Communications in Mathematics **33** (2025), no. 1, Paper no. 10
DOI: https://doi.org/10.46298/cm.15949

The geometric classification of symmetric Leibniz algebras*

Renato Fehlberg Júnior, Ivan Kaygorodov, Azamat Saydaliyev

Abstract. This paper is devoted to the complete geometric classification of complex 5-dimensional solvable symmetric Leibniz algebras. As a corollary, we have the complete geometric classification of complex 5-dimensional symmetric Leibniz algebras.

Contents

Introduction

In the past years, Leibniz algebras have been under active research (see, for example, [2, 3, 8, 9, 11, 12, 14, 17, 19] and references therein). In the present paper, we give the geometric classification of complex 5-dimensional symmetric Leibniz algebras. The algebraic classification (up to isomorphism) of algebras of dimension n from a certain variety defined by a certain family of polynomial identities is a classic problem in the theory of

*The work is supported by FCT 2023.08031.CEECIND and UID/00212/2023.
MSC 2020: 17A32 (primary); 17A30, 14D06, 14L30 (secondary).
Keywords: Symmetric Leibniz algebra, Lie algebra, geometric classification.
Contact information:
Renato Fehlberg Júnior:
Affiliation: Federal University of Espirito Santo, Vitória, Brazil.
Email: renato.fehlberg@ufes.br
Ivan Kaygorodov:
Affiliation: CMA-UBI, University of Beira Interior, Covilhã, Portugal.
Email: kaygorodov.ivan@gmail.com
Azamat Saydaliyev:
Affiliation: Institute of Mathematics Academy of Sciences of Uzbekistan, Tashkent, Uzbekistan; National University of Uzbekistan, Tashkent, Uzbekistan.
Email: azamatsaydaliyev6@gmail.com

non-associative algebras. There are many results related to the algebraic classification of small-dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel, and many other algebras [15]. Deformations and geometric properties of a variety of algebras defined by a family of polynomial identities have been an object of study since the 1970s, see [1, 2, 4, 7, 13, 14] and references in [15, 16, 18].

The variety of symmetric Leibniz algebras is a principal subvariety of the variety of weakly associative algebras [2] and of the variety of $\mathfrak{CD}$-algebras. Symmetric Leibniz algebras and anticommutative $\mathfrak{CD}$-algebras have the following common property: they are central extensions (in a suitable variety) of Lie algebras [5]. At the same time, the variety of symmetric Leibniz algebras is in the intersection of right Leibniz and left Leibniz algebras, and it plays an important role in one-sided Leibniz algebras. So, each quadratic (i.e., endowed with a bilinear, symmetric, and non-degenerate associative form) Leibniz algebra is a symmetric Leibniz algebra [5]. Every symmetric Leibniz algebra is flexible, power-associative, nil-algebra with nilindex 3, and they are also Lie-admissible algebras (about Lie-admissible algebras see [10]). It satisfies the following identities:

$$x(yz) = (xy)z + y(xz) \quad \text{and} \quad (xy)z = (xz)y + x(yz).$$

1 Definitions and notation

Given an n-dimensional vector space $\mathbb{V}$, the set $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V}) \cong \mathbb{V}^* \otimes \mathbb{V}^* \otimes \mathbb{V}$ is a vector space of dimension n^3. This space has the structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1, \ldots, e_n$ of $\mathbb{V}$. Then any $\mu \in \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum\limits_{k=1}^{n} c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1 \le i, j, k \le n)$.

Let T be a set of polynomial identities. The set of algebra structures on $\mathbb{V}$ satisfying polynomial identities from T forms a Zariski-closed subset of the variety $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbb{V}$, $\mu \in \mathbb{L}(T) \subset \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ and $g \in \mathrm{GL}(\mathbb{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\mathrm{GL}(\mathbb{V})$-orbits that correspond to the isomorphism classes of algebras. Let $\mathcal{O}(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $\mathrm{GL}(\mathbb{V})$ and $\overline{\mathcal{O}(\mu)}$ denote the Zariski closure of $\mathcal{O}(\mu)$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two n-dimensional algebras satisfying the identities from T, and let $\mu, \lambda \in \mathbb{L}(T)$ represent $\mathbf{A}$ and $\mathbf{B}$, respectively. We say that $\mathbf{A}$ degenerates to $\mathbf{B}$ and write $\mathbf{A} \to \mathbf{B}$ if $\lambda \in \overline{\mathcal{O}(\mu)}$. Note that in this case we have $\overline{\mathcal{O}(\lambda)} \subset \overline{\mathcal{O}(\mu)}$. Hence, the definition of degeneration does not depend on the choice of μ and λ. If $\mathbf{A} \not\cong \mathbf{B}$, then the assertion $\mathbf{A} \to \mathbf{B}$ is called a *proper degeneration*. We write $\mathbf{A} \not\to \mathbf{B}$ if $\lambda \notin \overline{\mathcal{O}(\mu)}$.

Let $\mathbf{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathbf{A}$ is *rigid* in $\mathbb{L}(T)$ if $\mathcal{O}(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components uniquely. The algebra $\mathbf{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{\mathcal{O}(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

Method of the description of degenerations of algebras. In the present work, we use the methods applied to Lie algebras in [13]. First of all, if $\mathbf{A} \to \mathbf{B}$ and $\mathbf{A} \not\cong \mathbf{B}$, then $\mathfrak{Der}(\mathbf{A}) < \mathfrak{Der}(\mathbf{B})$, where $\mathfrak{Der}(\mathbf{A})$ is the algebra of derivations of $\mathbf{A}$. We compute the dimensions of algebras of derivations and check the assertion $\mathbf{A} \to \mathbf{B}$ only for such $\mathbf{A}$ and $\mathbf{B}$ that $\mathfrak{Der}(\mathbf{A}) < \mathfrak{Der}(\mathbf{B})$.

To prove degenerations, we construct families of matrices parametrized by t. Namely, let $\mathbf{A}$ and $\mathbf{B}$ be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let $e_1, \dots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \leq i, j, k \leq n)$ be the structure constants of λ in this basis. If there exist $a_i^j(t) \in \mathbb{C}$ $(1 \leq i, j \leq n,\ t \in \mathbb{C}^*)$ such that $E_i^t = \sum\limits_{j=1}^{n} a_i^j(t) e_j$ $(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of μ in the basis $E_1^t, \dots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathbf{A} \to \mathbf{B}$. In this case $E_1^t, \dots, E_n^t$ is called a *parametrized basis* for $\mathbf{A} \to \mathbf{B}$. In case of $E_1^t, E_2^t, \dots, E_n^t$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, it will be denoted by $\mathbf{A} \xrightarrow{(E_1^t, E_2^t, \dots, E_n^t)} \mathbf{B}$. To simplify our equations, we will use the notation $A_i = \langle e_i, \dots, e_n \rangle,\ i = 1, \dots, n$ and write simply $A_p A_q \subset A_r$ instead of $c_{ij}^k = 0$ $(i \geq p,\ j \geq q,\ k < r)$.

Let $\mathbf{A}(*) := \{\mathbf{A}(\alpha)\}_{\alpha \in I}$ be a series of algebras, and let $\mathbf{B}$ be another algebra. Suppose that for $\alpha \in I$, $\mathbf{A}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathbb{L}(T)$ and $\mathbf{B}$ is represented by the structure $\lambda \in \mathbb{L}(T)$. Then we say that $\mathbf{A}(*) \to \mathbf{B}$ if $\lambda \in \overline{\{\mathcal{O}(\mu(\alpha))\}_{\alpha \in I}}$, and $\mathbf{A}(*) \not\to \mathbf{B}$ if $\lambda \notin \overline{\{\mathcal{O}(\mu(\alpha))\}_{\alpha \in I}}$.

Let $\mathbf{A}(*)$, $\mathbf{B}$, $\mu(\alpha)$ $(\alpha \in I)$ and λ be as above. To prove $\mathbf{A}(*) \to \mathbf{B}$ it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) \in \mathrm{GL}(\mathbb{V})$. Namely, let $e_1, \dots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \leq i, j, k \leq n)$ be the structure constants of λ in this basis. If we construct $a_i^j : \mathbb{C}^* \to \mathbb{C}$ $(1 \leq i, j \leq n)$ and $f : \mathbb{C}^* \to I$ such that $E_i^t = \sum\limits_{j=1}^{n} a_i^j(t) e_j$ $(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu(f(t))$ in the basis $E_1^t, \dots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathbf{A}(*) \to \mathbf{B}$. In this case $E_1^t, \dots, E_n^t$ and $f(t)$ are called a parametrized basis and a *parametrized index* for $\mathbf{A}(*) \to \mathbf{B}$, respectively.

We now explain how to prove $\mathbf{A}(*) \not\to \mathbf{B}$. Note that if $\mathfrak{Der}\,\mathbf{A}(\alpha) > \mathfrak{Der}\,\mathbf{B}$ for all $\alpha \in I$ then $\mathbf{A}(*) \not\to \mathbf{B}$. One can also use the following Lemma, whose proof is the same as the proof of [13, Lemma 1.5].

Lemma. *Let $\mathfrak{B}$ be a Borel subgroup of $\mathrm{GL}(\mathbb{V})$ and $\mathcal{R} \subset \mathbb{L}(T)$ be a $\mathfrak{B}$-stable closed subset. If $\mathbf{A}(*) \to \mathbf{B}$ and for any $\alpha \in I$ the agebra $\mathbf{A}(\alpha)$ can be represented by a structure $\mu(\alpha) \in \mathcal{R}$, then there is $\lambda \in \mathcal{R}$ representing $\mathbf{B}$.*

2 Our strategy

Let us remember the principal results, that will be used in our main results.

($\mathfrak{a}$) The algebraic classification of indecomposable 5-dimensional symmetric non-nilpotent Leibniz algebras is given in [9].

($\mathfrak{b}$) The geometric classification of 5-dimensional Lie algebras is given in [7].

($\mathfrak{c}$) The geometric classification of 5-dimensional nilpotent symmetric Leibniz algebras is given in [2].

($\mathfrak{d}$) The geometric classification of 4-dimensional Lie algebras is given in [6], and irreducible non-Lie components of 4-dimensional symmetric Leibniz algebras are given in [2].

Hence, our proof of the main statement will be based on the description of irreducible components of varieties given in ($\mathfrak{b}$), ($\mathfrak{c}$), ($\mathfrak{d}$), and given above.

Thanks to [7], all irreducible components of 5-dimensional Lie algebras are given below[†].

$\mathfrak{sl}_2 \oplus \mathfrak{r}_2$	:	$e_1e_2 = 2e_1$	$e_1e_3 = -e_2$	$e_2e_1 = -2e_1$	$e_2e_3 = 2e_3$
		$e_3e_1 = e_2$	$e_3e_2 = -2e_3$	$e_4e_5 = e_5$	$e_5e_4 = -e_5$
$\mathfrak{sl}_2 \ltimes \mathrm{V}_2$	:	$e_1e_2 = 2e_1$	$e_1e_3 = -e_2$	$e_1e_4 = e_5$	$e_2e_1 = -2e_1$
		$e_2e_3 = 2e_3$	$e_2e_4 = e_4$	$e_2e_5 = -e_5$	$e_3e_1 = e_2$
		$e_3e_2 = -2e_3$	$e_3e_5 = e_4$	$e_4e_1 = -e_5$	
		$e_4e_2 = -e_4$	$e_5e_2 = e_5$	$e_5e_3 = -e_4$	
$\overline{\mathrm{R}}^0_5(\mathfrak{n}_4)$	:	$e_1e_2 = e_2$	$e_1e_3 = \alpha e_3$	$e_1e_4 = (\alpha+1)e_4$	$e_1e_5 = (\alpha+2)e_5$
		$e_2e_1 = -e_2$	$e_2e_3 = e_4$	$e_2e_4 = e_5$	$e_3e_1 = -\alpha e_3$
		$e_3e_2 = -e_4$	$e_4e_1 = -(\alpha+1)e_4$	$e_4e_2 = -e_5$	$e_5e_1 = -(\alpha+2)e_5$
$\overline{\mathrm{R}}^0_5(\mathfrak{n}_3 \oplus \mathbb{C})$	:	$e_1e_2 = \beta e_2$	$e_1e_3 = e_3$	$e_1e_4 = \alpha e_4$	$e_1e_5 = (\alpha+1)e_5$
		$e_2e_1 = -\beta e_2$	$e_3e_1 = -e_3$	$e_3e_4 = e_5$	
		$e_4e_1 = -\alpha e_4$	$e_4e_3 = -e_5$	$e_5e_1 = -(\alpha+1)e_5$	
$\overline{\mathrm{R}}^0_5(\mathfrak{n}_3)$	:	$e_1e_3 = e_3$	$e_1e_5 = e_5$	$e_2e_4 = e_4$	$e_2e_5 = e_5$
		$e_3e_1 = -e_3$	$e_3e_4 = e_5$	$e_4e_2 = -e_4$	
		$e_4e_3 = -e_5$	$e_5e_1 = -e_5$	$e_5e_2 = -e_5$	
$\overline{\mathrm{R}}^0_5(\mathbb{C}^4)$	:	$e_1e_2 = e_2$	$e_1e_3 = \alpha e_3$	$e_1e_4 = \beta e_4$	$e_1e_5 = \gamma e_5$
		$e_2e_1 = -e_2$	$e_3e_1 = -\alpha e_3$	$e_4e_1 = -\beta e_4$	$e_5e_1 = -\gamma e_5$
$\overline{\mathrm{R}}^0_5(\mathbb{C}^3)$	:	$e_1e_3 = e_3$	$e_1e_5 = \alpha e_5$	$e_2e_4 = e_4$	$e_2e_5 = \beta e_5$
		$e_3e_1 = -e_3$	$e_5e_1 = -\alpha e_5$	$e_4e_2 = -e_4$	$e_5e_2 = -\beta e_5$

Thanks to [2], all irreducible components of 5-dimensional nilpotent symmetric Leibniz algebras are given below.

[†]The multiplication tables of suitable algebras were found in [20].

$\mathbb{S}_{21}^{\alpha,\beta}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3 + e_4 + \beta e_5$	$e_1e_3 = e_5$	$e_2e_1 = -e_3$	$e_2e_2 = e_5$
		$e_2e_3 = e_4$	$e_3e_1 = -e_5$	$e_3e_2 = -e_4$		
$\mathbb{S}_{22}^{\alpha}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = -e_3$	$e_2e_2 = \alpha e_5$
		$e_2e_4 = e_5$	$e_3e_1 = -e_5$	$e_4e_4 = e_5$		
$\mathbb{S}_{41}^{\alpha}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = -e_3$	$e_2e_2 = \alpha e_5$
		$e_2e_3 = e_4$	$e_2e_4 = e_5$	$e_3e_1 = -e_5$	$e_3e_2 = -e_4$	$e_4e_2 = -e_5$
$\mathfrak{V}_{4+1}$	:	$e_1e_2 = e_5$	$e_2e_1 = \lambda e_5$	$e_3e_4 = e_5$	$e_4e_3 = \mu e_5$	
$\mathfrak{V}_{3+2}$	:	$e_1e_1 = e_4$	$e_1e_2 = \mu_1 e_5$	$e_1e_3 = \mu_2 e_5$	$e_2e_1 = \mu_3 e_5$	$e_2e_2 = \mu_4 e_5$
		$e_2e_3 = \mu_5 e_5$	$e_3e_1 = \mu_6 e_5$	$e_3e_2 = \lambda e_4 + \mu_7 e_5$		$e_3e_3 = e_5$
$\mathfrak{V}_{2+3}$	:	$e_1e_1 = e_3 + \lambda e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_2e_2 = e_5$	

Thanks to [6], all irreducible components of 4-dimensional Lie algebras are given below.

$\mathfrak{r}_2 \oplus \mathfrak{r}_2$	:	$e_1e_2 = e_2$	$e_2e_1 = -e_2$	$e_3e_4 = e_4$	$e_4e_3 = -e_4$
$\mathfrak{sl}_2 \oplus \mathbb{C}$	:	$e_1e_2 = e_2$	$e_1e_3 = -e_3$	$e_2e_3 = e_1$	
		$e_2e_1 = -e_2$	$e_3e_1 = e_3$	$e_3e_2 = -e_1$	
$\mathfrak{g}_5(\alpha)$	:	$e_1e_2 = e_2$	$e_1e_3 = e_2 + \alpha e_3$	$e_1e_4 = (\alpha+1)e_4$	$e_2e_3 = e_4$
		$e_2e_1 = -e_2$	$e_3e_1 = -e_2 - \alpha e_3$	$e_4e_1 = -(\alpha+1)e_4$	$e_3e_2 = -e_4$
$\mathfrak{g}_4(\alpha,\beta)$	:	$e_1e_2 = e_2$	$e_1e_3 = e_2 + \alpha e_3$	$e_1e_4 = e_3 + \beta e_4$	
		$e_2e_1 = -e_2$	$e_3e_1 = -e_2 - \alpha e_3$	$e_4e_1 = -e_3 - \beta e_4$	

Thanks to [2], all irreducible non-Lie components of 4-dimensional symmetric Leibniz algebras are given below.

$\mathfrak{N}_2(\alpha)$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_2e_1 = -\alpha e_3$	$e_2e_2 = -e_4$	
$\mathfrak{N}_3(\alpha)$	:	$e_1e_1 = e_4$	$e_1e_2 = \alpha e_4$	$e_2e_1 = -\alpha e_4$	$e_2e_2 = e_4$	$e_3e_3 = e_4$
$\mathfrak{L}_{02}$	:	$e_1e_1 = e_4$	$e_1e_2 = -e_2$	$e_1e_3 = e_3$	$e_2e_1 = e_2$	
		$e_2e_3 = e_4$	$e_3e_1 = -e_3$	$e_3e_2 = -e_4$		
$\mathfrak{L}_{15}^{\alpha}$	:	$e_1e_1 = \alpha e_4$	$e_1e_2 = e_4$	$e_1e_3 = -e_3$	$e_2e_2 = e_4$	$e_3e_1 = e_3$
$\mathfrak{L}_{24}^{\alpha}$	:	$e_1e_1 = e_4$	$e_1e_2 = -e_2$	$e_1e_3 = -\alpha e_3$	$e_2e_1 = e_2$	$e_3e_1 = \alpha e_3$

Let us note that algebras $\mathfrak{N}_2(\alpha)$ and $\mathfrak{N}_3(\alpha)$, considered as 5-dimensional algebras, are nilpotent algebras, and they are in the suitable irreducible components of nilpotent 5-dimensional symmetric Leibniz algebras. Hence, they will be omitted.

From [9] we obtain the list of complex 5-dimensional solvable (non-split, non-nilpotent, non-Lie) symmetric Leibniz algebras.

$\mathcal{L}_{01}^{\alpha}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_4$
		$e_2e_1 = -e_3$	$e_2e_2 = -e_5$	$e_3e_1 = -e_5$	$e_4e_1 = -e_4$
$\mathcal{L}_{02}^{\alpha}$	:	$e_1e_2 = -e_3 + \alpha e_5$	$e_1e_3 = e_5$	$e_1e_4 = e_4$	
		$e_2e_1 = e_3 + \alpha e_5$	$e_3e_1 = -e_5$	$e_4e_1 = -e_4$	
$\mathcal{L}_{03}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_4$
		$e_2e_1 = -e_3$	$e_3e_1 = -e_5$	$e_4e_1 = -e_4$	

$\mathcal{L}_{04}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=e_5$	$e_1e_3=e_3+e_4$	$e_1e_4=e_4$
		$e_2e_2=e_5$	$e_2e_1=-e_5$	$e_3e_1=-e_3-e_4$	$e_4e_1=-e_4$
$\mathcal{L}_{05}^{\alpha}$	:	$e_1e_2=(\alpha+1)e_5$	$e_1e_3=e_3+e_4$	$e_1e_4=e_4$	
		$e_2e_1=(\alpha-1)e_5$	$e_3e_1=-e_3-e_4$	$e_4e_1=-e_4$	
$\mathcal{L}_{06}$	:	$e_1e_1=e_5$	$e_1e_2=e_5$	$e_1e_3=e_3+e_4$	$e_1e_4=e_4$
		$e_2e_1=-e_5$	$e_3e_1=-e_3-e_4$	$e_4e_1=-e_4$	
$\mathcal{L}_{07}^{\alpha,\beta}$	:	$e_1e_1=\beta e_5$	$e_1e_2=e_5$	$e_1e_3=e_3$	$e_1e_4=\alpha e_4$
		$e_2e_1=-e_5$	$e_2e_2=e_5$	$e_3e_1=-e_3$	$e_4e_1=-\alpha e_4$
$\mathcal{L}_{08}^{\alpha,\beta}$	:	$e_1e_2=(\beta+1)e_5$	$e_1e_3=e_3$	$e_1e_4=\alpha e_4$	
		$e_2e_1=(\beta-1)e_5$	$e_3e_1=-e_3$	$e_4e_1=-\alpha e_4$	
$\mathcal{L}_{09}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=e_5$	$e_1e_3=e_3$	$e_1e_4=\alpha e_4$
		$e_2e_1=-e_5$	$e_3e_1=-e_3$	$e_4e_1=-\alpha e_4$	
$\mathcal{L}_{10}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=e_4$	$e_1e_4=e_5$	$e_2e_1=-e_4$
		$e_2e_3=-e_3$	$e_3e_2=e_3$	$e_4e_1=-e_5$	
$\mathcal{L}_{11}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=e_3$	$e_1e_4=e_4$	$e_2e_1=-e_3$
		$e_2e_2=\alpha e_5$	$e_2e_3=e_5$	$e_3e_2=-e_5$	$e_4e_1=-e_4$
$\mathcal{L}_{12}^{\alpha,\beta}$	:	$e_1e_1=\beta e_5$	$e_1e_2=e_3-e_5$	$e_1e_4=e_4$	$e_2e_1=-e_3-e_5$
		$e_2e_2=\alpha e_5$	$e_2e_3=e_5$	$e_3e_2=-e_5$	$e_4e_1=-e_4$
$\mathcal{L}_{13}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=e_5$	$e_1e_3=e_3$	$e_1e_4=-e_4$
		$e_2e_1=-e_5$	$e_2e_2=e_5$	$e_3e_1=-e_3$	
		$e_3e_4=-e_5$	$e_4e_1=e_4$	$e_4e_3=e_5$	
$\mathcal{L}_{14}^{\alpha}$	:	$e_1e_2=(\alpha-1)e_5$	$e_1e_3=e_3$	$e_1e_4=-e_4$	$e_2e_1=(\alpha+1)e_5$
		$e_3e_1=-e_3$	$e_3e_4=e_5$	$e_4e_1=e_4$	$e_4e_3=-e_5$
$\mathcal{L}_{15}$	:	$e_1e_1=e_5$	$e_1e_2=-e_5$	$e_1e_3=e_3$	
		$e_1e_4=-e_4$	$e_2e_1=e_5$	$e_3e_1=-e_3$	
		$e_3e_4=e_5$	$e_4e_1=e_4$	$e_4e_3=-e_5$	
$\mathcal{L}_{16}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2$	$e_1e_3=e_3$	
		$e_1e_4=-\alpha e_4$	$e_2e_1=e_2$	$e_2e_3=-e_5$	
		$e_3e_1=-e_3$	$e_3e_2=e_5$	$e_4e_1=\alpha e_4$	
$\mathcal{L}_{17}$	:	$e_1e_1=e_5$	$e_1e_2=e_2$	$e_1e_3=-e_3-e_4$	
		$e_1e_4=-e_4$	$e_2e_1=-e_2$	$e_2e_3=-e_5$	
		$e_3e_1=e_3+e_4$	$e_3e_2=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{18}$	:	$e_1e_1=e_5$	$e_1e_2=-e_5$	$e_1e_3=-e_3$	
		$e_1e_4=-e_4$	$e_2e_1=e_5$	$e_2e_3=e_4$	
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{19}$	:	$e_1e_3=e_3$	$e_1e_4=e_4$	$e_2e_1=-2e_5$	$e_2e_3=e_4$
		$e_3e_1=-e_3$	$e_3e_2=-e_4$	$e_4e_1=-e_4$	
$\mathcal{L}_{20}$	:	$e_1e_1=e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	$e_2e_1=2e_5$
		$e_2e_3=e_4$	$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$
$\mathcal{L}_{21}$	:	$e_1e_2=-e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	
		$e_2e_1=e_5$	$e_2e_2=e_5$	$e_2e_3=e_4$	
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{22}$	:	$e_1e_1=e_5$	$e_1e_2=-e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$
		$e_2e_1=e_5$	$e_2e_2=e_5$	$e_2e_3=e_4$	$e_3e_1=e_3$

		$e_3e_2=-e_4$	$e_4e_1=e_4$		
$\mathcal{L}_{23}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	
		$e_2e_1=2e_5$	$e_2e_2=e_5$	$e_2e_3=e_4$	
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{24}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2$	$e_1e_3=2e_3$	$e_1e_4=-e_4$
		$e_2e_1=e_2$	$e_2e_3=e_4$	$e_2e_4=e_5$	$e_3e_1=-2e_3$
		$e_3e_2=-e_4$	$e_4e_1=e_4$	$e_4e_2=-e_5$	
$\mathcal{L}_{25}^{\alpha,\beta,\gamma}$	:	$e_1e_1=\gamma e_5$	$e_1e_2=(\beta-1)e_5$	$e_1e_3=-e_3$	$e_2e_1=(\beta+1)e_5$
		$e_2e_2=\alpha e_5$	$e_2e_4=-e_4$	$e_3e_1=e_3$	$e_4e_2=e_4$
$\mathcal{L}_{26}^{\alpha,\beta}$	:	$e_1e_1=\beta e_4+e_5$	$e_1e_2=(\alpha-1)e_4$	$e_1e_3=-e_3$	
		$e_2e_1=(\alpha+1)e_4$	$e_2e_2=e_5$	$e_3e_1=e_3$	
$\mathcal{L}_{27}^{\alpha}$	:	$e_1e_1=e_4$	$e_1e_2=(\alpha-1)e_4$	$e_1e_3=-e_3$	
		$e_2e_1=(\alpha+1)e_4$	$e_2e_2=e_5$	$e_3e_1=e_3$	
$\mathcal{L}_{28}^{\alpha}$	:	$e_1e_2=(\alpha-1)e_4$	$e_1e_3=-e_3$	$e_2e_1=(\alpha+1)e_4$	
		$e_2e_2=e_5$	$e_3e_1=e_3$		
$\mathcal{L}_{29}^{\alpha}$	:	$e_1e_1=\alpha e_4$	$e_1e_2=-e_4+e_5$	$e_1e_3=-e_3$	
		$e_2e_1=e_4+e_5$	$e_2e_2=e_4$	$e_3e_1=e_3$	
$\mathcal{L}_{30}$	:	$e_1e_1=e_4$	$e_1e_2=-e_4+e_5$	$e_1e_3=-e_3$	
		$e_2e_1=e_4+e_5$	$e_3e_1=e_3$		
$\mathcal{L}_{31}$	:	$e_1e_2=-e_4+e_5$	$e_1e_3=-e_3$	$e_2e_1=e_4+e_5$	$e_3e_1=e_3$
$\mathcal{L}_{32}$	:	$e_1e_1=e_5$	$e_1e_2=-e_4$	$e_1e_3=-e_3$	
		$e_2e_1=e_4$	$e_2e_2=e_4$	$e_3e_1=e_3$	
$\mathcal{L}_{33}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=(\alpha-1)e_4$	$e_1e_3=-e_3$	
		$e_2e_1=(\alpha+1)e_4$	$e_3e_1=e_3$		
$\mathcal{L}_{34}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=-e_5$	$e_1e_4=-e_4$	$e_2e_1=e_5$
		$e_2e_2=e_5$	$e_3e_3=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{35}^{\alpha}$	:	$e_1e_2=(\alpha-1)e_5$	$e_1e_4=-e_4$	$e_2e_1=(\alpha+1)e_5$	
		$e_3e_3=e_5$	$e_4e_1=e_4$		
$\mathcal{L}_{36}$	:	$e_1e_1=e_5$	$e_1e_2=-e_5$	$e_1e_4=-e_4$	
		$e_2e_1=e_5$	$e_3e_3=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{37}$	:	$e_1e_2=-e_5$	$e_1e_4=-e_4$	$e_2e_1=e_5$	
		$e_2e_3=e_5$	$e_3e_2=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{38}$	:	$e_1e_1=e_5$	$e_1e_2=-e_5$	$e_1e_4=-e_4$	$e_2e_1=e_5$
		$e_2e_3=e_5$	$e_3e_2=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{39}$	:	$e_1e_2=-e_5$	$e_1e_3=e_5$	$e_1e_4=-e_4$	
		$e_2e_1=e_5$	$e_3e_1=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{40}$	:	$e_1e_2=-e_5$	$e_1e_3=e_5$	$e_1e_4=-e_4$	$e_2e_1=e_5$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_4e_1=e_4$	
$\mathcal{L}_{41}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2-e_3$	$e_1e_3=-e_3-e_4$	$e_1e_4=-e_4$
		$e_2e_1=e_2+e_3$	$e_3e_1=e_3+e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{42}^{\alpha,\beta}$	:	$e_1e_1=e_5$	$e_1e_2=-\alpha e_2$	$e_1e_3=-\beta e_3$	$e_1e_4=-e_4$
		$e_2e_1=\alpha e_2$	$e_3e_1=\beta e_3$	$e_4e_1=e_4$	
$\mathcal{L}_{43}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2-e_4$	$e_1e_3=-\alpha e_3$	$e_1e_4=-e_4$
		$e_2e_1=e_2+e_4$	$e_3e_1=\alpha e_3$	$e_4e_1=e_4$	

$\mathcal{L}_{44}$[‡]	:	$e_1e_1=e_5$	$e_1e_2=-e_2$	$e_1e_3=e_3$	$e_2e_1=e_2$
		$e_2e_3=e_4$	$e_3e_1=-e_3$	$e_3e_2=-e_4$	
$\mathcal{L}_{45}$	:	$e_1e_1=e_5$	$e_2e_3=e_3$	$e_2e_4=-e_4$	$e_3e_2=-e_3$
		$e_3e_4=-e_5$	$e_4e_2=e_4$	$e_4e_3=e_5$	
$\mathcal{L}_{46}$	:	$e_1e_1=e_5$	$e_2e_2=e_5$	$e_2e_3=e_3$	$e_2e_4=-e_4$
		$e_3e_2=-e_3$	$e_3e_4=-e_5$	$e_4e_2=e_4$	$e_4e_3=e_5$
$\mathcal{L}_{47}$	:	$e_1e_2=e_5$	$e_2e_1=e_5$	$e_2e_3=e_3$	$e_2e_4=-e_4$
		$e_3e_2=-e_3$	$e_3e_4=-e_5$	$e_4e_2=e_4$	$e_4e_3=e_5$
$\mathcal{L}_{48}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2$	$e_1e_3=-\alpha e_3$	
		$e_1e_4=-(\alpha+1)e_4$	$e_2e_1=e_2$	$e_2e_3=e_4$	
		$e_3e_1=\alpha e_3$	$e_3e_2=-e_4$	$e_4e_1=(\alpha+1)e_4$	
$\mathcal{L}_{49}$	:	$e_1e_1=e_5$	$e_1e_2=-e_2-e_3$	$e_1e_3=-e_3$	
		$e_1e_4=-2e_4$	$e_2e_1=e_2+e_3$	$e_2e_3=-e_4$	
		$e_3e_1=e_3$	$e_3e_2=e_4$	$e_4e_1=2e_4$	
$\mathcal{L}_{50}$	:	$e_1e_1=e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	$e_2e_3=e_4$
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{51}$	:	$e_1e_2=e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	$e_2e_1=e_5$
		$e_2e_3=e_4$	$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$
$\mathcal{L}_{52}$	:	$e_1e_1=e_5$	$e_1e_2=e_5$	$e_1e_3=-e_3$	
		$e_1e_4=-e_4$	$e_2e_1=e_5$	$e_2e_3=e_4$	
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{53}$	:	$e_1e_3=-e_3$	$e_1e_4=-e_4$	$e_2e_2=e_5$	$e_2e_3=e_4$
		$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$	
$\mathcal{L}_{54}$	:	$e_1e_1=e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$	$e_2e_2=e_5$
		$e_2e_3=e_4$	$e_3e_1=e_3$	$e_3e_2=-e_4$	$e_4e_1=e_4$
$\mathcal{L}_{55}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=e_5$	$e_1e_3=-e_3$	$e_1e_4=-e_4$
		$e_2e_1=e_5$	$e_2e_2=e_5$	$e_2e_3=e_4$	$e_3e_1=e_3$
		$e_3e_2=-e_4$	$e_4e_1=e_4$		
$\mathcal{L}_{56}$	:	$e_1e_1=e_5$	$e_1e_3=-e_3$	$e_2e_4=-e_4$	
		$e_3e_1=e_3$	$e_4e_2=e_4$		
$\mathcal{L}_{57}^{\alpha}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=e_5$	$e_1e_3=-e_3$	$e_2e_1=e_5$
		$e_2e_4=-e_4$	$e_3e_1=e_3$	$e_4e_2=e_4$	
$\mathcal{L}_{58}^{\alpha,\beta}$	:	$e_1e_1=\alpha e_5$	$e_1e_2=\beta e_5$	$e_1e_3=-e_3$	$e_2e_1=\beta e_5$
		$e_2e_2=e_5$	$e_2e_4=-e_4$	$e_3e_1=e_3$	$e_4e_2=e_4$
$\mathcal{L}_{59}$	:	$e_1e_1=e_5$	$e_1e_2=e_2+e_3$	$e_1e_3=e_3$	$e_1e_4=-e_4$
		$e_2e_1=-e_2-e_3$	$e_3e_1=-e_3$	$e_4e_1=-e_4$	
$\mathcal{L}_{60}$	:	$e_1e_2=e_5$	$e_2e_1=-e_5$	$e_2e_3=e_3+e_4$	
		$e_2e_4=e_4$	$e_3e_2=-e_3-e_4$	$e_4e_2=-e_4$	
$\mathcal{L}_{61}$	:	$e_1e_1=e_5$	$e_2e_2=e_5$	$e_2e_3=e_3+e_4$	
		$e_2e_4=e_4$	$e_3e_2=-e_3-e_4$	$e_4e_2=-e_4$	
$\mathcal{L}_{62}^{\alpha}$	:	$e_1e_1=e_5$	$e_1e_2=-\alpha e_2$	$e_1e_3=-e_3$	$e_1e_4=e_4$
		$e_2e_1=\alpha e_2$	$e_3e_1=e_3$	$e_4e_1=e_4$	

[‡] $\mathcal{L}_{44}=\mathcal{L}_{48}^{-1}$

$\mathcal{L}_{63}^{\alpha}$	:	$e_1e_2=e_5$ $e_2e_1=e_5$	$e_1e_3=-e_3$ $e_3e_1=e_3$	$e_1e_4=-\alpha e_4$ $e_4e_1=\alpha e_4$	
$\mathcal{L}_{64}^{\alpha}$	:	$e_1e_1=e_5$ $e_2e_2=e_5$	$e_1e_3=-e_3$ $e_3e_1=e_3$	$e_1e_4=-\alpha e_4$ $e_4e_1=\alpha e_4$	
$\mathcal{L}_{65}$	:	$e_1e_1=e_4$ $e_2e_3=-e_3$	$e_1e_2=e_5$ $e_3e_2=e_3$	$e_2e_1=e_5$	
$\mathcal{L}_{66}$	:	$e_1e_1=e_4$ $e_2e_2=e_4$	$e_1e_2=e_5$ $e_3e_2=e_3$	$e_2e_1=e_5$ $e_2e_3=-e_3$	
$\mathcal{L}_{67}$	:	$e_1e_2=e_5$ $e_3e_3=e_5$	$e_2e_1=e_5$ $e_4e_2=e_4$	$e_2e_4=-e_4$	
$\mathcal{L}_{68}$	:	$e_1e_3=e_5$ $e_3e_1=e_5$	$e_2e_2=e_5$ $e_4e_2=e_4$	$e_2e_4=-e_4$	
$\mathcal{L}_{69}^{\alpha,\beta}$	:	$e_1e_1=\beta e_5$ $e_2e_2=\alpha e_5$	$e_1e_2=e_5$ $e_2e_3=-e_3$	$e_1e_4=-e_4$ $e_3e_2=e_3$	$e_2e_1=e_5$ $e_4e_1=e_4$
$\mathcal{L}_{70}^{\alpha}$	:	$e_1e_1=e_5$ $e_2e_3=-e_3$	$e_1e_4=-e_4$ $e_3e_2=e_3$	$e_2e_2=\alpha e_5$ $e_4e_1=e_4$	
$\mathcal{L}_{71}^{\alpha}$	:	$e_1e_3=(\alpha-1)e_5$ $e_3e_1=(\alpha+1)e_5$	$e_2e_2=e_5$ $e_4e_2=e_4$	$e_2e_4=-e_4$	
$\mathcal{L}_{72}^{\alpha}$	:	$e_1e_2=e_5$ $e_2e_4=-e_4$	$e_1e_3=(\alpha-1)e_5$ $e_3e_1=(\alpha+1)e_5$	$e_2e_1=e_5$ $e_4e_2=e_4$	
$\mathcal{L}_{73}^{\alpha}$	:	$e_1e_2=e_5$ $e_2e_4=-e_4$	$e_1e_3=(\alpha-1)e_5$ $e_3e_1=(\alpha+1)e_5$	$e_2e_1=e_5$ $e_4e_2=e_4$	$e_2e_2=e_5$
$\mathcal{L}_{74}^{\alpha,\beta}$	:	$e_1e_2=e_5$ $e_2e_2=\beta e_5$ $e_3e_1=(\alpha+1)e_5$	$e_1e_3=(\alpha-1)e_5$ $e_2e_3=e_5$ $e_3e_2=e_5$	$e_2e_1=e_5$ $e_2e_4=-e_4$ $e_4e_2=e_4$	
$\mathcal{L}_{75}$	:	$e_1e_1=e_5$ $e_2e_4=-e_4$	$e_1e_3=-e_5$ $e_3e_1=e_5$	$e_2e_2=e_5$ $e_4e_2=e_4$	
$\mathcal{L}_{76}^{\alpha}$	:	$e_1e_1=e_5$ $e_2e_4=-e_4$	$e_1e_3=-e_5$ $e_3e_1=e_5$	$e_2e_2=\alpha e_5$ $e_3e_2=e_5$	$e_2e_3=e_5$ $e_4e_2=e_4$
$\mathcal{L}_{77}^{\alpha}$	:	$e_1e_1=e_5$ $e_2e_2=\alpha e_5$	$e_1e_2=e_5$ $e_2e_4=-e_4$	$e_1e_3=-e_5$ $e_3e_1=e_5$	$e_2e_1=e_5$ $e_4e_2=e_4$

3 Results and proof

Proposition. *Algebras $\mathfrak{r}_2\oplus\mathfrak{r}_2$, $\mathfrak{sl}_2\oplus\mathbb{C}$, $\mathfrak{g}_4(\alpha,\beta)$, and $\mathfrak{g}_5(\alpha)$ give irreducible components in the variety of complex 4-dimensional symmetric Leibniz algebras. Algebras $\mathfrak{sl}_2\oplus\mathfrak{r}_2$, $\mathfrak{sl}_2\ltimes \mathrm{V}_2$, $\overline{\mathrm{R}}_5^0(\mathfrak{n}_4)$, $\overline{\mathrm{R}}_5^0(\mathfrak{n}_3\oplus\mathbb{C})$, $\overline{\mathrm{R}}_5^0(\mathfrak{n}_3)$, $\overline{\mathrm{R}}_5^0(\mathbb{C}^4)$, and $\overline{\mathrm{R}}_5^0(\mathbb{C}^3)$ give irreducible components in the variety of complex 5-dimensional symmetric Leibniz algebras.*

Proof. It is easy to see that each algebra, excepting $\mathfrak{sl}_2\oplus\mathbb{C}$, from the presented list, has a trivial annihilator. On the other side, it is known that each non-Lie symmetric Leibniz algebra is a central extension of a suitable Lie algebra [5], and it has a nontrivial annihilator. Hence, no non-Lie symmetric Leibniz algebras can degenerate to algebras from our list. $\mathfrak{sl}_2\oplus\mathbb{C}$ is rigid due to uniqueness of non-solvable 4-dimensional symmetric algebra. □

Summarizing the result from the Proposition above and proof from Theorem E[§] in [2], we have the following statement.

Statement. *The variety of complex 4-dimensional symmetric Leibniz algebras has dimension 13 and it has 9 irreducible components defined by*

$$\mathcal{C}_1=\overline{\mathcal{O}(\mathfrak{r}_2\oplus\mathfrak{r}_2)},\ \ \mathcal{C}_2=\overline{\mathcal{O}(\mathfrak{sl}_2\oplus\mathbb{C})},\ \ \mathcal{C}_3=\overline{\mathcal{O}(\mathfrak{L}_{02})},\ \ \mathcal{C}_4=\overline{\mathcal{O}(\mathfrak{g}_5(\alpha))},\ \ \mathcal{C}_5=\overline{\mathcal{O}(\mathfrak{N}_2(\alpha))},$$
$$\mathcal{C}_6=\overline{\mathcal{O}(\mathfrak{N}_3(\alpha))},\ \ \mathcal{C}_7=\overline{\mathcal{O}(\mathfrak{L}_{15}^{\alpha})},\ \ \mathcal{C}_8=\overline{\mathcal{O}(\mathfrak{L}_{24}^{\alpha})},\ \ \textit{and } \mathcal{C}_9=\overline{\mathcal{O}(\mathfrak{g}_4(\alpha,\beta))}.$$

In particular, there are only 3 rigid algebras in this variety.

Theorem. *The variety of complex 5-dimensional solvable symmetric Leibniz algebras has dimension 24 and it has 20 irreducible components defined by*

$$\mathcal{C}_{01}=\overline{\mathcal{O}(\mathcal{L}_{17})},\ \ \mathcal{C}_{02}=\overline{\mathcal{O}(\mathcal{L}_{24})},\ \ \mathcal{C}_{03}=\overline{\mathcal{O}(\mathcal{L}_{46})},\ \ \mathcal{C}_{04}=\overline{\mathcal{O}(\overline{\mathrm{R}}_5^0(\mathfrak{n}_3))},\ \ \mathcal{C}_{05}=\overline{\mathcal{O}(\overline{\mathrm{R}}_5^0(\mathfrak{n}_4))},$$
$$\mathcal{C}_{06}=\overline{\mathcal{O}(\overline{\mathrm{R}}_5^0(\mathfrak{n}_3\oplus\mathbb{C}))},\ \ \mathcal{C}_{07}=\overline{\mathcal{O}(\overline{\mathrm{R}}_5^0(\mathbb{C}^4))},\ \ \mathcal{C}_{08}=\overline{\mathcal{O}(\overline{\mathrm{R}}_5^0(\mathbb{C}^3))},\ \ \mathcal{C}_{09}=\overline{\mathcal{O}(\mathfrak{V}_{2+3})},$$
$$\mathcal{C}_{10}=\overline{\mathcal{O}(\mathfrak{V}_{3+2})},\ \ \mathcal{C}_{11}=\overline{\mathcal{O}(\mathfrak{V}_{4+1})},\ \ \mathcal{C}_{12}=\overline{\mathcal{O}(\mathbb{S}_{21}^{\alpha,\beta})},\ \ \mathcal{C}_{13}=\overline{\mathcal{O}(\mathcal{L}_{13}^{\alpha})},\ \ \mathcal{C}_{14}=\overline{\mathcal{O}(\mathcal{L}_{16}^{\alpha})},$$
$$\mathcal{C}_{15}=\overline{\mathcal{O}(\mathcal{L}_{25}^{\alpha,\beta,\gamma})},\ \ \mathcal{C}_{16}=\overline{\mathcal{O}(\mathcal{L}_{26}^{\alpha,\beta})},\ \ \mathcal{C}_{17}=\overline{\mathcal{O}(\mathcal{L}_{42}^{\alpha,\beta})},\ \ \mathcal{C}_{18}=\overline{\mathcal{O}(\mathcal{L}_{48}^{\alpha})},\ \ \mathcal{C}_{19}=\overline{\mathcal{O}(\mathcal{L}_{62}^{\alpha})},\ \ \textit{and}$$
$$\mathcal{C}_{20}=\overline{\mathcal{O}(\mathcal{L}_{74}^{\alpha,\beta})}.$$

In particular, there are only 4 rigid algebras in this variety.

Proof. The proof of our Theorem will be based on two big parts: first, we give the list of all necessary degenerations, and then, we present all reasons for non-degeneration.

Degenerations. Let us give some useful degenerations for our proof.

$\mathcal{L}_{25}^{t,\ 1,\ \frac{1}{t}-\alpha t}$	$\xrightarrow{te_1-e_2,\ te_2-\frac{1}{t}e_3,\ e_3-t^2e_5,\ e_4,\ -t^3e_5}$	$\mathcal{L}_{01}^{\alpha}$
$\mathcal{L}_{25}^{0,\ \alpha t,\ 2\alpha}$	$\xrightarrow{te_1-e_2,\ te_2+e_3,\ te_3+t^2e_5,\ te_4+t^4e_5,\ t^3e_5}$	$\mathcal{L}_{02}^{\alpha}$
$\mathcal{L}_{25}^{0,\ 0,\ -t}$	$\xrightarrow{te_1-e_2,\ te_2+e_3,\ -te_3-t^2e_5,\ e_4-t^4e_5,\ -t^3e_5}$	$\mathcal{L}_{03}$
$\mathcal{L}_{25}^{\frac{1}{t^2},\ -\frac{1+t}{t^2},\ \frac{1+2t+(1+\alpha t^2)t^2}{t^2}}$	$\xrightarrow{-e_1+(-1-t)e_2+e_4,\ t^2e_2+te_3,\ e_3+te_4,\ t^2e_4,\ t^2e_5}$	$\mathcal{L}_{04}^{\alpha}$
$\mathcal{L}_{25}^{0,\ -\alpha,\ 2\alpha(1-t)}$	$\xrightarrow{-e_1+(t-1)e_2,\ t^2e_2+te_3+te_4,\ e_3+(1-t)e_4,\ t(t-1)e_4,\ t^2e_5}$	$\mathcal{L}_{05}^{\alpha}$
$\mathcal{L}_{25}^{-\frac{t^2}{(t-1)^2},\ \frac{t^2}{1-t},\ 0}$	$\xrightarrow{-e_1+(t-1)e_2,\ t^2e_2+te_4,\ e_3-\frac{1}{t}e_4,\ e_4,\ t^2e_5}$	$\mathcal{L}_{06}$
$\mathcal{L}_{25}^{\frac{1}{t},\ -\frac{\alpha}{t},\ \frac{\alpha^2+\beta t^2}{t}}$	$\xrightarrow{-e_1-\alpha e_2,\ te_2+te_3,\ e_3,\ e_4+\frac{t^2}{\alpha}e_5,\ te_5}$	$\mathcal{L}_{07}^{\alpha,\beta}$
$\mathcal{L}_{25}^{0,\ -\beta,\ 2\alpha\beta}$	$\xrightarrow{-e_1-\alpha e_2+e_4,\ te_2+te_3-\frac{t}{\alpha}e_4,\ e_3,\ e_4+t^2e_5,\ te_5}$	$\mathcal{L}_{08}^{\alpha,\beta}$
$\mathcal{L}_{25}^{0,\ 0,\ t}$	$\xrightarrow{-e_1-\alpha e_2-\frac{\alpha}{t}e_4,\ te_2+e_4,\ e_3+te_4+(\alpha-1)t^3e_5,\ e_4+t^2e_5,\ te_5}$	$\mathcal{L}_{09}^{\alpha}$

[§] In general, due to omitted Lie algebras in the variety of symmetric Leibniz algebras, [2, Theorem E] is not correct. But it gives a description of irreducible non-Lie components in 4-dimensional symmetric Leibniz algebras.

$\mathcal{L}_{25}^{0,\ 0,\ -\alpha}$	$\xrightarrow{te_1+te_2,\ e_2-\frac{1}{t}e_3,\ e_4,\ e_3-te_5,\ -t^2e_5}$	$\mathcal{L}_{10}^{\alpha}$
$\mathcal{L}_{25}^{\alpha,\ \alpha t^2,\ t^2+\alpha t^4}$	$\xrightarrow{-e_1+t^2e_2,\ te_2+te_3-e_4,\ t^2e_4+te_5,\ e_3,\ t^2e_5}$	$\mathcal{L}_{11}^{\alpha}$
$\mathcal{L}_{25}^{\alpha,\ t,\ \beta t^2}$	$\xrightarrow{-e_1+\frac{1}{t}e_4,\ te_2+te_3,\ e_4+te_5,\ e_3,\ t^2e_5}$	$\mathcal{L}_{12}^{\alpha,\beta}$
$\mathcal{L}_{13}^{-\alpha^2}$	$\xrightarrow{e_1-\alpha e_2,\ te_2+t^2e_4,\ e_3,\ te_4,\ -te_5}$	$\mathcal{L}_{14}^{\alpha}$
$\mathcal{L}_{13}^{-t}$	$\xrightarrow{e_1,\ te_2+te_4,\ te_3,\ e_4,\ -te_5}$	$\mathcal{L}_{15}$
$\mathcal{L}_{25}^{\alpha,\ -\alpha,\ t+\alpha}$	$\xrightarrow{e_1+e_2,\ te_2+te_3-e_4,\ e_3-\frac{1}{t}e_4,\ e_4,\ te_5}$	$\mathcal{L}_{18}$
$\mathcal{L}_{25}^{0,\ 1,\ -2}$	$\xrightarrow{-e_1-e_2,\ te_2,\ e_3-e_4,\ te_4,\ te_5}$	$\mathcal{L}_{19}$
$\mathcal{L}_{25}^{0,\ 1,\ -2+t}$	$\xrightarrow{e_1+e_2+e_4,\ te_2+te_3,\ e_3-e_4,\ te_4,\ te_5}$	$\mathcal{L}_{20}$
$\mathcal{L}_{25}^{\frac{1}{t},\ -\frac{1}{t},\ \frac{1}{t}}$	$\xrightarrow{e_1+e_2,\ te_2+t^2e_3-t^2e_4,\ e_3-e_4,\ te_4,\ te_5}$	$\mathcal{L}_{21}$
$\mathcal{L}_{25}^{\frac{1}{t},\ -\frac{1}{t},\ \frac{1+t^2}{t}}$	$\xrightarrow{e_1+e_2,\ te_2+te_3+te_4,\ e_3+e_4,\ -te_4+t^2e_5,\ te_5}$	$\mathcal{L}_{22}$
$\mathcal{L}_{25}^{\frac{1}{t},\ \frac{t-1}{t},\ \frac{\alpha t^2-2t+1}{t}}$	$\xrightarrow{e_1+e_2,\ te_2,\ e_3+e_4,\ -te_4,\ te_5}$	$\mathcal{L}_{23}^{\alpha}$
$\mathcal{L}_{26}^{\alpha,\ \frac{(1-\alpha)t^2+1+\alpha}{t}}$	$\xrightarrow{\qquad\qquad}$	$\mathcal{L}_{27}^{\alpha}$
$E_1^t=e_1+te_2$ $E_3^t=e_3+(1+\alpha)(1+t^2)e_4+(t+t^3)e_5$ $E_5^t=\frac{(1+\alpha)^2(1+t^2)^2}{t^2}e_5$	$E_2^t=\frac{(1+\alpha)(1+t^2)}{t}e_2$ $E_4^t=\frac{(1+\alpha)(1+t^2)}{t}e_4+(1+t^2)e_5$	
$\mathcal{L}_{26}^{\alpha,\ 0}$	$\xrightarrow{e_1,\ \frac{1}{t}e_2,\ e_3+e_4,\ \frac{1}{t}e_4,\ \frac{1}{t^2}e_5}$	$\mathcal{L}_{28}^{\alpha}$
$\mathcal{L}_{26}^{1+\frac{1}{t},\ \frac{2+4t+(1+\alpha)t^2}{t\sqrt{-1-2t}}}$	$\xrightarrow{e_1+\frac{t-1}{\sqrt{2t-1}}e_2,\ \frac{-t}{\sqrt{2t-1}}e_2,\ e_3,\ \frac{t^2}{2t-1}e_5,\ \frac{1}{\sqrt{2t-1}}e_4+\frac{t}{2t-1}e_5}$	$\mathcal{L}_{29}^{\alpha}$
$\mathcal{L}_{29}^{t}$	$\xrightarrow{e_1,\ te_2,\ e_3+t^2e_4+t^2e_5,\ te_4,\ te_5}$	$\mathcal{L}_{30}$
$\mathcal{L}_{01}^{\frac{1}{t^2}}$	$\xrightarrow{-e_1-\frac{1}{t}e_2,\ te_2,\ e_4,\ te_3,\ e_5}$	$\mathcal{L}_{31}$
$\mathcal{L}_{26}^{-2,\ \frac{1+8t}{\sqrt{-t(1+4t)}}}$	$\xrightarrow{\qquad\qquad}$	$\mathcal{L}_{32}$
$E_1^t=e_1-\frac{2t}{\sqrt{t(1-4t)}}e_2$ $E_4^t=\frac{t}{1-4t}e_5$	$E_2^t=\frac{-t}{\sqrt{t(1-4t)}}e_2$ $E_5^t=\frac{1}{\sqrt{t(1-4t)}}e_4+\frac{1}{1-4t}e_5$	$E_3^t=e_3$
$\mathcal{L}_{26}^{\alpha,\ -\frac{2\alpha(1+\alpha)\sqrt{t}}{\sqrt{1-t(1+\alpha)^2}}}$	$\xrightarrow{\qquad\qquad}$	$\mathcal{L}_{33}^{\alpha}$
$E_1^t=e_1+\frac{(1+\alpha)\sqrt{t}}{\sqrt{1-t(1+\alpha)^2}}e_2$ $E_3^t=e_3+\frac{\sqrt{t^3}}{\sqrt{1-t(1+\alpha)^2}}e_4+\frac{t^2}{1-t(1+\alpha)^2}e_5$ $E_5^t=\frac{1}{1-t(1+\alpha)^2}e_5$	$E_2^t=\frac{\sqrt{t}}{\sqrt{1-t(1+\alpha)^2}}e_2$ $E_4^t=\frac{\sqrt{t}}{\sqrt{1-t(1+\alpha)^2}}e_4+\frac{t}{1-t(1+\alpha)^2}e_5$	
$\mathcal{L}_{74}^{-\frac{1}{t},\ -\frac{2t}{1+\alpha t^2}}$	$\xrightarrow{te_1+e_2+\frac{t-\alpha t^3}{1+\alpha t^2}e_3,\ te_2+\frac{2t^2}{1+\alpha t^2}e_3,\ ite_2,\ e_4,\ \frac{2t^3}{1+\alpha t^2}e_5}$	$\mathcal{L}_{34}^{\alpha}$
$\mathcal{L}_{34}^{-\alpha^2}$	$\xrightarrow{e_1+\alpha e_2,\ t^2e_2+te_4,\ te_3,\ e_4,\ t^2e_5}$	$\mathcal{L}_{35}^{\alpha}$
$\mathcal{L}_{25}^{-\frac{2}{t},\ \frac{2-t^2}{t^2},\ -\frac{2}{t^3}}$	$\xrightarrow{te_1+e_2,\ e_3+2e_5,\ te_2,\ e_4,-2te_5}$	$\mathcal{L}_{36}$

$\mathcal{L}_{34}^{0}$	$\xrightarrow{e_1,\ -\frac{1}{t}e_2+\frac{i}{t}e_3+te_4,\ ie_3,\ e_4,\ -\frac{1}{t}e_5}$	$\mathcal{L}_{37}$
$\mathcal{L}_{34}^{-\frac{1}{t}}$	$\xrightarrow{e_1,\ -\frac{1}{t}e_2+\frac{i}{t}e_3+te_4,\ ie_3,\ e_4,\ -\frac{1}{t}e_5}$	$\mathcal{L}_{38}$
$\mathcal{L}_{25}^{\frac{2}{t},\ -\frac{1}{t^2},0}$	$\xrightarrow{te_1+e_2-e_4,\ e_3-\frac{1}{t}e_5,\ te_2,\ e_4+te_5,\ e_5}$	$\mathcal{L}_{39}$
$\mathcal{L}_{74}^{\frac{1}{t^2}+1,\ \frac{2t(2t+1)\left(t^2+1\right)}{2t^4+t^3+4t^2+2t+2}}$ $E_1^t=-e_1+e_2-\frac{t^2\left(2t^2+t+2\right)}{2t^4+t^3+4t^2+2t+2}e_3$ $\quad E_2^t=te_2-\frac{2\left(t^5+t^3\right)}{2t^4+t^3+4t^2+2t+2}e_3$ $E_3^t=-\frac{2t^5\left(t^2+1\right)}{2t^4+t^3+4t^2+2t+2}e_3+te_4$ $\quad E_4^t=e_4+\frac{2t^4\left(t^2+1\right)}{2t^4+t^3+4t^2+2t+2}e_5$ $E_5^t=\frac{2\left(t^5+t^3\right)}{2t^4+t^3+4t^2+2t+2}e_5$	$\xrightarrow{\qquad\qquad}$	$\mathcal{L}_{40}$
$\mathcal{L}_{42}^{\frac{1}{1+t^2},\ \frac{1+t}{1+t^2}}$	$\xrightarrow{(1+t^2)e_1,\ e_2+(1+t)e_3+(1+t)e_4,\ te_3+t^2e_4,\ (t^4-t^3)e_4,\ (1+t^2)^2e_5}$	$\mathcal{L}_{41}$
$\mathcal{L}_{42}^{\frac{1}{1+t},\ \frac{\alpha}{1+t}}$	$\xrightarrow{(1+t)e_1,\ e_2+\frac{1}{t}e_4,\ e_3,\ e_4,\ (1+t)^2e_5}$	$\mathcal{L}_{43}$
$\mathcal{L}_{13}^{\alpha}$	$\xrightarrow{\frac{1}{t}e_2,\ e_1,\ \frac{1}{t^2}e_3,\ e_4,\ \frac{1}{t^2}e_5}$	$\mathcal{L}_{45}$
$\mathcal{L}_{13}^{-\frac{1}{t^4}}$	$\xrightarrow{e_2+\frac{1}{t}e_4,\ e_1+\frac{1}{t^2}e_2,\ e_3,\ \frac{1}{t^2}e_4,\ \frac{1}{t^2}e_5}$	$\mathcal{L}_{47}$
$\mathcal{L}_{48}^{1+t}$	$\xrightarrow{e_1,\ e_2+e_3,\ te_3,\ -te_4,\ e_5}$	$\mathcal{L}_{49}$
$\mathcal{L}_{25}^{0,\ 0,\ 1}$	$\xrightarrow{e_1+e_2,\ te_2,\ e_3+e_4,\ -te_4,\ e_5}$	$\mathcal{L}_{50}$
$\mathcal{L}_{25}^{\frac{1+t^2}{t^2},\ 0,\ -\frac{1}{t^2}}$	$\xrightarrow{e_1+e_2,\ te_2,\ e_3+e_4,\ -te_4+e_5,\ \frac{1}{t}e_5}$	$\mathcal{L}_{51}$
$\mathcal{L}_{25}^{0,\ \frac{1}{t^2},\ \frac{t-2}{t^2}}$	$\xrightarrow{e_1+e_2+te_4,\ te_2,\ e_3+e_4,\ -te_4+e_5,\ \frac{1}{t}e_5}$	$\mathcal{L}_{52}$
$\mathcal{L}_{25}^{\frac{1}{t^2},\ -\frac{1+t^2}{t^2},\ \frac{1+2t^2}{t^2}}$	$\xrightarrow{e_1+e_2,\ te_2,\ e_3+e_4,\ -te_4,\ e_5}$	$\mathcal{L}_{53}$
$\mathcal{L}_{25}^{\frac{1}{t^2},\ -\frac{1+t^2}{t^2},\ \frac{1+3t^2}{t^2}}$	$\xrightarrow{e_1+e_2,\ te_2,\ e_3+e_4,\ -te_4,\ e_5}$	$\mathcal{L}_{54}$
$\mathcal{L}_{25}^{\frac{1}{t^3},\ \frac{t-1}{t^3},\ \frac{1-2t+\alpha t^2}{t^3}}$	$\xrightarrow{e_1+e_2+te_4,\ te_2,\ e_3+e_4,\ -te_4+e_5,\ \frac{1}{t}e_5}$	$\mathcal{L}_{55}^{\alpha}$
$\mathcal{L}_{25}^{0,\ 1,\ \frac{1}{t}}$	$\xrightarrow{e_1,\ e_2,\ e_3,\ e_4,\frac{1}{t}e_5}$	$\mathcal{L}_{56}$
$\mathcal{L}_{25}^{0,\ \frac{1}{t^2},\ \frac{\alpha}{t^2}}$	$\xrightarrow{e_1-te_4,\ e_2,\ e_3,\ e_4+\frac{1}{t}e_5,\ \frac{1}{t^2}e_5}$	$\mathcal{L}_{57}^{\alpha}$
$\mathcal{L}_{25}^{\frac{1}{t^2},\ \frac{\beta}{t^2},\ \frac{\alpha}{t^2}}$	$\xrightarrow{e_1-te_4,\ e_2,\ e_3,\ e_4+\frac{1}{t}e_5,\ \frac{1}{t^2}e_5}$	$\mathcal{L}_{58}^{\alpha,\beta}$
$\mathcal{L}_{42}^{-\frac{1}{1+t},\ -1}$	$\xrightarrow{(1+t)e_1,\ e_2+e_3,\ te_3,\ e_4,\ (1+t)^2e_5}$	$\mathcal{L}_{59}$
$\mathcal{L}_{25}^{0,\ 0,\ 0}$	$\xrightarrow{t^2e_2,\ -e_1-(1+t)e_2,\ e_3+e_4,\ te_4+t^3e_5,\ -t^2e_5}$	$\mathcal{L}_{60}$
$\mathcal{L}_{25}^{\frac{1}{t^4},\ -\frac{1+t+t^4}{t^4},\ \frac{1+2t+t^2+3t^4+2t^5}{t^4}}$	$\xrightarrow{t^2e_2,\ -e_1-(1+t)e_2,\ e_3+e_4,\ te_4,\ e_5}$	$\mathcal{L}_{61}$
$\mathcal{L}_{25}^{0,\ \frac{1}{t^2},\ -\frac{2\alpha}{t^2}}$	$\xrightarrow{e_1+\alpha e_2-e_4,\ te_2,\ e_3,\ e_4+e_5,\ \frac{1}{t}e_5}$	$\mathcal{L}_{63}^{\alpha}$
$\mathcal{L}_{25}^{\frac{1}{t^2},\ -\frac{\alpha+t^2}{t^2},\ \frac{(1+2\alpha)t^2+\alpha^2}{t^2}}$	$\xrightarrow{e_1+\alpha e_2,\ te_2,\ e_3,\ e_4,\ e_5}$	$\mathcal{L}_{64}^{\alpha}$
$\mathcal{L}_{26}^{\frac{1}{t^2},\ -\frac{2i}{t^2}}$	$\xrightarrow{\qquad\qquad}$	$\mathcal{L}_{65}$

$E_1^t = \frac{it}{1-t^2}e_2 + te_3 \qquad E_2^t = e_1 + ie_2 \qquad E_3^t = e_3 + \frac{i}{1-t^2}e_4 - \frac{t^2}{(t^2-1)^2}e_5$

$E_4^t = -\frac{t^2}{(t^2-1)^2}e_5 \qquad E_5^t = \frac{i}{t-t^3}e_4 - \frac{t}{(t^2-1)^2}e_5$

$$\mathcal{L}_{26}^{\frac{1}{t^2},\ -\frac{2\sqrt{3}}{t^2}} \xrightarrow{2e_2+te_3,\ e_1+\sqrt{3}e_2,\ e_3+\frac{2}{t}e_4,\ 4e_5,\ \frac{2}{t^2}e_4+2\sqrt{3}e_5} \mathcal{L}_{66}$$

$$\mathcal{L}_{74}^{1,\ 0} \xrightarrow{-2t^2e_3,\ e_2,\ te_1+te_2-te_3,\ e_4-2t^3e_5,\ -2t^2e_5} \mathcal{L}_{67}$$

$$\mathcal{L}_{74}^{\frac{2}{t^2}-1,\ -\frac{2\left(t^7-t^6-2t^5+4t^4+2t^3-4t^2+4\right)}{t^3\left(t^6-4t^4+4t^2-4\right)}} \longrightarrow \mathcal{L}_{68}$$

$E_1^t = \frac{1}{t}e_1 + te_2 + \frac{1}{2}t^2\left(\frac{2t\left(t^4-t^2+2\right)}{t^6-4t^4+4t^2-4} - 1\right)e_3 \qquad E_2^t = e_2 + \frac{t^2\left(t^4-2t^2+2\right)}{t^6-4t^4+4t^2-4}e_3$

$E_3^t = e_3 + te_4 \qquad E_4^t = e_4 + \frac{1}{t}e_5 \qquad E_5^t = \frac{2}{t^3}e_5$

$$\mathcal{L}_{25}^{\frac{\beta}{t^2},\ \frac{1}{t^2},\ \frac{\alpha}{t^2}} \xrightarrow{e_2,\ e_1+te_4,\ e_3,\ e_4+\frac{1}{t}e_5,\ \frac{1}{t^2}e_5} \mathcal{L}_{69}^{\alpha,\beta}$$

$$\mathcal{L}_{25}^{\frac{1}{t^2},\ 1,\ \frac{\alpha}{t^2}} \xrightarrow{e_2,\ e_1,\ e_3,\ e_4,\ \frac{1}{t^2}e_5} \mathcal{L}_{70}^{\alpha}$$

$$\mathcal{L}_{73}^{\alpha} \xrightarrow{te_1,\ e_2,\ \frac{1}{t}e_3,\ e_4,\ e_5} \mathcal{L}_{71}^{\alpha}$$

$$\mathcal{L}_{73}^{\frac{\alpha}{1+t^8}} \longrightarrow \mathcal{L}_{72}^{\alpha}$$

$E_1^t = \frac{1+2\alpha-\alpha^2}{t^4((1+t^8)^2+2\alpha(1+t^8)-\alpha^2)}e_1 - te_2 + \frac{2t+(2+3\alpha-\alpha^2)t^3+4t^8+(1+\alpha)t^{11}+2t^{16}}{2\alpha(1+t^8)}e_4$

$E_2^t = e_2 + \frac{t^5(2+4\alpha-2\alpha^2+(2+5\alpha+4\alpha^2-\alpha^3)t^3+(6+8\alpha-2\alpha^2)t^8+(3+5\alpha+2\alpha^2)t^{11}+(6+4\alpha)t^{16}+(1+\alpha)t^{19}+2t^{24})}{2\alpha(1+2\alpha-\alpha^2)}e_3$

$E_3^t = \frac{(1+t^8)((1+t^8)^2+2\alpha(1+t^8)-\alpha^2)}{1+2\alpha-\alpha^2}e_3 + t^4e_4 \qquad E_4^t = e_4 - \frac{1}{t}e_5 \qquad E_5^t = \frac{1}{t^4}e_5$

$$\mathcal{L}_{74}^{\alpha,\ -\frac{1}{t^2}} \xrightarrow{-\frac{1}{t^2}e_1,\ e_2,\ e_3+te_4,\ e_4-\frac{1}{t}e_5,\ -\frac{1}{t^2}e_5} \mathfrak{L}_{73}^{\alpha}$$

$$\mathcal{L}_{74}^{t^2,\ \frac{t^2}{t^2-1}} \longrightarrow \mathcal{L}_{75}$$

$E_1^t = \frac{1-t^2}{2t}e_1 - te_2 - \frac{1+t^4}{t-t^5}e_3 \qquad E_2^t = e_2 - \frac{1}{1+t^2}e_3 \qquad E_3^t = \frac{2t}{(t^2-1)(1+t^4)}e_3$

$E_4^t = e_4 + \frac{t}{t^2-1}e_5 \qquad E_5^t = \frac{1}{t^2-1}e_5$

$$\mathcal{L}_{74}^{-\frac{t^2}{2+t^2},\ \frac{1}{2}\alpha t^2(2+t^2)} \longrightarrow \mathfrak{L}_{76}^{\alpha}$$

$E_1^t = \frac{1}{2}(2+t^2)e_1 + t^2e_2 - \frac{1}{2}(2+t^2)(1+\alpha t^4)e_3 \qquad E_2^t = e_2$

$E_3^t = (t^2+\frac{t^4}{2})e_3 + te_4 \qquad E_4^t = e_4 \qquad E_5^t = (t^2+\frac{t^4}{2})e_5$

$$\mathcal{L}_{74}^{-t^2,\ \frac{2\alpha}{1-\alpha t^2}} \longrightarrow \mathcal{L}_{77}^{\alpha}$$

$E_1^t = \frac{1}{t}e_1 + te_2 - \frac{\alpha t^2-2t+1}{t-\alpha t^3}e_3 \qquad E_2^t = e_2 \qquad E_3^t = -\frac{2t}{\alpha t^2-1}e_3$

$E_4^t = e_4 \qquad E_5^t = \frac{2}{1-\alpha t^2}e_5$

$$\mathcal{L}_{13}^{\frac{1}{t}} \xrightarrow{-e_1,\ -\frac{1}{t}e_3,\ e_4,\ \frac{1}{t}e_5,\ e_2} \mathfrak{L}_{02}$$

$$\mathcal{L}_{25}^{-\frac{2}{t},\ -1,\ -2\alpha t} \xrightarrow{e_1,\ te_2,\ e_3,\ -2te_5,\ e_4} \mathfrak{L}_{15}^{\alpha}$$

$$\mathcal{L}_{25}^{\frac{1}{t},\ -1-\frac{1}{\alpha t},\ \frac{1+t+2\alpha t}{\alpha^2 t}} \xrightarrow{\alpha e_1+e_2,\ e_4,\ e_3,\ e_5,\ te_2} \mathfrak{L}_{24}^{\alpha}$$

$$\mathcal{L}_{74}^{-\frac{4\alpha+\sqrt{1-4\alpha}-1}{\sqrt{1-4\alpha}-1},\ \frac{8\left(\sqrt{1-4\alpha}-1\right)}{\left(4\alpha+\sqrt{1-4\alpha}-1\right)\left(t^2-4\right)}} \longrightarrow \mathbb{S}_{22}$$

$E_1^t=-\frac{(\sqrt{1-4\alpha}-1)t}{4\alpha+\sqrt{1-4\alpha}-1}e_1-te_2+\frac{(\sqrt{1-4\alpha}-1)t(t^2+4)}{(4\alpha+\sqrt{1-4\alpha}-1)(t^2-4)}e_3-\frac{4}{(\sqrt{1-4\alpha}-1)t^2}e_4$		
$E_2^t=-\frac{1}{4}\left(\sqrt{1-4\alpha}-1\right)t^2e_2-\frac{2t^2}{t^2-4}e_3$	$E_3^t=e_4+\frac{(\sqrt{1-4\alpha}-1)^3t^3}{(2\alpha+\sqrt{1-4\alpha}-1)(4\alpha+\sqrt{1-4\alpha}-1)(t^2-4)}e_5$	
$E_4^t=\frac{t^2}{2}e_2+\frac{2}{(\sqrt{1-4\alpha}-1)t}e_4$	$E_5^t=-\frac{(\sqrt{1-4\alpha}-1)^3t^4}{(2\alpha+\sqrt{1-4\alpha}-1)(4\alpha+\sqrt{1-4\alpha}-1)(t^2-4)}e_5$	
$\mathcal{L}_{13}^{\alpha t^4}$	$\xrightarrow{t^3e_2-\frac{1}{t}e_3+\frac{t^6}{2}e_4,\ te_1+\frac{2}{t^2}e_3,\ e_3+\frac{t^7}{2}e_4,\ -t^8e_4-t^5e_5,\ t^6e_5}$	$\mathbb{S}_{41}^{\alpha}$

Non-degenerations. To prove all non-degenerations, we will work with the algebras listed in the table below. The present table also contains some useful information about the nilpotent radical, dimension of orbits, and the dimension of derived algebras. The present information can justify some obvious non-degeneration reasons, which will be omitted without detailed explication.

Algebra	Rad	dim Rad	dim A^2	dim Orb
$\mathfrak{V}_{3+2}$	$\mathfrak{V}_{3+2}$	5	2	24
$\mathcal{L}_{17}$	$\mathfrak{n}_3\oplus\mathbb{C}$	4	4	22
$\mathcal{L}_{25}^{\alpha,\beta,\gamma}$	$\mathbb{C}^3$	3	3	22
$\mathcal{L}_{74}^{\alpha,\beta}$	$\mathbb{C}^3$	3	2	22
$\mathcal{L}_{24}$	$\mathfrak{n}_4$	4	4	21
$\mathcal{L}_{13}^{\alpha}$	$\mathfrak{l}_2$	4	3	21
$\mathcal{L}_{26}^{\alpha,\beta}$	$\mathfrak{l}_1\oplus\mathbb{C}^2$	4	3	21
$\mathbb{S}_{21}^{\alpha,\beta}$	$\mathbb{S}_{21}^{\alpha,\beta}$	5	3	21
$\mathcal{L}_{16}^{\alpha}$	$\mathfrak{n}_3\oplus\mathbb{C}$	4	4	20
$\mathcal{L}_{48}^{\alpha}$	$\mathfrak{n}_3\oplus\mathbb{C}$	4	4	20
$\mathcal{L}_{62}^{\alpha}$	$\mathbb{C}^4$	4	4	20
$\mathcal{L}_{42}^{\alpha,\beta}$	$\mathbb{C}^4$	4	4	20
$\mathcal{L}_{46}$	$\mathfrak{n}_3$	3	3	20
$\mathfrak{V}_{4+1}$	$\mathfrak{V}_{4+1}$	5	1	20
$\mathfrak{V}_{2+3}$	$\mathfrak{V}_{2+3}$	5	3	18

Here, we are using the following low-dimensional algebras:

$\mathfrak{l}_1$	:	$e_1e_1=e_2$			
$\mathfrak{l}_2$	:	$e_1e_1=e_4$	$e_2e_3=e_4$	$e_3e_2=-e_4$	
$\mathfrak{n}_3$	:	$e_1e_2=e_3$	$e_2e_1=-e_3$		
$\mathfrak{n}_4$	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_2e_1=-e_3$	$e_3e_1=-e_4$

It is known that $\mathfrak{n}_3\oplus\mathbb{C}\not\to\left\{\mathfrak{n}_4,\ \mathfrak{l}_2,\ \mathfrak{l}_1\oplus\mathbb{C}^2\right\}$.

The list below comprises reasons for all the necessary non-degenerations.

$\mathcal{L}_{17}$	$\not\to$	$\left\{\mathbb{S}_{21}^{\alpha,\beta},\ \mathcal{L}_{16}^{\alpha},\ \mathcal{L}_{42}^{\alpha,\beta},\ \mathcal{L}_{48}^{\alpha},\ \mathcal{L}_{62}^{\alpha},\ \mathfrak{V}_{2+3},\ \mathfrak{V}_{4+1}\right\}$
$\mathcal{R}$	$=$	$\left\{\begin{array}{l} A_3^2+A_4A_2+A_2A_4+A_5A_1+A_1A_5=0, \\ A_1A_2+A_2A_1\subseteq A_2,\ A_1A_3+A_3A_1\subseteq A_3,\ A_1A_4+A_4A_1\subseteq A_4, \\ c_{11}^1=c_{11}^2=c_{11}^3=c_{11}^4=0,\ c_{22}^1=c_{22}^2=c_{22}^3=c_{22}^4=c_{22}^5=0, \\ c_{12}^2=-c_{13}^3=-c_{14}^4,\ c_{13}^5=-c_{31}^5,\ c_{14}^5=-c_{41}^5,\ c_{23}^5=-c_{32}^5 \end{array}\right\}$
$\mathcal{L}_{25}^{\alpha,\beta,\gamma}$	$\not\to$	$\left\{\mathbb{S}_{21}^{\alpha,\beta},\ \mathfrak{V}_{2+3},\ \mathfrak{V}_{4+1}\right\}$
$\mathcal{R}$	$=$	$\left\{\begin{array}{l} A_3^2+A_5A_1+A_1A_5=0, \\ A_1^2\subseteq A_2,\ A_1A_3+A_3A_1\subseteq A_3,\ A_1A_4+A_4A_1\subseteq A_4, \\ c_{11}^1=c_{11}^2=c_{11}^3=c_{11}^4=0,\ c_{22}^1=c_{22}^2=c_{22}^3=c_{22}^4=0, \\ c_{12}^4=-c_{21}^4,\ c_{13}^5=-c_{31}^5,\ c_{14}^5=-c_{41}^5,\ c_{23}^5=-c_{32}^5 \end{array}\right\}$
$\mathcal{L}_{74}^{\alpha,\beta}$	$\not\to$	$\mathfrak{V}_{4+1}$
$\mathcal{R}$	$=$	$\left\{\ A_3^2+A_5A_1+A_1A_5=0,\ c_{14}^5=-c_{41}^5,\ c_{24}^5=-c_{42}^5\ \right\}$
$\mathcal{L}_{24}$	$\not\to$	$\left\{\mathcal{L}_{16}^{\alpha},\ \mathcal{L}_{42}^{\alpha,\beta},\ \mathcal{L}_{48}^{\alpha},\ \mathcal{L}_{62}^{\alpha},\ \mathfrak{V}_{2+3},\ \mathfrak{V}_{4+1}\right\}$
$\mathcal{R}$	$=$	$\left\{\begin{array}{l} A_1^2\subseteq A_2,\ A_3^2+A_5A_1+A_1A_5=0,\ A_1A_2+A_2A_1\subseteq A_2, \\ A_1A_3+A_3A_1\subseteq A_3,\ A_1A_4+A_3A_2+A_2A_3+A_4A_1\subseteq A_4, \\ 2c_{12}^2=-c_{13}^3=2c_{14}^4,\ c_{13}^4=-c_{31}^4, \\ c_{14}^5=-c_{41}^5,\ c_{13}^5=-c_{31}^5,\ c_{23}^5=-c_{32}^5,\ c_{24}^5=-c_{42}^5, \\ c_{11}^1=c_{11}^2=c_{11}^3=c_{11}^4=0,\ c_{22}^1=c_{22}^2=c_{22}^3=c_{22}^4=c_{22}^5=0 \end{array}\right\}$
$\mathcal{L}_{13}^{\alpha}$	$\not\to$	$\left\{\mathfrak{V}_{2+3},\ \mathfrak{V}_{4+1}\right\}$
$\mathcal{R}$	$=$	$\left\{\begin{array}{l} A_1A_5+A_5A_1=0,\ c_{34}^5=-c_{43}^5,\ c_{24}^5=-c_{42}^5,\ c_{34}^5=-c_{43}^5, \\ c_{kk}^j=0,\ 1\le k\le 5\ \&\ 1\le j\le 4 \end{array}\right\}$
$\mathcal{L}_{26}^{\alpha,\beta}$	$\not\to$	$\left\{\mathfrak{V}_{2+3},\ \mathfrak{V}_{4+1}\right\}$
$\mathcal{R}$	$=$	$\left\{\ A_3^2+A_1A_4+A_4A_1=0,\ c_{11}^3=c_{22}^3=c_{22}^4=0,\ c_{12}^3=-c_{21}^3\ \right\}$
$\left\{\mathcal{L}_{16}^{\alpha},\ \mathcal{L}_{42}^{\alpha,\beta},\ \mathcal{L}_{46}^{\alpha},\ \mathcal{L}_{48}^{\alpha},\ \mathcal{L}_{62}^{\alpha}\right\}\not\to\mathfrak{V}_{2+3}$		
$\mathcal{R}$	$=$	$\left\{\ c_{kk}^j=0,\ 1\le k\le 5\ \&\ 1\le j\le 4\ \right\}$

$\square$

Corollary. *The variety of complex* 5*-dimensional symmetric Leibniz algebras has dimension* 24 *and it has* 23 *irreducible components defined by solvable algebras given in Theorem and*

$$\mathcal{C}_{21}=\overline{\mathcal{O}(\mathfrak{sl}_2\oplus\mathfrak{r}_2)},\ \ \mathcal{C}_{22}=\overline{\mathcal{O}(\mathfrak{sl}_2\ltimes \mathrm{V}_2)},\ \ \mathcal{C}_{23}=\overline{\mathcal{O}(\mathfrak{sl}_2\oplus\mathfrak{l}_1)}.$$

In particular, there are only 7 *rigid algebras in this variety.*

Proof. It is easy to see that there is only one non-Lie non-solvable symmetric Leibniz algebra:

$$\mathfrak{sl}_2\oplus\mathfrak{l}_1\quad:\quad e_1e_2=2e_1\qquad e_1e_3=-e_2\quad e_2e_3=2e_3$$

$$e_2e_1 = -2e_1 \quad e_3e_1 = e_2 \qquad e_3e_2 = -2e_3 \quad e_4e_4 = e_5$$

Obviously, $\dim \mathfrak{Der}(\mathfrak{sl}_2 \oplus \mathfrak{l}_1) = 5$. Hence it can degenerate only to $\mathfrak{V}_{2+3}$, but it is not the case due to the following condition $\mathcal{R} = \left\{ c_{kk}^{j} = 0,\ 1 \leq k \leq 5 \ \&\ 1 \leq j \leq 4 \right\}$. It follows that all irreducible components are defined by all irreducible components from solvable symmetric Leibniz algebras, $\mathfrak{sl}_2 \oplus \mathfrak{r}_2$, $\mathfrak{sl}_2 \ltimes \mathrm{V}_2$, and $\mathfrak{sl}_2 \oplus \mathfrak{l}_1$.

□

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Received: June 27, 2025
Accepted for publication: June 27, 2025
Communicated by: Ivan Kaygorodov