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On polynomial equations over split octonions

Artem Lopatin, Alexander Rybalov

Abstract. Working over the split octonions over an algebraically closed field, we solve all polynomial equations in which all the coefficients but the constant term are scalar. As a consequence, we calculate the $n^{-\text{th}}$ roots of an octonion.

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1 Introduction

Assume that \mathbb{F} is a field of an arbitrary characteristic $p = \operatorname{char} \mathbb{F} \ge 0$. All vector spaces and algebras are over \mathbb{F} .

The problem of solving polynomial equations was historically considered as one of key problems in mathematics, which influenced the creation of algebraic geometry and other branches of mathematics. Polynomial equations were considered not only over fields, but also over matrix algebras, algebras of quaternions, octonions, etc. Rodríguez-Ordóñez [30]

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proved that every polynomial equation over the algebra $\mathbf{A}_{\mathbb{R}}$ of *Cayley numbers* (i.e., the division algebra of real octonions) of positive degree with the only term of the highest degree has a solution. An explicit algorithm for a solution of the quadratic equations $x^2 + bx + c = 0$ over $\mathbf{A}_{\mathbb{R}}$ was obtained by Wag, Zhang and Zhang [32] together with criterions whether this equation has one, two or infinitely many solutions.

In general, an octonion algebra \mathbf{C} (or a Cayley algebra) over the field \mathbb{F} is a nonassociative alternative unital algebra of dimension 8, endowed with a non-singular quadratic multiplicative form $n : \mathbf{C} \to \mathbb{F}$, which is called the *norm*. The norm *n* is called *isotropic* if n(a) = 0 for some non-zero $a \in \mathbf{C}$, otherwise the norm *n* is anisotropic. In case *n* is anisotropic, the octonion algebra \mathbf{C} is a division algebra. In case *n* is isotropic, there exists a unique octonion algebra $\mathbf{O}_{\mathbb{F}}$ over \mathbb{F} with isotropic norm (see Theorem 1.8.1 of [31]). This algebra is called the *split octonion algebra*. Note that, if \mathbb{F} is algebraically closed, then any octonion algebra is isomorphic to the split octonion algebra $\mathbf{O}_{\mathbb{F}}$ (for example, see Lemma 2.2 from [28]). Since Artin's theorem claims that in an alternative algebra every subalgebra generated by two elements is associative, then any octonion algebra is power-associative, i.e., the subalgebra generated by a single element is associative. Therefore, given $a \in \mathbf{C}$, we can write down a^n without specifying the brackets in the product.

Flaut and Shpakivskyi [17] considered the equation $x^n = a$ over real octonion division algebras. For an octonion division algebra **C** over an arbitrary field \mathbb{F} , Chapman [7] presented a complete method for finding the solutions of the polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ over **C**. Moreover, Chapman and Vishkautsan [9], working over a division algebra **C**, determined the solutions of the polynomial equation $(a_nc)x^n + (a_{n-1}c)x^{n-1} + \cdots + (a_1c)x + (a_0c) = 0$ and discussed the solutions of the polynomial equation $(a_nc)x^n + (a_{n-1}c)x^{n-1} + \cdots + (a_1c)x + (a_0c) = 0$ and discussed the solutions of the polynomial equation $(ca_n)x^n + (ca_{n-1})x^{n-1} + \cdots + (ca_1)x + (ca_0) = 0$. Chapman and Levin [8] described a method for finding so-called "alternating roots" of polynomials over an arbitrary division Cayley-Dickson algebra. Chapman and Vishkautsan [10] examined the conditions under which, for a root a of a polynomial f(x) over a general Cayley–Dickson algebra, there exists a factorization f(x) = g(x)(x - a) for some polynomial g(x). The case of polynomial equations over an arbitrary algebra has recently been considered by Illmer and Netzer in [22], where some conditions were determined that imply the existence of a common solution of n polynomial equations in n variables, with an application to polynomial equations over $\mathbf{A}_{\mathbb{R}}$.

Assume that \mathcal{A} is a unital algebra over \mathbb{F} . Consider a general polynomial equation $f(a_1, \ldots, a_m, x) = 0$ over \mathcal{A} , i.e., $f(a_1, \ldots, a_m, x)$ is an element of the absolutely free unital algebra $\mathbb{F}\langle a_1, \ldots, a_m, x \rangle$ with free generators a_1, \ldots, a_m, x . Here $x \in \mathcal{A}$ is a variable and $a_1, \ldots, a_m \in \mathcal{A}$ are coefficients. We aim to describe a method for calculating x when a_1, \ldots, a_m are given. Note that we can also consider $f(a_1, \ldots, a_m, x)$ as an element of the relatively free algebra $\mathbb{F}_{\mathcal{A}}\langle a_1, \ldots, a_m, x \rangle := \mathbb{F}\langle a_1, \ldots, a_m, x \rangle/\operatorname{Id}(\mathcal{A})$ for \mathcal{A} , where $\operatorname{Id}(\mathcal{A})$ stands for the T-ideal of all polynomial identities for \mathcal{A} . By knowing an \mathbb{F} -basis for $\mathbb{F}_{\mathcal{A}}\langle a_1, \ldots, a_m, x \rangle$, we can obtain a canonical form for f(x). Therefore, a problem of solving a polynomial equation over an algebra \mathcal{A} is tightly connected with the problem of explicit description of polynomial identities for \mathcal{A} . Although the theory of algebras with polynomial identities is a well-developed area of algebra with many deep results, there

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are still few results offering an explicit description of generators of T-ideals of polynomial identities for particular finite-dimensional algebras (see [11-14, 16]

for recent results). Due to the difficulty of solving a general equation $f(a_1, \ldots, a_m, x) = 0$ over \mathcal{A} , it may be interesting to consider this equation over some vector subspace $\mathcal{V} \subset \mathcal{A}$ generating the algebra \mathcal{A} , i.e., to assume that $a_1, \ldots, a_m, x \in \mathcal{V}$. In this case, instead of polynomial identities, we should use so-called weak polynomial identities for the pair $(\mathcal{A}, \mathcal{V})$ (see survey [15] and papers [23, 25–27] for recent results on weak polynomial identities).

The split-octonions have numerous applications to physics. As an example, the Dirac equation, which describes the motion of a free spin 1/2 particle, such as an electron or a proton, can be represented by the split-octonions (see [18–20]). There exist applications of split-octonions to electromagnetic theory (see [4–6]), geometrodynamics (see [3]), unified quantum theories (see [1, 2, 24]), special relativity (see [21]).

In a recent paper, Lopatin and Zubkov [28] considered the linear equations ax = c, (ax)b = c, a(bx) = c over the split octonion algebra **O** in case \mathbb{F} is algebraically closed. Note that, over a division octonion algebra, these equations can easily be solved, and for non-zero a, b there is a unique solution. However, in the case of an algebraically closed field, the situation is drastically different. Specifically, the set X of all solutions of one of the above linear equations with non-zero a, b is either empty, or contains the only element, or the affine variety X has dimension r, where

- r = 4 in case we consider the equation ax = c;
- $r \in \{4, 5, 7\}$ in case we consider the equation (ax)b = c;
- $r \in \{4, 6, 8\}$ in case we consider the equation a(bx) = c.

In Sections 2 and 3 we assume that the field \mathbb{F} is algebraically closed. In Section 2 we explicitly define the octonion algebra \mathbf{O} and its group of automorphisms $\operatorname{Aut}(\mathbf{O}) = G_2$. In Section 3 we solve every equation

$$\alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_1 x = c \tag{1.1}$$

with scalar $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ and possibly non-scalar constant term $c \in \mathbf{O}$ with respect to the variable $x \in \mathbf{O}$ (see Theorem 3.2). The solution of equation (1.1) is obtained modulo solution of polynomial equations over \mathbb{F} . In Corollary 3.3 we explicitly describe the number of solutions of equation (1.1). In Corollary 3.4 we apply the obtained general result to the $n^{-\text{th}}$ roots of $c \in \mathbf{O}$, i.e., to the solutions of the equation $x^n = c$.

2 Octonions

In this section we assume that the field \mathbb{F} is algebraically closed.

2.1 Split-octonions

The split octonion algebra $\mathbf{O} = \mathbf{O}(\mathbb{F})$, also known as the split Cayley algebra, is the vector space of all matrices

$$a = \begin{pmatrix} \alpha & \mathbf{u} \\ \mathbf{v} & \beta \end{pmatrix}$$
 with $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$,

together with the multiplication given by the following formula:

$$aa' = \begin{pmatrix} \alpha\alpha' + \mathbf{u} \cdot \mathbf{v}' & \alpha\mathbf{u}' + \beta'\mathbf{u} - \mathbf{v} \times \mathbf{v}' \\ \alpha'\mathbf{v} + \beta\mathbf{v}' + \mathbf{u} \times \mathbf{u}' & \beta\beta' + \mathbf{v} \cdot \mathbf{u}' \end{pmatrix}, \text{ where } a' = \begin{pmatrix} \alpha' & \mathbf{u}' \\ \mathbf{v}' & \beta' \end{pmatrix},$$

 $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ and $\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$. For short, we denote $\mathbf{c}_1 = (1, 0, 0)$, $\mathbf{c}_2 = (0, 1, 0)$, $\mathbf{c}_3 = (0, 0, 1)$, $\mathbf{0} = (0, 0, 0)$ from \mathbb{F}^3 . Consider the following basis of \mathbf{O} :

$$e_1 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \ \mathbf{u}_i = \begin{pmatrix} 0 & \mathbf{c}_i \\ \mathbf{0} & 0 \end{pmatrix}, \ \mathbf{v}_i = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{c}_i & 0 \end{pmatrix}$$

for i = 1, 2, 3. The unity of **O** is denoted by $1_{\mathbf{O}} = e_1 + e_2$. We identify octonions

$$\alpha \mathbf{1}_{\mathbf{0}}, \left(\begin{array}{cc} 0 & \mathbf{u} \\ \mathbf{0} & 0 \end{array}\right), \left(\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{v} & 0 \end{array}\right)$$

with $\alpha \in \mathbb{F}$, $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$, respectively. Note that $\mathbf{u}_i \mathbf{u}_j = (-1)^{\epsilon_{ij}} \mathbf{v}_k$ and $\mathbf{v}_i \mathbf{v}_j = (-1)^{\epsilon_{ji}} \mathbf{u}_k$, where $\{i, j, k\} = \{1, 2, 3\}$ and ϵ_{ij} is the parity of the permutation $\begin{pmatrix} 1 & 2 & 3 \\ k & i & j \end{pmatrix}$.

The algebra \mathbf{O} is endowed with the linear involution

$$\overline{a} = \left(\begin{array}{cc} \beta & -\mathbf{u} \\ -\mathbf{v} & \alpha \end{array}\right),$$

which satisfies the equality $\overline{aa'} = \overline{a'a}$, a norm $n(a) = a\overline{a} = \alpha\beta - \mathbf{u}\cdot\mathbf{v}$, and a non-degenerate symmetric bilinear form $q(a, a') = n(a+a') - n(a) - n(a') = \alpha\beta' + \alpha'\beta - \mathbf{u}\cdot\mathbf{v}' - \mathbf{u}'\cdot\mathbf{v}$. Define the linear function trace by $\operatorname{tr}(a) = a + \overline{a} = \alpha + \beta$. The subspace of traceless octonions is denoted by $\mathbf{O}_0 = \{a \in \mathbf{O} \mid \operatorname{tr}(a) = 0\}$ and the affine variety of octonions with zero norm is denoted by $\mathbf{O}_{\#} = \{a \in \mathbf{O} \mid n(a) = 0\}$. Notice that

$$tr(aa') = tr(a'a) \text{ and } n(aa') = n(a)n(a').$$
 (2.1)

The next quadratic equation holds:

$$a^{2} - \operatorname{tr}(a)a + n(a) = 0.$$
(2.2)

The algebra **O** is a simple *alternative* algebra, i.e., the following identities hold for $a, b \in \mathbf{O}$:

$$a(ab) = (aa)b, (ba)a = b(aa).$$
 (2.3)

Moreover,

$$\overline{a}(ab) = n(a)b, \quad (ba)\overline{a} = n(a)b. \tag{2.4}$$

The following remark is well-known and can easily be proven.

Remark 2.1. Given an $a \in \mathbf{O}$, one of the following cases holds:

- if $n(a) \neq 0$, then there exist unique $b, c \in \mathbf{O}$ such that $ba = 1_{\mathbf{O}}$ and $ac = 1_{\mathbf{O}}$; moreover, in this case we have $b = c = \overline{a}/n(a)$ and we denote $a^{-1} := b = c$.
- if n(a) = 0, then a does not have a left inverse as well as a right inverse.

Equalities (2.4) imply that for $a \notin \mathbf{O}_{\#}$ we have

$$a^{-1}(ab) = b, \ (ba)a^{-1} = b.$$
 (2.5)

2.2 The group G_2

The group $\operatorname{Aut}(\mathbf{O})$ of all automorphisms of the algebra \mathbf{O} is the exceptional simple group $\operatorname{G}_2 = \operatorname{G}_2(\mathbb{F})$. The group G_2 contains a Zariski closed subgroup $\operatorname{SL}_3 = \operatorname{SL}_3(\mathbb{F})$. Namely, every $g \in \operatorname{SL}_3$ defines the following automorphism of \mathbf{O} :

$$a \to \left(\begin{array}{cc} \alpha & \mathbf{u}g \\ \mathbf{v}g^{-T} & \beta \end{array} \right),$$

where g^{-T} stands for $(g^{-1})^T$ and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$ are considered as row vectors. For every $\mathbf{u}, \mathbf{v} \in \mathbf{O}$ define $\delta_1(\mathbf{u}), \delta_2(\mathbf{v})$ from Aut(\mathbf{O}) as follows:

$$\delta_{1}(\mathbf{u})(a') = \begin{pmatrix} \alpha' - \mathbf{u} \cdot \mathbf{v}' & (\alpha' - \beta' - \mathbf{u} \cdot \mathbf{v}')\mathbf{u} + \mathbf{u}' \\ \mathbf{v}' - \mathbf{u}' \times \mathbf{u} & \beta' + \mathbf{u} \cdot \mathbf{v}' \end{pmatrix},$$

$$\delta_{2}(\mathbf{v})(a') = \begin{pmatrix} \alpha' + \mathbf{u}' \cdot \mathbf{v} & \mathbf{u}' + \mathbf{v}' \times \mathbf{v} \\ (-\alpha' + \beta' - \mathbf{u}' \cdot \mathbf{v})\mathbf{v} + \mathbf{v}' & \beta' - \mathbf{u}' \cdot \mathbf{v} \end{pmatrix}.$$

The group G₂ is generated by SL₃ and $\delta_1(t\mathbf{u}_i), \delta_2(t\mathbf{v}_i)$ for all $t \in \mathbb{F}$ and i = 1, 2, 3 (for example, see Section 3 of [33]). By straightforward calculations, it is easy to see that

$$\hbar: \mathbf{O} \to \mathbf{O}, \text{ defined by } a \to \begin{pmatrix} \beta & -\mathbf{v} \\ -\mathbf{u} & \alpha \end{pmatrix},$$
 (2.6)

belongs to G_2 .

The action of G_2 on **O** satisfies the next properties:

$$\overline{ga} = \overline{ga}, \text{ tr}(ga) = \text{tr}(a), n(ga) = n(a), q(ga, ga') = q(a, a').$$

Thus, G_2 acts also on \mathbf{O}_0 and $\mathbf{O}_{\#}$. The group G_2 acts diagonally on the vector space $\mathbf{O}^n = \mathbf{O} \oplus \cdots \oplus \mathbf{O}$ (*n* times) by $g(a_1, \ldots, a_n) = (ga_1, \ldots, ga_n)$ for all $g \in G_2$ and $a_1, \ldots, a_n \in \mathbf{O}$.

We fix a binary relation < on the field \mathbb{F} such that for each pair $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$ exactly one of $\alpha < \beta$ or $\beta < \alpha$ holds. Note that, we do not assume that < is transitive, and we do not assume compatibility with the field operations.

Proposition 2.2 (Part 1 of Proposition 3.3 from [29]). The following set is a minimal set of representatives of G_2 -orbits on O:

1. $\alpha 1_0$,

2. $\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_2 \end{pmatrix}$ with $\alpha_1 < \alpha_2$, 3. $\begin{pmatrix} \alpha & (1,0,0) \\ \mathbf{0} & \alpha \end{pmatrix}$,

where $\alpha, \alpha_1, \alpha_2 \in \mathbb{F}$. In other words, **O** is the disjoint union of the following G₂-orbits:

$$\alpha 1_{\mathbf{O}}, \ O_2(\alpha_1, \alpha_2) := G_2(\alpha_1 e_1 + \alpha_2 e_2), \ O_3(\alpha) := G_2(\alpha 1_{\mathbf{O}} + \mathbf{u}_1),$$

where $\alpha_1 < \alpha_2, \ \alpha, \alpha_1, \alpha_2 \in \mathbb{F}$.

Definition 2.3. • The elements from Proposition 2.2 are called *canonical octonions*.

• Given $a \in \mathbf{O}$, the diagonal elements of the canonical octonion from G_2a are called eigenvalues $(\alpha_1, \alpha_2) \in \mathbb{F}^2$ for a, where $\alpha_1 \leq \alpha_2$. Note that both eigenvalues are solutions of the equation $\alpha^2 - \operatorname{tr}(a)\alpha + n(a) = 0$.

The following remark is a consequence of Part 2 of Proposition 3.3 from [29].

Remark 2.4. Assume $\alpha, \alpha_i, \beta, \beta_i, \gamma_i \in \mathbb{F}$ for i = 1, 2.

1. Two octonions $\alpha_1 \mathbf{1}_{\mathbf{O}} + \beta_1 \mathbf{u}_1$ and $\alpha_2 \mathbf{1}_{\mathbf{O}} + \beta_2 \mathbf{u}_1$ belong to the same G₂-orbit on **O** if and only if

- $\alpha_1 = \alpha_2$,
- either $\beta_1 = \beta_2 = 0$, or β_1 , β_2 are non-zero.

2. If $\gamma_1 \neq \gamma_2$, then $\alpha \mathbf{1}_{\mathbf{O}} + \beta \mathbf{u}_1 \notin O_2(\gamma_1, \gamma_2)$.

3 Polynomial equations

In this section we assume that the field \mathbb{F} is algebraically closed. Given a commutative associative polynomial $f(\xi) = \alpha_n \xi^n + \cdots + \alpha_1 \xi + \alpha_0 \in \mathbb{F}[\xi]$, where $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$, we write $f'(\xi)$ for its derivative. For each $x \in \mathbf{O}$ we naturally define the substitution

$$f(x) = \alpha_n x^n + \dots + \alpha_1 x + \alpha_0 \mathbf{1}_{\mathbf{O}} \in \mathbf{O}.$$

Assume that $f(\xi) \in \mathbb{F}[\xi]$ is a non-zero polynomial without constant term and $\gamma \in \mathbb{F}$. The multiplicity of a root $\xi_1 \in \mathbb{F}$ for the equation $f(\xi) = \gamma$, where $\xi \in \mathbb{F}$ is a variable, is k > 0 such that $f(\xi) - \gamma = (\xi - \xi_1)^k g(\xi)$ for some $g(\xi) \in \mathbb{F}[\xi]$ with $g(\xi_1) \neq 0$. If k = 1, then root ξ_1 is called *simple*. If $k \geq 2$, then the root ξ_1 is called *multiple*. It is trivial that

• ξ_1 is a simple root if and only if $f(\xi_1) = \gamma$ and $f'(\xi_1) \neq 0$;

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• ξ_1 is a multiple root if and only if $f(\xi_1) = \gamma$ and $f'(\xi_1) = 0$.

Lemma 3.1. Assume that $f(\xi) \in \mathbb{F}[\xi]$ and $\alpha, \beta \in \mathbb{F}$. Then

$$f(\alpha \mathbf{1}_{\mathbf{O}} + \beta \mathbf{u}_1) = f(\alpha)\mathbf{1}_{\mathbf{O}} + f'(\alpha)\beta \mathbf{u}_1.$$

Proof. For short, denote $a = \alpha \mathbf{1}_{\mathbf{0}} + \beta \mathbf{u}_{\mathbf{1}}$.

1. Assume that $f(\xi) = \xi^n$ for some n > 0. We prove by induction on n that

$$a^n = \alpha^n \mathbf{1}_{\mathbf{O}} + n\alpha^{n-1}\beta \mathbf{u}_1. \tag{3.1}$$

In case n = 1 claim (3.1) is trivial.

Assume that claim (3.1) holds for some n > 1. Then

$$a^{n+1} = (\alpha^n \mathbf{1}_{\mathbf{O}} + n\alpha^{n-1}\beta \mathbf{u}_1)(\alpha \mathbf{1}_{\mathbf{O}} + \beta \mathbf{u}_1) = \alpha^{n+1}\mathbf{1}_{\mathbf{O}} + n\alpha^n\beta \mathbf{u}_1 + \alpha^n\beta \mathbf{u}_1 =$$
$$= \alpha^{n+1}\mathbf{1}_{\mathbf{O}} + (n+1)\alpha^n\beta \mathbf{u}_1.$$

Therefore, claim (3.1) holds for every n > 0.

2. Assume that $f(\xi) = \alpha_n \xi^n + \cdots + \alpha_1 \xi + \alpha_0 \in \mathbb{F}[\xi]$ for some $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$ and $n \ge 0$. Note that in case n = 0 we have $f(a) = \alpha_0 \mathbf{1}_{\mathbf{0}}$ and the claim of the lemma holds.

For n > 0 we apply part 1 to obtain that

$$f(a) = \sum_{i=1}^{n} \alpha_i a^i + \alpha_0 \mathbf{1}_{\mathbf{O}} = \sum_{i=1}^{n} \alpha_i (\alpha^i \mathbf{1}_{\mathbf{O}} + i\alpha^{i-1}\beta \mathbf{u}_1) + \alpha_0 \mathbf{1}_{\mathbf{O}}$$

and the required statement is proven.

Theorem 3.2. Assume that $f(\xi) \in \mathbb{F}[\xi]$ is a non-zero polynomial without constant term and $c \in \mathbf{O}$. Acting by G_2 on the equation f(x) = c, where $x \in \mathbf{O}$ is a variable, we can assume that c is a canonical octonion from Proposition 2.2. Let $X \subset \mathbf{O}$ be the set of all solutions of the equation f(x) = c. Then

- 1. in case $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{F}$, we have
 - $X = \begin{cases} \xi_1 \mathbf{1}_{\mathbf{0}} \mid \xi_1 \in \mathbb{F} \text{ satisfies } f(\xi_1) = \gamma \} \bigcup \\ \{O_2(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \mathbb{F} \text{ satisfy } f(\xi_1) = f(\xi_2) = \gamma \text{ and } \xi_1 < \xi_2 \} \bigcup \\ \{O_3(\xi_1) \mid \xi_1 \in \mathbb{F} \text{ is a multiple root for } f(\xi) = \gamma \}; \end{cases}$

2. in case $c = \gamma_1 e_1 + \gamma_2 e_2$ for some $\gamma_1, \gamma_2 \in \mathbb{F}$ with $\gamma_1 < \gamma_2$, we have

$$X = \{\xi_1 e_1 + \xi_2 e_2 \mid \xi_1, \xi_2 \in \mathbb{F} \text{ satisfy } f(\xi_1) = \gamma_1 \text{ and } f(\xi_2) = \gamma_2\};\$$

3. in case $c = \gamma 1_{\mathbf{O}} + \mathbf{u}_1$ for some $\gamma \in \mathbb{F}$, we have

$$X = \left\{ \xi_1 \mathbf{1}_{\mathbf{O}} + \frac{1}{f'(\xi_1)} \mathbf{u}_1 \, \middle| \, \xi_1 \in \mathbb{F} \text{ is a simple root for } f(\xi) = \gamma \right\}.$$

Proof. Assume that there exists some $x \in X$. Consider $g \in G_2$ such that gx is canonical. Note that

$$f(gx) = gc. \tag{3.2}$$

1. Assume that $c = \gamma \mathbf{1}_{\mathbf{0}}$ for some $\gamma \in \mathbb{F}$. Then we can rewrite equality (3.2) as

$$f(gx) = \gamma 1_{\mathbf{O}}.\tag{3.3}$$

One of the following possibilities holds:

- (a) $gx = \xi_1 \mathbf{1}_{\mathbf{0}}$ for some $\xi_1 \in \mathbb{F}$. Equality (3.3) is equivalent to $f(\xi_1) = \gamma$. Thus, $x = g^{-1}\xi_1 \mathbf{1}_{\mathbf{0}} = \xi_1 \mathbf{1}_{\mathbf{0}}$.
- (b) $gx = \xi_1 e_1 + \xi_2 e_2$ for some $\xi_1, \xi_2 \in \mathbb{F}$ with $\xi_1 < \xi_2$. Equality (3.3) is equivalent to $f(\xi_1) = f(\xi_2) = \gamma$. Therefore, $x \in O_2(\xi_1, \xi_2) \subset X$.
- (c) $gx = \xi_1 \mathbf{1}_{\mathbf{0}} + \mathbf{u}_1$ for some $\xi_1 \in \mathbb{F}$. Lemma 3.1 implies that equality (3.3) is equivalent to $f(\xi_1) = \gamma$ and $f'(\xi_1) = 0$. Therefore, $x \in O_3(\xi_1) \subset X$.

2. Assume that $c = \gamma_1 e_1 + \gamma_2 e_2$ for some $\gamma_1, \gamma_2 \in \mathbb{F}$ with $\gamma_1 < \gamma_2$. We have that one of the following possibilities holds:

(a) $gx = \xi_1 \mathbf{1}_{\mathbf{0}}$ for some $\xi_1 \in \mathbb{F}$. Equality (3.2) implies that

$$f(\xi_1)1_{\mathbf{O}} \in O_2(\gamma_1, \gamma_2);$$

a contradiction to Proposition 2.2.

(b) $gx = \xi_1 e_1 + \xi_2 e_2$ for some $\xi_1, \xi_2 \in \mathbb{F}$ with $\xi_1 < \xi_2$. Equality (3.2) is equivalent to

$$f(\xi_1)e_1 + f(\xi_2)e_2 = g(\gamma_1e_1 + \gamma_2e_2).$$

Therefore,

$$f(\xi_1)e_1 + f(\xi_2)e_2 \in O_2(\gamma_1, \gamma_2).$$

Let $f(\xi_1) < f(\xi_2)$. Then Proposition 2.2 implies that $f(\xi_1) = \gamma_1$ and $f(\xi_2) = \gamma_2$. Hence, $g \in \text{St}_{G_2}(\gamma_1 e_1 + \gamma_2 e_2)$. Since $\text{St}_{G_2}(\gamma_1 e_1 + \gamma_2 e_2) = \text{SL}_3$ by Lemma 2.1 of [29], we have that $x = g^{-1}(\xi_1 e_1 + \xi_2 e_2) = \xi_1 e_1 + \xi_2 e_2$.

In case $f(\xi_1) = f(\xi_2)$ we obtain a contradiction with Proposition 2.2.

Let $f(\xi_1) > f(\xi_2)$. The equality $\hbar(f(\xi_1)e_1 + f(\xi_2)e_2) = \hbar g(\gamma_1 e_1 + \gamma_2 e_2)$ implies $f(\xi_2)e_1 + f(\xi_1)e_2 = \hbar g(\gamma_1 e_1 + \gamma_2 e_2)$. Thus, it follows from Proposition 2.2 that $f(\xi_1) = \gamma_2$ and $f(\xi_2) = \gamma_1$. Hence, $\hbar g \in \text{St}_{G_2}(\gamma_1 e_1 + \gamma_2 e_2)$. Since $\text{St}_{G_2}(\gamma_1 e_1 + \gamma_2 e_2) = \text{SL}_3$ by Lemma 2.1 of [29], we have that $x = (\hbar g)^{-1}\hbar(\xi_1 e_1 + \xi_2 e_2) = \xi_2 e_1 + \xi_1 e_2$.

(c) $gx = \xi_1 \mathbf{1}_{\mathbf{0}} + \mathbf{u}_1$ for some $\xi_1 \in \mathbb{F}$. Lemma 3.1 together with equality (3.2) implies that

$$f(\xi_1)\mathbf{1}_{\mathbf{O}} + f'(\xi_1)\mathbf{u}_1 \in O_2(\gamma_1, \gamma_2);$$

a contradiction by part 2 of Remark 2.4.

3. Assume that $c = \gamma \mathbf{1}_{\mathbf{0}} + \mathbf{u}_1$ for some $\gamma \in \mathbb{F}$. We have that one of the following possibilities holds:

(a) $gx = \xi_1 \mathbf{1}_{\mathbf{0}}$ for some $\xi_1 \in \mathbb{F}$. Equality (3.2) implies that

$$f(\xi_1)1_{\mathbf{O}} \in O_3(\gamma);$$

a contradiction to Proposition 2.2.

(b) $gx = \xi_1 e_1 + \xi_2 e_2$ for some $\xi_1, \xi_2 \in \mathbb{F}$ with $\xi_1 < \xi_2$. Equality (3.2) implies that

$$f(\xi_1)e_1 + f(\xi_2)e_2 \in O_3(\gamma);$$

a contradiction to Proposition 2.2.

(c) $gx = \xi_1 \mathbf{1}_{\mathbf{0}} + \mathbf{u}_1$ for some $\xi_1 \in \mathbb{F}$. Lemma 3.1 implies that equality (3.2) is equivalent to

$$f(\xi_1)\mathbf{1}_{\mathbf{O}} + f'(\xi_1)\mathbf{u}_1 = g(\gamma \mathbf{1}_{\mathbf{O}} + \mathbf{u}_1).$$
(3.4)

It follows from part 1 of Remark 2.4 that equality (3.4) is equivalent to $f(\xi_1) = \gamma$, $f'(\xi_1) \neq 0$, and $g\mathbf{u}_1 = f'(\xi_1)\mathbf{u}_1$. Thus,

$$x = g^{-1}(\xi_1 \mathbf{1}_{\mathbf{O}} + \mathbf{u}_1) = \xi_1 \mathbf{1}_{\mathbf{O}} + g^{-1} \mathbf{u}_1 = \xi_1 \mathbf{1}_{\mathbf{O}} + \frac{1}{f'(\xi_1)} \mathbf{u}_1.$$

On the other hand, Lemma 3.1 implies that for every $\xi_1 \in \mathbb{F}$ with $f(\xi_1) = \gamma$ and $f'(\xi_1) \neq 0$ we have

$$f\left(\xi_1 \mathbf{1}_{\mathbf{O}} + \frac{1}{f'(\xi_1)}\mathbf{u}_1\right) = \gamma \mathbf{1}_{\mathbf{O}} + \mathbf{u}_1.$$

The following corollaries are immediate consequences of Theorem 3.2.

Corollary 3.3. Assume that $f(\xi) \in \mathbb{F}[\xi]$ is a non-zero polynomial of degree n > 1 without constant term and $c \in \mathbf{O}$. Denote by $X \subset \mathbf{O}$ the set of all solutions of the equation f(x) = c. Let $(\gamma_1, \gamma_2) \in \mathbb{F}^2$ be the eigenvalues of c. Then

- (a) X is infinite if and only if $\gamma_1 = \gamma_2$ and $c = \gamma_1 \mathbf{1}_{\mathbf{O}}$;
- (b) X is empty if and only if $\gamma_1 = \gamma_2$, $c \neq \gamma_1 \mathbf{1}_{\mathbf{0}}$, and the equation $f(\xi) = \gamma_1$, where $\xi \in \mathbb{F}$ is a variable, does not have any simple root;

- (c) $|X| \leq n$ in case $\gamma_1 = \gamma_2$ and $c \neq \gamma_1 \mathbf{1}_{\mathbf{O}}$;
- (d) $1 \leq |X| \leq n^2$ in case $\gamma_1 \neq \gamma_2$.

Corollary 3.4. Assume that n > 1 is an integer and $c \in \mathbf{O}$. Acting by G_2 on the equation $x^n = c$, where $x \in \mathbf{O}$ is a variable, we can assume that c is a canonical octonion from Proposition 2.2. Let $X \subset \mathbf{O}$ be the set of all solutions of the equation $x^n = c$. Then

1. in case $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{F}$, we have

$$\begin{aligned} X &= \begin{cases} \xi_1 \mathbf{1}_{\mathbf{0}} \mid \xi_1 \in \mathbb{F} \text{ satisfies } \xi_1^n = \gamma \} \bigcup \\ \{O_2(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \mathbb{F} \text{ satisfy } \xi_1^n = \xi_2^n = \gamma \text{ and } \xi_1 < \xi_2 \} \bigcup Y, \text{ where} \\ Y &= \begin{cases} O_3(\xi_1) \mid \xi_1 \in \mathbb{F} \text{ satisfies } \xi_1^n = \gamma \end{cases} \text{ if } p \mid n \text{ or } \gamma = 0; \text{ and } Y = \emptyset, \text{ otherwise}; \end{aligned}$$

2. in case $c = \gamma_1 e_1 + \gamma_2 e_2$ for some $\gamma_1, \gamma_2 \in \mathbb{F}$ with $\gamma_1 < \gamma_2$, we have

$$X = \{\xi_1 e_1 + \xi_2 e_2 \mid \xi_1, \xi_2 \in \mathbb{F} \text{ satisfy } \xi_1^n = \gamma_1 \text{ and } \xi_2^n = \gamma_2\};$$

3. in case $c = \gamma \mathbf{1}_{\mathbf{0}} + \mathbf{u}_1$ for some $\gamma \in \mathbb{F}$, we have

$$X = \left\{ \xi_1 \mathbf{1}_{\mathbf{O}} + \frac{\xi_1}{n\gamma} \mathbf{u}_1 \, \middle| \, \xi_1 \in \mathbb{F} \text{ satisfies } \xi_1^n = \gamma \right\}, \quad \text{if } p \not| n \text{ and } \gamma \neq 0;$$
$$X = \emptyset, \quad \text{if } p \mid n \text{ or } \gamma = 0.$$

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