 0,\end{aligned}\tag{57}$$

and this completes the proof of the part (i).

Let us now prove the part (ii). Recall the notation $\Lambda_{x,\alpha}(a,b)$ from the beginning of the proof of Theorem 2.4. Considering events in the sequence $\big(\Lambda_{x,\alpha}(e^{-n}, e^{-(n-1)}), n \in \mathbb{Z}\big)$ are independent, let us define the events

$$\Upsilon_n = \big\{\Phi_x(\alpha) \in (e^{-n}, e^{-(n-1)}]\big\},$$

and let $\sigma_{x,\alpha} = \lfloor -\ln \Phi_x(\alpha) \rfloor + 1$, i.e., $\sigma_{x,\alpha}$ is the only integer k for which Υ_k occurs. Let $\widehat{W}^{x,(n)}$ be the bi-infinite trajectory defined in the following way: let $\widehat{W}_0^{x,(n)}$ be chosen uniformly at random on $\partial\mathsf{B}(x, e^{-(n-1)})$; then $(\widehat{W}_t^{x,(n)}, t < 0)$ is distributed as $\widehat{W}^{x,e^{-(n-1)}}$ and $(\widehat{W}_t^{x,(n)}, t > 0)$ is distributed as $\widehat{W}^{x,e^{-n}}$. That is, from a random point on $\partial\mathsf{B}(x, e^{-(n-1)})$ we draw one trajectory conditioned on (immediately) escaping from $\mathsf{B}(x, e^{-(n-1)})$ and another one conditioned on never hitting $\mathsf{B}(x, e^{-n})$. Then, Lemma 3.9 of [3] implies that the cell $\mathfrak{A}^{x,(n)}$ formed by $\widehat{W}^{x,(n)}$ around x is the standard Brownian amoeba rescaled by $\inf_{t>0} \big|\widehat{W}_t^{x,(n)} - x\big|$, and then shifted to x.

An important observation is that all BRI's trajectories are actually ordered by their α-coordinates (see e.g. Remark 2.7 of [3]), so it may make sense to speak about the *first* trajectory belonging to some set of trajectories (in the case when we are able to find one with minimal α-coordinate among them). By Lemma 3.7 (and also assuring that the entrance measure of that BRI's trajectory to $\partial\mathsf{B}(x, e^{-(n-1)})$ is not far from uniform), we can have a coupling $\widehat{W}^{x,(n)}$ with the first BRI's trajectory (for all $\alpha > 0$) that intersects $\mathsf{B}(x, e^{-(n-1)})$ but does not intersect $\mathsf{B}(x, e^{-n})$, such that the traces of these coincide with high probability in the vicinity of $\mathsf{B}(x, e^{-n})$ (more precisely, in $\mathsf{B}(x, e^{-n}n^{\theta})$, where $\theta := \max(2, \gamma_2^{-1})$ with γ_2 of Corollary 3.9). Now, we are interested in proving that, with high probability, $\mathfrak{C}_x(\alpha) = \mathfrak{A}^{x,(\sigma_{x,\alpha})}$. Let us define three families of events

$$\begin{aligned} G_1^{(n)} &= \big\{\mathfrak{A}^{x,(n)} \subset \mathsf{B}(x, e^{-n}n^{\theta})\big\}, \\ G_2^{(n)} &= \big\{\Phi_x^{(2)}(\alpha) > e^{-n}n^{\theta}\big\}, \\ G_3^{(n)} &= \big\{\widehat{W}_{[-\infty,\infty]}^{x,(n)} \cap \mathsf{B}(x, e^{-n}n^{\theta}) = \widehat{W}_{[-\infty,\infty]}^{*,(n)} \cap \mathsf{B}(x, e^{-n}n^{\theta})\big\}, \end{aligned}$$

where $\widehat{W}^{*,(n)}$ stands for the first BRI's trajectory that intersects $\mathsf{B}(x, e^{-(n-1)})$ but does not intersect $\mathsf{B}(x, e^{-n})$; as suggested above, for each n, we assume that it is coupled with the $\widehat{W}^{x,(n)}$ in such a way that the probability that the corresponding traces coincide is maximized.

Recall the notation $r_b = \exp\big(-2\alpha b^{-1} \ln^2 |x|\big)$, so that $\ln r_b^{-1} = 2\alpha b^{-1} \ln^2 |x|$. Also, abbreviate $b_0 := 3\ln(\alpha \ln^2 |x|) > 1$ and $n_0 = \ln r_{b_0}^{-1} = \frac{2\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}$. Note that Theorem 2.4 implies that

$$\mathbb{P}\big[\sigma_{x,\alpha} \le n_0\big] \le O\big(\tfrac{1}{(\alpha \ln^2 |x|)^3}\big). \tag{58}$$

Then, Corollary 3.9 implies that, for $n \ge n_0$

$$\mathbb{P}\big[G_1^{(n)}\big] \ge 1 - 2n^{-\theta\gamma_2} \ge 1 - 2n^{-1} \ge 1 - \frac{3\ln(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}. \tag{59}$$

Observe that if N is a Poisson-distributed random variable, then an elementary computation implies that $\mathbb{P}[N = 1 \mid N \geq 1] \geq \mathbb{P}[N = 0]$. Therefore, quite analogously to the proof of Theorem 2.4, using (56) after some elementary calculations we can obtain for $n \geq n_0$ (note that $\Psi_1^{(h)}$ and $\Psi_3^{(h)}$ grow in h and $\Psi_2^{(h)}$ grows in h at least when $\alpha \ln^2 |x|$ is large enough)

$$\begin{aligned}
&\mathbb{P}\big[G_2^{(n)} \mid \Upsilon_n\big] \\
&= \mathbb{P}\big[\operatorname{card}\big(\Lambda_{x,\alpha}(e^{-n}, e^{-(n-1)})\big) = 1 \mid \Upsilon_n\big]\mathbb{P}\big[\Lambda_{x,\alpha}(e^{-(n-1)}, e^{-n}n^{\theta}) = \emptyset\big] \\
&\geq \mathbb{P}\big[\Lambda_{x,\alpha}(e^{-n}, e^{-(n-1)}) = \emptyset\big]\mathbb{P}\big[\Lambda_{x,\alpha}(e^{-(n-1)}, e^{-n}n^{\theta}) = \emptyset\big] \\
&= \mathbb{P}\big[\Lambda_{x,\alpha}(e^{-n}, e^{-n}n^{\theta}) = \emptyset\big] \\
&\geq 1 - O\big(\Psi^* + b_0(H_1 + H_2 + H_3)\big),
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
\Psi^* &= \frac{\ln^3(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}, \\
H_1 &= \frac{\tilde{\ell}_x \ln(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}, \\
H_2 &= \frac{\alpha \ln^2 |x| + \ln |x| \times \ln(\alpha \ln^2 |x|)}{|x| \exp\big(\frac{2\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}\big) \ln |x|}, \\
H_3 &= \frac{1 + H_1}{(|x| - 1) \exp\big(\frac{2\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}\big)}.
\end{aligned}$$

We now intend to simplify the term $O(\Psi^* + b_0(H_1 + H_2 + H_3))$. First, it can be easily seen that, as $\alpha \ln^2 |x| \to \infty$, the term H_2 is negligible in comparison to H_1. Then, assuming that $H_1 \leq 1$, we have

$$H_3 \leq \frac{2}{(|x| - 1) \exp\big(\frac{\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}\big)} \times \exp\Big(-\frac{\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}\Big);$$

since

$$\ln\Big((|x| - 1) \exp\big(\frac{\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)}\big)\Big) \geq \frac{\alpha \ln^2 |x|}{3\ln(\alpha \ln^2 |x|)} - |\ln(|x| - 1)| \geq 0, \tag{61}$$

if H_1 is small enough, we see that H_3 is at most of order of H_1. Therefore, since both Ψ^* and $b_0 H_1$ are bounded from above by $\frac{\tilde{\ell}_x \ln^3(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}$, for $n \geq n_0$ we obtain that

$$\mathbb{P}\big[G_2^{(n)} \mid \Upsilon_n\big] \geq 1 - O\big(\tfrac{\tilde{\ell}_x \ln^3(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}\big). \tag{62}$$

Next, we deal with the event $G_3^{(n)}$. To achieve a successful coupling of $\widehat{W}^{x,(n)}$ with the actual BRI(α) trajectory, we first have to estimate the probability that their entrance points to $\partial\mathsf{B}(x, e^{-(n-1)})$ can be successfully coupled. Here substituting for the moment $r := e^{-(n-1)}$ and recall Lemma 3.10 of [3]: the entrance measure of a BRI's trajectory to $\mathsf{B}(x,r)$ is the harmonic (i.e., uniform) measure $\mathrm{hm}_{\mathsf{B}(x,r)}$ biased by the logarithm of the norm of the entrance point (and obviously that, it holds $\frac{\ln|y|}{\ln|z|} = (1 + O(\frac{r}{|x|\ln|x|}))$ for $y,z \in \partial\mathsf{B}(x,r)$). Then, the entrance measure of a trajectory of $\Lambda_{x,\alpha}(r/e,r)$ is the preceding one additionally biased by the probability of not hitting $\mathsf{B}(x,r/e)$. We note that, for $y \in \partial\mathsf{B}(x,r)$, Lemma 3.5 gives us that

$$\mathbb{P}_y[\widehat{\tau}(x,r/e)=\infty] = \frac{1}{\ln\frac{e}{r} + \ell_x + \ln|x|}\Big(1 + O\big(\tfrac{r\tilde{\ell}_x}{(|x|-1)(\ln r^{-1}+\ln|x|)^2}\big)\Big);$$

this implies that for any $y,z \in \partial\mathsf{B}(x,r)$,

$$\left|\frac{\mathbb{P}_y[\widehat{\tau}(x,r/e)=\infty]}{\mathbb{P}_z[\widehat{\tau}(x,r/e)=\infty]} - 1\right| \le O\big(\tfrac{r\tilde{\ell}_x}{(|x|-1)(\ln r^{-1}+\ln|x|)^2}\big). \tag{63}$$

Therefore, the probability of successfully coupling the entrance points is at least

$$1 - O\big(\tfrac{r}{|x|\ln|x|} + \tfrac{r\tilde{\ell}_x}{(|x|-1)(\ln r^{-1}+\ln|x|)^2}\big) = 1 - O\big(\tfrac{r\tilde{\ell}_x}{(|x|-1)(\ln r^{-1}+\ln|x|)^2}\big). \tag{64}$$

Next, to couple the actual traces with Lemma 3.7, we have to assure that (44) is verified for $n \ge n_0$. It is clear that it holds for large enough $\alpha\ln^2|x|$ in the case $|x| \ge 2$, since it would be enough to check that[2] $r_{b_0}\ln^{2\theta} r_{b_0}^{-1} \le \frac{1}{2}$, which is clearly the case. Assume now that $|x| < 2$; essentially, we need to verify that $r_{b_0}\ln^{2\theta} r_{b_0}^{-1} < (|x|-1)^{1+\delta_0}$. Let $\alpha\ln^2|x|$ be large enough so that $r_{b_0}\ln^{2\theta} r_{b_0}^{-1} \le r_{b_0}^{1/2}$. Then, as in (61), we have

$$\ln\frac{(|x|-1)^{1+\delta_0}}{r_{b_0}^{1/2}} \ge \frac{\alpha\ln^2|x|}{3\ln(\alpha\ln^2|x|)} - (1+\delta_0)\big|\ln(|x|-1)\big| > 0 \tag{65}$$

if $\frac{\tilde{\ell}_x\ln(\alpha\ln^2|x|)}{\alpha\ln^2|x|}$ is small enough, so (44) indeed holds.

Then, using Lemma 3.7 (for both positive and negative sides of the trajectories, which are independent except for their starting point), we obtain for $n \ge n_0$ (recall (64))

$$\mathbb{P}\big[G_3^{(n)} \mid \Upsilon_n\big] \ge 1 - O\big(\tfrac{\ln(\alpha b_0^{-1}\ln^2|x|)}{\ln|x|+\alpha b_0^{-1}\ln^2|x|} + \tfrac{\tilde{\ell}_x r_{b_0}\ln^{2\theta} r_{b_0}^{-1}}{|x|-1} + \tfrac{r\tilde{\ell}_x}{(|x|-1)(\ln r^{-1}+\ln|x|)^2}\big) \tag{66}$$

(again, it is not difficult to see that the second term with generic $b \in (0,b_0)$ is majorized by the term with b_0, in the case when $\alpha\ln^2|x|$ is large enough). We have that

$$\frac{\ln(\alpha b_0^{-1}\ln^2|x|)}{\ln|x| + \alpha b_0^{-1}\ln^2|x|} \le \frac{3\ln^2(\alpha\ln^2|x|)}{\alpha\ln^2|x|}.$$

[2]Observe that the function $f(x) = x\ln^{2\theta} x^{-1}$ is increasing for $x \in (0, e^{-2\theta})$.

As for the second term, it is clearly of smaller order when $|x| \geq 2$. If $1 < |x| < 2$, we have $\tilde{\ell}_x \leq O\big((|x|-1)^{-1}\big)$ so we can write

$$\frac{\tilde{\ell}_x r_{b_0} \ln^{2\theta} r_{b_0}^{-1}}{|x|-1} \leq O\Big(\frac{r_{b_0}^{1/2}}{(|x|-1)^2}\Big) \leq r_{b_0}^{1/4} \times O\Big(\frac{r_{b_0}^{1/4}}{(|x|-1)^2}\Big);$$

analogously to (61) and (65), we can show that $\frac{r_{b_0}^{1/4}}{(|x|-1)^2} \leq 1$ (assuming that $\frac{\tilde{\ell}_x \ln(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}$ is small enough), which allows us to conclude that the second term is of a smaller order than the first one in that case as well. The third term can be treated quite analogously to the second one, so we obtain

$$\mathbb{P}\big[G_3^{(n)} \mid \Upsilon_n\big] \geq 1 - O\big(\tfrac{\tilde{\ell}_x \ln^2(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}\big) \tag{67}$$

for $n \geq n_0$.

We can now estimate the probability of the event of interest. It remains to write using (58), (59), (62), (67)

$$\begin{aligned}
\mathbb{P}\big[\mathfrak{C}_x(\alpha) = \mathfrak{A}^{x,(\sigma_{x,\alpha})}\big] &= \mathbb{E}\big(\mathbb{P}\big[\mathfrak{C}_x(\alpha) = \mathfrak{A}^{x,(\sigma_{x,\alpha})} \mid \sigma_{x,\alpha}\big]\big) \\
&\geq \mathbb{E}\big(\mathbb{P}\big[\mathfrak{C}_x(\alpha) = \mathfrak{A}^{x,(\sigma_{x,\alpha})} \mid \sigma_{x,\alpha}\big]\mathbf{1}\{\sigma_{x,\alpha} \geq n_0\}\big) \\
&\geq \mathbb{P}\big[\sigma_{x,\alpha} \geq n_0\big] \times \min_{n \geq n_0} \mathbb{P}\big[G_1^{(n)} \cap G_2^{(n)} \cap G_3^{(n)} \mid \Upsilon_n\big] \\
&\geq 1 - O\big(\tfrac{\tilde{\ell}_x \ln^3(\alpha \ln^2 |x|)}{\alpha \ln^2 |x|}\big).
\end{aligned}$$

This concludes the proof of Theorem 2.6. □

Proof of Theorem 2.7. We need to prove that for any $\delta > 0$ it holds that

$$(1-2\delta)\sqrt{\frac{\ln \alpha}{2\alpha}} \leq M_\alpha \leq (1+2\delta)\sqrt{\frac{\ln \alpha}{2\alpha}} \tag{68}$$

eventually for all large enough α.

We start by showing that the second inequality in (68) (with δ on the place of 2δ) is verified with high probability. Let us place $n_{\delta,\alpha} := \lceil \frac{6\pi}{\delta}\sqrt{\frac{2\alpha}{\ln \alpha}}\rceil$ points $x_1, \ldots, x_{n_{\delta,\alpha}}$ on $\partial\mathsf{B}(1)$ in such a way that every closed arc of length $\frac{\delta}{3}\sqrt{\frac{\ln \alpha}{2\alpha}}$ contains at least one of these points. Then, we have the following inclusion:

$$\Big\{M_\alpha > (1+\delta)\sqrt{\frac{\ln \alpha}{2\alpha}}\Big\} \subset \bigcup_{k=1}^{n_{\delta,\alpha}} \Big\{\Phi_{x_k}(\alpha) > \Big(1+\frac{\delta}{3}\Big)\sqrt{\frac{\ln \alpha}{2\alpha}}\Big\}. \tag{69}$$

Next, we recall that Lemma 3.12 of [3] states that, for $x \in \partial\mathsf{B}(1)$ and small enough $r > 0$,

$$\widehat{\text{cap}}\,\big(\mathsf{B}(x,r)\big) = \text{cap}\,\big(\mathsf{B}(1) \cup \mathsf{B}(x,r)\big) = \frac{r^2}{\pi}\big(1+O(r)\big), \tag{70}$$

so, by (7)

$$\mathbb{P}\Big[\Phi_{x_k}(\alpha) > \Big(1+\frac{\delta}{3}\Big)\sqrt{\frac{\ln\alpha}{2\alpha}}\Big] = \exp\Big(-\frac{(1+\frac{\delta}{3})^2\ln\alpha}{2}\big(1+o(1)\big)\Big) \le O\big(\alpha^{-\frac{1}{2}(1+\frac{2\delta}{3})}\big);$$

by the union bound, this shows that

$$\mathbb{P}\Big[M_\alpha > (1+\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}\Big] \le O\big(\alpha^{-\delta/3}\big). \tag{71}$$

Now, we need to obtain an upper-bound on the probability that M_α does not exceed $(1-\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}$. Denote $m_{\delta,\alpha} = \lceil \alpha^{\frac{1}{2}(1-\frac{\delta}{3})}\rceil$, and let $y_1,\dots,y_{m_{\delta,\alpha}}$ be points placed on $\partial\mathsf{B}(1)$ with equal spacing between neighbouring ones (so that the distance between the neighbouring ones is of order $\alpha^{-\frac{1}{2}(1-\frac{\delta}{3})}$). Again, by (7) and (70) for large enough α we have that for any k

$$\begin{aligned}\mathbb{P}\Big[\Phi_{y_k}(\alpha) \le (1-\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}\Big] &= 1-\exp\Big(-\frac{(1-\delta)^2\ln\alpha}{2}\big(1+o(1)\big)\Big)\\ &\le 1-\alpha^{-\frac{1}{2}(1-\delta)}.\end{aligned} \tag{72}$$

Next, abbreviate $A_k = \mathsf{B}\big(y_k,(1-\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}\big)\setminus\mathsf{B}(1)$, and consider all the interlacement trajectories that has an intersection with $\bigcup_{k=1}^{m_{\delta,\alpha}} A_k$. We say that such a trajectory is of type 1 if it intersects only one of the A_k's, and of type 2 if it intersects several of them. By construction, the numbers of type-1 trajectories that intersect A_k are *independent* Poisson random variables for $k=1,\dots,m_{\delta,\alpha}$, and they are also independent of the process of type-2 trajectories. Note also that, by (70), the total number of trajectories (i.e., of both types) that intersect A_k is Poisson with parameter $\frac{1}{2}(1-\delta)^2\ln\alpha\times\big(1+o(1)\big)$; this means that the number of type-1 trajectories that intersect A_k is dominated by a Poisson random variable with that parameter. Therefore, from (72) we obtain

$$\begin{aligned}&\mathbb{P}\big[\text{there exists } j \text{ such that no type-1 trajectory hits } A_j\big]\\ &\qquad\ge 1-\big(1-\alpha^{-\frac{1}{2}(1-\delta)}\big)^{m_{\delta,\alpha}}\\ &\qquad\ge 1-\exp(-\alpha^{\delta/3}).\end{aligned} \tag{73}$$

Now, we work with the process of type-2 trajectories. First, we claim that, for any k and any $x\in A_k$

$$\mathbb{P}_x\Big[\widehat{W} \text{ hits } \bigcup_{\ell\neq k} A_\ell\Big] \le O\big(\alpha^{-\delta/3}\ln\alpha\big). \tag{74}$$

To show the above, we use Proposition 2.3 together with the fact that the Bes(3) process can be represented as the norm of the 3-dimensional Brownian motion. Assume without restricting generality that $k=1$ in (74), and that $y_1=1$. Denote $r_{\delta,\alpha}=(1-\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}$ and let A'_j be the pre-image of A_j with respect to the exponential map. It is clear that, for fixed δ and large enough α we have $A'_j \subset [0, 2r_{\delta,\alpha}] \times I_j$, where I_j is the interval of length $4r_{\delta,\alpha}$ centred at the pre-image of y_j. For $j \in \mathbb{Z}$, denote also

$$B'_j = [0, 2r_{\delta,\alpha}] \times \Big[\frac{2\pi}{m_{\delta,\alpha}} j - 2r_{\delta,\alpha}, \frac{2\pi}{m_{\delta,\alpha}} j + 2r_{\delta,\alpha}\Big] \subset \mathbb{R}_+ \times \mathbb{R}.$$

Let $\varphi : \mathbb{R}^4 \to \mathbb{R}_+ \times \mathbb{R}$ be defined by $\varphi(x, y, z, t) = (\sqrt{x^2 + y^2 + z^2}, t)$, and let $W^{(4)}$ be the standard Brownian motion in $\mathbb{R}^4$ started at the origin. We then have, for $x' \in A'_1$

$$\mathbb{P}_{x'}\Big[(Z, B) \text{ hits } \bigcup_{2\leq \ell \leq m_{\delta,\alpha}} A'_\ell\Big] \leq \mathbb{P}_{\varphi^{-1}(x')}\Big[W^{(4)} \text{ hits } \bigcup_{\ell \in \mathbb{Z}\setminus\{0\}} \varphi^{-1}(B'_\ell)\Big].$$

Recall that, in $\mathbb{R}^4$, we have (momentarily "lifting" the notation $\mathsf{B}(\cdot,\cdot)$ to that dimension) $\mathbb{P}_x[W^{(4)} \text{ hits } \mathsf{B}(y, r)] \asymp \big(\frac{r}{|x-y|}\big)^2$. Then, $\varphi^{-1}(B'_\ell)$ is a cylinder of the linear size of the order $\sqrt{\frac{\ln\alpha}{\alpha}}$ (so it fits into a ball of the size of the same order), and the distance between $\varphi^{-1}(x')$ and $\varphi^{-1}(B'_\ell)$ is of order $|\ell|\alpha^{-\frac{1}{2}(1-\frac{\delta}{3})}$. This indeed shows that (74) holds (note that $\sqrt{\frac{\ln\alpha}{\alpha}}/\alpha^{-\frac{1}{2}(1-\frac{\delta}{3})} = \alpha^{-\delta/6}\sqrt{\ln\alpha}$, and we also have to use the fact that $\sum_{\ell\in\mathbb{Z}\setminus\{0\}} |\ell|^{-2} < \infty$).

Now, (74) shows that the number of type-2 trajectories that hits a set A_k has expectation at most $O\big(\alpha^{-\delta/3}\ln^2\alpha\big)$ (recall that the total number of trajectories hitting A_k is Poisson with parameter $O(\ln\alpha)$). Using the fact that the process of type-2 trajectories is independent from all the type-1 ones, we find that if $\zeta \in \{1, \dots, m_{\delta,\alpha}\}$ is a (random) index such that no type-1 trajectory touches A_ζ (by (73), such ζ exists with probability at least $1 - \exp(-\alpha^{\delta/3})$), then also no type-2 trajectory touches A_ζ with probability at least $1 - O\big(\alpha^{-\delta/3}\ln^2\alpha\big)$. This means that

$$\mathbb{P}\Big[M_\alpha < (1-\delta)\sqrt{\frac{\ln\alpha}{2\alpha}}\Big] \leq O\big(\alpha^{-\delta/3}\ln^2\alpha\big). \tag{75}$$

Together with (71), this implies the convergence in probability in (17); now, we will deduce the a.s. convergence using the monotonicity of M_α. Indeed, let $\alpha_k = e^{\sqrt{k}}$. By the Borel-Cantelli lemma, for any fixed $\delta > 0$ the estimates (71) and (75) imply that

$$(1-\delta)\sqrt{\frac{\ln\alpha_k}{2\alpha_k}} \leq M_{\alpha_k} \leq (1+\delta)\sqrt{\frac{\ln\alpha_k}{2\alpha_k}} \tag{76}$$

for all but finitely many k. We have that $M_{\alpha_{k+1}} \leq M_\alpha \leq M_{\alpha_k}$ for all $\alpha \in [\alpha_k, \alpha_{k+1}]$, and it is also elementary to verify that

$$\frac{\sqrt{\frac{\ln\alpha_{k+1}}{2\alpha_{k+1}}}}{\sqrt{\frac{\ln\alpha_k}{2\alpha_k}}} \to 1 \text{ as } k \to \infty.$$

This allows us to deduce (68) from (76) and therefore concludes the proof of Theorem 2.7. □

Acknowledgements

The authors were partially supported by CMUP, member of LASI, which is financed by national funds through FCT — Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020. The authors thank the anonymous referees for their helpful comments and suggestions (in particular, for observing that one can obtain the a.s. convergence in Theorem 2.7 using the monotonicity of M_α).

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Received: October 15, 2024
Accepted for publication: January 30, 2025
Communicated by: Ivan Kaygorodov