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The geometric classification of nilpotent Lie–Yamaguti, Bol and compatible Lie algebras*

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Abstract. The geometric classifications of complex 4-dimensional nilpotent Lie–Yamaguti algebras, 4-dimensional nilpotent Bol algebras, and 4-dimensional nilpotent compatible Lie algebras are given.

Introduction

The study of algebras endowed with two or more multiplication operations is not a recent development; nevertheless, it continues to attract considerable attention within the mathematical community due to its theoretical richness and potential applications [6, 7, 10, 24, 40]. The class of Poisson algebras is the most well-known example of algebras featuring two multiplication operations.

The notion of the Lie triple system was formally introduced as an algebraic object by Jacobson in connection with problems arising from quantum mechanics. The example of a Lie triple system that arose from quantum mechanics is known as the Meson field, as

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described by Jacobson [21]. The notion of Lie–Yamaguti algebras is a generalization of Lie triple systems and Lie algebras, derived from Nomizu's work on the invariant affine connections on homogeneous spaces in the 1950s [34]. The notion of a Lie–Yamaguti algebra is a natural abstraction made by Yamaguti of Nomizu's considerations. Yamaguti called these systems general Lie triple systems [41]. Kikkawa renamed the notion of a general Lie triple system as Lie triple algebra and showed how to "integrate" these algebras to nonassociative multiplications on reductive homogeneous spaces [25]. Kinyon and Weinstein observed that Lie triple algebras, which they called "Lie–Yamaguti algebras" in their paper, can be constructed from Leibniz algebras [26]. The interplay between their binary and ternary operations reflects the geometric and algebraic intricacies of the spaces from which they originate, making Lie–Yamaguti algebras a significant object of study in both mathematics and theoretical physics. Lie–Yamaguti algebras have been studied by several authors in [1,17,19,38,39,41,42]. In particular, Sagle constructed some remarkable examples of Lie–Yamaguti algebras arising from reductive homogeneous spaces in differential geometry. In [42], the deformation and extension theory of Lie–Yamaguti algebras is studied, and the notion of Nijenhuis operators for Lie–Yamaguti algebras is introduced to describe trivial deformations. The algebraic classification of the nilpotent Lie–Yamaguti algebras up to dimension four was given in [1].

In 1937, Bol introduced the notion of a left (right) Bol loop [8], as a loop satisfying the identity $a(b(ac)) = (a(ba))c$, (resp. $((ca)b)a = c((ab)a)$). Recall that a quasigroup is a non-empty set with a binary operation such that each element occurs exactly once in each row and exactly once in each column of the quasigroup multiplication table, and a quasigroup with an identity element is called a loop. The class of Bol algebras plays a similar role concerning Bol loops, as Lie algebras play concerning Lie groups or Malcev algebras concerning Moufang loops [37]. A connection of the Bol algebras with right-alternative algebras was established in [33], and it was demonstrated that the commutator algebra of an arbitrary right-alternative algebra is a Bol algebra. Filippov classified homogeneous Bol algebras and studied the relation between Bol algebras and Malcev algebras in [12]. Kuz'min and Zaĭdi investigated the solvability and semisimplicity of Bol algebras [28]. Pérez–Izquierdo constructed an envelope for Bol algebras and proved that any Bol algebra is located inside the generalized left alternative nucleus of the envelope [36]. Special identities for Bol algebras are investigated in [20], and it was shown that there are no special identities of degree less than 8. All the special identities of degree 8 in the partition six-two were obtained.

A compatible Lie algebra is a pair of Lie algebras such that any linear combination of these two Lie brackets is still a Lie bracket. Such structures are closely related to the (infinitesimal) deformations of Lie algebras. Compatible Lie algebras are considered in many fields in mathematics and mathematical physics, such as the study of the classical Yang–Baxter equation [16], integrable equations of the principal chiral model type [14], elliptic theta functions [35], and loop algebras over Lie algebras [15]. Compatible Lie algebras have a natural one-to-one correspondence to compatible linear Poisson structures, whereas the latter are important Poisson structures related to bi-Hamiltonian structures. Recall that a bi-Hamiltonian structure is a pair of Poisson structures on a manifold which are compatible, i.e., their sum is again a Poisson structure [27].

Degenerations and geometric properties of a variety of algebras have been an object of study since 1970's. Gabriel [13] described the irreducible components of the variety of 4-dimensional unital associative algebras. Mazzola [32] classified, both algebraically and geometrically, the variety of unital associative algebras of dimension 5. Burde and Steinhoff [9] constructed the graphs of degenerations for the varieties of 3-dimensional and 4-dimensional Lie algebras over $\mathbb{C}$. Grunewald and O'Halloran [18] calculated the degenerations for the nilpotent Lie algebras of dimension up to 5.

The study of degenerations of algebras is very rich and closely related to deformation theory. It offers an insightful geometric perspective on the subject and has been the object of a lot of research. One of the main problems of the geometric classification of a variety of algebras is the description of its irreducible components. In the case of finitely many orbits (i.e., isomorphism classes), the irreducible components are determined by the rigid algebras — algebras whose orbit closure is an irreducible component of the variety under consideration. There are many results related to the deformation and geometric classification of algebras in different varieties of associative and non-associative algebras, superalgebras and Poisson algebras, (see [3,4,7,9,11,22,23,30,31] and references in [22,23]).

In the present paper, we continue the study of the geometric classification of algebras and describe all irreducible components of the variety of four-dimensional nilpotent Lie–Yamaguti algebras, four-dimensional nilpotent Bol algebras, three- and four-dimensional nilpotent compatible Lie algebras. Our main results are summarized below.

Theorem A. *The variety of complex four-dimensional nilpotent Lie–Yamaguti algebras has dimension* 14*. It is defined by one rigid algebra and two* 1*-parametric families of algebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the algebras given in Theorem 2.7.*

Theorem B. *The variety of complex four-dimensional nilpotent Bol algebras has dimension* 15*. It is defined by one rigid algebra and one* 1*-parametric family of algebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the algebras given in Theorem 2.8.*

Theorem C. *The variety of complex four-dimensional nilpotent compatible Lie algebras has dimension* 13*. It is defined by one* 1*-parametric family of algebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the algebras given in Theorem 2.10.*

1 Preliminaries: the algebraic classification

All the algebras below will be over $\mathbb{C}$ and all the linear maps will be $\mathbb{C}$-linear. For simplicity, every time we write the multiplication table of an algebra the products of basis elements whose values are zero or can be recovered from the commutativity or from the anticommutativity are omitted. The notion of a nontrivial algebra means that the multiplication is nonzero. In this section, we present the necessary concepts related to

Lie–Yamaguti algebras, Bol algebras, and compatible Lie algebras. We present the algebraic classification of four-dimensional nilpotent Lie–Yamaguti algebras, four-dimensional nilpotent Bol algebras.

Definition 1.1. Let $\mathcal{L}$ be an algebra with one bilinear multiplication $[\cdot,\cdot]$ and one trilinear multiplication $[\cdot,\cdot,\cdot]$ $\big($resp., two bilinear multiplications $[\cdot,\cdot]$ and $\{\cdot,\cdot\}\big)$. Then $\mathfrak{D}$ is a derivation if it satisfies

$$\mathfrak{D}[x,y] = [\mathfrak{D}(x),y] + [x,\mathfrak{D}(y)],\ \mathfrak{D}[x,y,z] = [\mathfrak{D}(x),y,z] + [x,\mathfrak{D}(y),z] + [x,y,\mathfrak{D}(z)].$$
$$\big(\text{resp., } \mathfrak{D}[x,y] = [\mathfrak{D}(x),y] + [x,\mathfrak{D}(y)],\ \mathfrak{D}\{x,y\} = \{\mathfrak{D}(x),y\} + \{x,\mathfrak{D}(y)\}\big).$$

We denote the set of all derivations of the algebra $\mathcal{L}$ by $\mathfrak{Der}(\mathcal{L})$.

Definition 1.2. Let $\mathcal{L}$ be an algebra with one bilinear multiplication $[\cdot,\cdot]$ and one trilinear multiplication $[\cdot,\cdot,\cdot]$ $\big($resp., two bilinear multiplications $[\cdot,\cdot]$ and $\{\cdot,\cdot\}\big)$. Then $\mathcal{L}$ is nilpotent if $\mathcal{L}^{(m)} = 0$, for some $m \geq 2$, where

$$\mathcal{L}^{(1)} := \mathcal{L}, \quad \mathcal{L}^{(n)} := \sum_{i+j=n} [\mathcal{L}^{(i)}, \mathcal{L}^{(j)}] + \sum_{i+j+k=n+1} [\mathcal{L}^{(i)}, \mathcal{L}^{(j)}, \mathcal{L}^{(k)}],\ i,j,k \geq 1, n \geq 2.$$
$$\big(\text{resp., } \mathcal{L}^{(1)} := \mathcal{L}, \quad \mathcal{L}^{(n)} := \sum_{i+j=n} [\mathcal{L}^{(i)}, \mathcal{L}^{(j)}] + \sum_{i+j=n} \{\mathcal{L}^{(i)}, \mathcal{L}^{(j)}\},\ i,j \geq 1, n \geq 2\big).$$

Definition 1.3 (see [26])**.** A Lie–Yamaguti algebra is a vector space $\mathcal{L}$ with a bilinear multiplication $[-,-]$ and a trilinear multiplication $[-,-,-]$ satisfying:

(LY1) $[x,y] + [y,x] = 0$,
(LY2) $[x,y,z] + [y,x,z] = 0$,
(LY3) $[x,y,z] + [y,z,x] + [z,x,y] + [[x,y],z] + [[y,z],x] + [[z,x],y] = 0$,
(LY4) $[[x,y],z,u] + [[y,z],x,u] + [[z,x],y,u] = 0$,
(LY5) $[x,y,[u,v]] - [[x,y,u],v] - [u,[x,y,v]] = 0$,
(LY6) $[u,v,[x,y,z]] - [[u,v,x],y,z] - [x,[u,v,y],z] - [x,y,[u,v,z]] = 0$.

Example 1.4. The Lie–Yamaguti algebra with $[x,y] = 0$ for any $x,y \in \mathcal{L}$ is exactly a Lie triple system, closely related to symmetric spaces, while the Lie–Yamaguti algebra with $[x,y,z] = 0$ for any $x,y,z \in \mathcal{L}$ is a Lie algebra.

Theorem 1.5 (see [1])**.** *Let $\mathcal{L}$ be a 4-dimensional complex nilpotent Lie–Yamaguti algebra. Then $\mathcal{L}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{L}_{01}$	$: [e_1,e_2] = e_3,$	*(Lie algebra).*	
$\mathcal{L}_{02}$	$: [e_1,e_2,e_1] = e_3.$		
$\mathcal{L}_{03}$	$: [e_1,e_2] = e_3,$	$[e_1,e_2,e_1] = e_3.$	
$\mathcal{L}_{04}$	$: [e_1,e_3] = e_4,$	$[e_2,e_3,e_2] = e_4.$	
$\mathcal{L}_{05}$	$: [e_1,e_2] = e_4,$	$[e_2,e_3,e_2] = e_4.$	
$\mathcal{L}_{06}$	$: [e_2,e_3,e_2] = e_4,$	$[e_3,e_1,e_3] = e_4.$	
$\mathcal{L}_{07}$	$: [e_1,e_2] = e_4,$	$[e_2,e_3,e_2] = e_4,$	$[e_3,e_1,e_3] = e_4.$
$\mathcal{L}_{08}$	$: [e_1,e_3] = e_4,$	$[e_2,e_3,e_2] = e_4,$	$[e_3,e_1,e_3] = e_4.$

$\mathcal{L}_{09}$: $[e_2,e_3]=e_4,\quad [e_2,e_3,e_2]=e_4,\quad [e_3,e_1,e_3]=e_4.$
$\mathcal{L}_{10}$: $[e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4.\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{11}$: $[e_1,e_3]=e_4,\quad [e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4.\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{12}$: $[e_2,e_3]=e_4,\quad [e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4.\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{13}$: $[e_1,e_2]=e_4,\quad [e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4.\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{14}$: $[e_1,e_2]=e_4,\quad [e_2,e_3]=e_4,\quad [e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4.\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{15}$: $[e_1,e_2]=e_4,\quad [e_1,e_3]=e_4,\quad [e_2,e_3,e_1]=e_4,\quad [e_3,e_1,e_2]=e_4,\quad [e_2,e_3,e_2]=e_4,\quad [e_2,e_1,e_3]=2e_4.$
$\mathcal{L}_{16}^{\alpha}$: $[e_2,e_3,e_1]=\alpha e_4,\quad [e_1,e_2,e_3]=-(1+\alpha)e_4,\quad [e_3,e_1,e_2]=e_4.$
$\mathcal{L}_{17}^{\alpha}$: $[e_1,e_2]=e_4,\quad [e_1,e_2,e_3]=-(1+\alpha)e_4,\quad [e_2,e_3,e_1]=\alpha e_4,\quad [e_3,e_1,e_2]=e_4.$
$\mathcal{L}_{18}^{\alpha}$: $[e_1,e_2]=e_4,\quad [e_1,e_3]=e_4,\quad [e_1,e_2,e_3]=-(1+\alpha)e_4,\quad [e_2,e_3,e_1]=\alpha e_4,\quad [e_3,e_1,e_2]=e_4.$
$\mathcal{L}_{19}^{\alpha}$: $[e_1,e_2]=e_4,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_4,\quad [e_3,e_1,e_2]=e_4.\quad [e_2,e_3,e_1]=\alpha e_4,\quad [e_1,e_2,e_3]=-(1+\alpha)e_4,$
$\mathcal{L}_{20}^{\alpha}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=\alpha e_4.$
$\mathcal{L}_{21}$: $[e_1,e_2]=e_3,\quad [e_2,e_3]=e_4,\quad [e_1,e_2,e_1]=e_4.$
$\mathcal{L}_{22}$: $[e_1,e_2]=e_3,\quad [e_2,e_3,e_2]=e_4.$
$\mathcal{L}_{23}^{\alpha}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=\alpha e_4,\quad [e_1,e_2,e_1]=e_4,\quad [e_2,e_3,e_2]=e_4.$
$\mathcal{L}_{24}$: $[e_1,e_2]=e_3,\quad [e_2,e_3]=e_4,\quad [e_1,e_2,e_1]=e_4,\quad [e_2,e_3,e_2]=e_4.$
$\mathcal{L}_{25}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3,e_2]=e_4.$
$\mathcal{L}_{26}$: $[e_1,e_2]=e_3,\quad [e_2,e_3]=e_4,\quad [e_2,e_3,e_2]=e_4.$
$\mathcal{L}_{27}$: $[e_1,e_2]=e_3,\quad [e_1,e_3,e_2]=e_4,\quad [e_2,e_3,e_1]=e_4.$
$\mathcal{L}_{28}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4,\quad [e_2,e_3,e_1]=e_4.$
$\mathcal{L}_{29}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4,\quad [e_2,e_3,e_1]=e_4.$
$\mathcal{L}_{30}$: $[e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3.$
$\mathcal{L}_{31}$: $[e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4.$
$\mathcal{L}_{32}$: $[e_1,e_2]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4.$
$\mathcal{L}_{33}$: $[e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4.$
$\mathcal{L}_{34}$: $[e_1,e_2]=e_4,\quad [e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_3]=e_4,\quad [e_1,e_3,e_2]=e_4.$
$\mathcal{L}_{35}$: $[e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4,\quad [e_1,e_2,e_2]=e_4.$
$\mathcal{L}_{36}$: $[e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4,\quad [e_1,e_2,e_2]=e_4.$
$\mathcal{L}_{37}$: $[e_1,e_2]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4,\quad [e_1,e_2,e_2]=e_4.$
$\mathcal{L}_{38}$: $[e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4.$
$\mathcal{L}_{39}$: $[e_1,e_2]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4.$
$\mathcal{L}_{40}$: $[e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_3,e_1]=e_4.$
$\mathcal{L}_{41}$: $[e_1,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_2]=e_4.$
$\mathcal{L}_{42}$: $[e_1,e_2]=e_3,\quad [e_2,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,\quad [e_1,e_2,e_3]=-e_4,\quad [e_1,e_3,e_2]=-e_4.$
$\mathcal{L}_{43}$: $[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_4,\quad [e_1,e_2,e_1]=e_3,$

$$
\begin{array}{lllll}
 & [e_1,e_2,e_3]=-e_4, & [e_1,e_3,e_2]=-e_4. & & \\
\mathcal{L}_{44}^{\alpha} & :[e_1,e_2]=e_3+e_4, & [e_1,e_3]=\alpha e_4, & [e_2,e_3]=e_4, & [e_1,e_2,e_1]=e_3, \\
 & [e_1,e_2,e_3]=-e_4, & [e_1,e_3,e_2]=-e_4. & & \\
\mathcal{L}_{45}^{\alpha} & :[e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_1,e_2,e_1]=e_3, & [e_1,e_3,e_1]=\alpha e_4. \\
\mathcal{L}_{46}^{\alpha\neq 0} & :[e_1,e_2]=e_3+\alpha e_4, & [e_1,e_3]=e_4, & [e_1,e_2,e_1]=e_3, & [e_1,e_3,e_1]=e_4. \\
\mathcal{L}_{47}^{\alpha} & :[e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_1,e_2,e_1]=e_3, & [e_1,e_2,e_2]=e_4, \\
 & [e_1,e_3,e_1]=\alpha e_4. & & & \\
\mathcal{L}_{48} & :[e_1,e_2]=e_3, & [e_1,e_2,e_1]=e_3, & [e_1,e_3,e_1]=e_4. & \\
\mathcal{L}_{49} & :[e_1,e_2]=e_3, & [e_1,e_2,e_1]=e_3, & [e_1,e_2,e_2]=e_4, & [e_1,e_3,e_1]=e_4. \\
\mathcal{L}_{50} & :[e_1,e_2,e_1]=e_3, & [e_1,e_2,e_2]=e_4. & & \\
\mathcal{L}_{51} & :[e_1,e_2]=e_4, & [e_1,e_2,e_1]=e_3. & & \\
\mathcal{L}_{52} & :[e_1,e_2]=e_3, & [e_1,e_2,e_1]=e_3, & [e_1,e_2,e_2]=e_4. &
\end{array}
$$

Where $\mathcal{L}_{46}^{0}\cong\mathcal{L}_{45}^{1}$.

Remark 1.6. The algebras listed below form Lie triple systems.

$$\mathcal{L}_{02},\ \mathcal{L}_{06},\ \mathcal{L}_{10},\ \mathcal{L}_{16}^{\alpha},\ \mathcal{L}_{31},\ \mathcal{L}_{35},\ \mathcal{L}_{50}.$$

Definition 1.7 (see [12])**.** A Bol algebra is a vector space $\mathfrak{B}$ with a bilinear multiplication $[-,-]$ and a trilinear multiplication $[-,-,-]$ satisfying:

(B1) $[x,y]+[y,x]=0,$

(B2) $[x,y,z]+[y,x,z]=0,$

(B3) $[x,y,z]+[y,z,x]+[z,x,y]=0,$

(B4) $[u,v,[x,y,z]]-[[u,v,x],y,z]-[x,[u,v,y],z]-[x,y,[u,v,z]]=0,$

(B5) $[[x,y,z],t]-[[x,y,t],z]+[z,t,[x,y]]-[x,y,[z,t]]+[[x,y],[z,t]]=0.$

Identities (B2)–(B4) mean that a Bol algebra is a Lie triple system with respect to the trilinear operation $[-,-,-]$. Thus, Lie–Yamaguti algebras and Bol algebras are generalizations of the Lie triple systems.

Theorem 1.8 (see [2])**.** *Let $\mathfrak{B}$ be a 4-dimensional complex nilpotent Bol algebra. Then $\mathfrak{B}$ is isomorphic to one of the following Lie–Yamaguti algebras $\mathcal{L}_{01},\dots,\mathcal{L}_{41}$, $\mathcal{L}_{45}^{\alpha},\dots,\mathcal{L}_{52}$ or one of the following mutually non-isomorphic algebras:*

$$
\begin{array}{lllll}
\mathfrak{B}_{01} & :[e_1,e_2]=e_3, & [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_1]=-\frac{1}{2}e_4, & \\
 & [e_1,e_3,e_2]=\frac{1}{2}e_4, & [e_2,e_3,e_2]=e_4. & & \\
\mathfrak{B}_{02} & :[e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_1]=-\frac{1}{2}e_4, \\
 & [e_1,e_3,e_2]=\frac{1}{2}e_4, & [e_2,e_3,e_2]=e_4. & & \\
\mathfrak{B}_{03} & :[e_1,e_2]=e_3, & [e_2,e_3]=e_4, & [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_1]=-\frac{1}{2}e_4, \\
 & [e_1,e_3,e_2]=\frac{1}{2}e_4, & [e_2,e_3,e_2]=e_4. & & \\
\mathfrak{B}_{04}^{\alpha} & :[e_1,e_2]=e_3, & [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_1]=(\alpha-1)e_4, & [e_1,e_3,e_2]=\alpha e_4. \\
\mathfrak{B}_{05}^{\alpha} & :[e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_1,e_2,e_3]=e_4, & \\
 & [e_2,e_3,e_1]=(\alpha-1)e_4, & [e_1,e_3,e_2]=\alpha e_4. & & \\
\mathfrak{B}_{06}^{\alpha} & :[e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_2,e_3]=e_4, & [e_1,e_2,e_3]=e_4, \\
 & [e_2,e_3,e_1]=(\alpha-1)e_4, & [e_1,e_3,e_2]=\alpha e_4. & &
\end{array}
$$

$$\begin{array}{lllll}
\mathfrak{B}_{07}^{\alpha}: & [e_1,e_2]=e_3, & [e_1,e_3]=\alpha e_4, & [e_1,e_2,e_1]=e_4, & [e_1,e_2,e_3]=e_4,\\
& [e_2,e_3,e_1]=e_4, & [e_1,e_3,e_2]=2e_4. & & \\
\mathfrak{B}_{08}^{\alpha}: & [e_1,e_2]=e_3, & [e_1,e_3]=\alpha e_4, & [e_2,e_3]=e_4, & [e_1,e_2,e_1]=e_4,\\
& [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_1]=e_4, & [e_1,e_3,e_2]=2e_4. & \\
\mathfrak{B}_{09}: & [e_1,e_2]=e_3, & [e_1,e_2,e_1]=e_3, & [e_1,e_2,e_3]=e_4, & [e_1,e_3,e_2]=e_4.\\
\mathfrak{B}_{10}: & [e_1,e_2]=e_3, & [e_1,e_3]=e_4, & [e_1,e_2,e_1]=e_3, & \\
& [e_1,e_2,e_3]=e_4, & [e_1,e_3,e_2]=e_4. & &
\end{array}$$

Definition 1.9. A compatible Lie algebra is vector space $\mathfrak{g}$ with two bilinear multiplications $[-,-]$ and $\{-,-\}$, where $\mathfrak{g}_1=(\mathfrak{g},[-,-])$ and $\mathfrak{g}_2=(\mathfrak{g},\{-,-\})$ are Lie algebras and the two operations are required to satisfy the following identity

$$\{[x,y],z\}+\{[y,z],x\}+\{[z,x],y\}+[\{x,y\},z]+[\{y,z\},x]+[\{z,x\},y]=0.$$

Theorem 1.10 (see [29]). *Let $\mathfrak{g}$ be a complex 3-dimensional nilpotent compatible Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{lll}
\mathfrak{L}_1 & : & \{e_1,e_2\}=e_3.\\
\mathfrak{L}_2^{\alpha} & : & [e_1,e_2]=e_3, \quad \{e_1,e_2\}=\alpha e_3.
\end{array}$$

Theorem 1.11 (see [29]). *Let $\mathfrak{g}$ be a complex 4-dimensional nilpotent compatible Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{llllll}
L_{01} & : & \{e_1,e_2\}=e_3. & & & \\
L_{02}^{\alpha} & : & [e_1,e_2]=e_3, & \{e_1,e_2\}=\alpha e_3. & & \\
L_{03} & : & [e_2,e_3]=e_4, & \{e_1,e_3\}=e_4. & & \\
L_{04} & : & [e_1,e_2]=e_3, & [e_2,e_3]=e_4, & \{e_1,e_2\}=e_4. & \\
L_{05}^{\alpha} & : & [e_2,e_3]=\alpha e_4, & \{e_1,e_2\}=e_3, & \{e_2,e_3\}=e_4. & \\
L_{06} & : & [e_1,e_3]=e_4, & \{e_1,e_2\}=e_3, & \{e_2,e_3\}=e_4. & \\
L_{07} & : & [e_1,e_2]=e_4, & \{e_1,e_2\}=e_3, & \{e_2,e_3\}=e_4. & \\
L_{08}^{\alpha} & : & [e_1,e_2]=e_3, & \{e_1,e_2\}=\alpha e_3, & \{e_2,e_3\}=e_4. & \\
L_{09}^{\alpha,\beta} & : & [e_1,e_2]=e_3, & [e_2,e_3]=e_4, & \{e_1,e_2\}=\alpha e_3, & \{e_2,e_3\}=\beta e_4.\\
L_{10}^{\alpha} & : & [e_1,e_2]=e_3, & [e_2,e_3]=e_4, & \{e_1,e_2\}=\alpha e_3, & \{e_1,e_3\}=e_4.
\end{array}$$

2 The geometric classification

2.1 Preliminaries: geometric classification

Given a complex vector space $\mathbb{V}$ of dimension n, the set of bilinear and trilinear maps

$$\begin{array}{rcccl}
\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V}) & \cong & \mathrm{Hom}(\mathbb{V}^{\otimes 2},\mathbb{V}) & \cong & (\mathbb{V}^*)^{\otimes 2}\otimes\mathbb{V},\\
\mathrm{Tril}(\mathbb{V}\times\mathbb{V}\times\mathbb{V},\mathbb{V}) & \cong & \mathrm{Hom}(\mathbb{V}^{\otimes 3},\mathbb{V}) & \cong & (\mathbb{V}^*)^{\otimes 3}\otimes\mathbb{V}
\end{array}$$

are vector spaces of dimension n^3 and n^4, respectively. The set of pairs of bilinear and trilinear maps

$$\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V})\oplus\mathrm{Tril}(\mathbb{V}\times\mathbb{V}\times\mathbb{V},\mathbb{V})\cong((\mathbb{V}^*)^{\otimes 2}\otimes\mathbb{V})\oplus((\mathbb{V}^*)^{\otimes 3}\otimes\mathbb{V})$$

is a vector space of dimension n^4+n^3. This vector space has the structure of the affine space $\mathbb{C}^{n^4+n^3}$ in the following sense: fixed a basis $e_1,\dots,e_n$ of $\mathbb{V}$, any pair $(\mu,\mu')\in\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V})\oplus\mathrm{Tril}(\mathbb{V}\times\mathbb{V}\times\mathbb{V},\mathbb{V})$ is determined by some parameters $c_{i,j}^k, c_{i,j,k}^l\in\mathbb{C}$, called structural constants, such that

$$\mu(e_i,e_j)=\sum_{k=1}^{n}c_{i,j}^k e_k \text{ and } \mu'(e_i,e_j,e_k)=\sum_{l=1}^{n}c_{i,j,k}^l e_l$$

which correspond to a point in the affine space $\mathbb{C}^{n^4+n^3}$. Then a subset $\mathcal{S}$ of $\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V})\oplus\mathrm{Tril}(\mathbb{V}\times\mathbb{V}\times\mathbb{V},\mathbb{V})$ corresponds to an algebraic variety, i.e., a Zariski closed set, if there are some polynomial equations in variables $c_{i,j}^k, c_{i,j,k}^l$ with zero locus equal to the set of structural constants of the pairs in $\mathcal{S}$.

Given the identities defining Lie–Yamaguti (Bol) algebras, we can obtain a set of polynomial equations in the variables $c_{i,j}^k, c_{i,j,k}^l$. This class of n-dimensional Lie–Yamaguti (Bol) algebras is a variety. Denote it by $\mathcal{T}_n$. Now, consider the following action of $\mathrm{GL}(\mathbb{V})$ on $\mathcal{T}_n$:

$$(g*\mu)(x,y):=g\mu(g^{-1}x,g^{-1}y),\quad (g*\mu')(x,y,z):=g\mu'(g^{-1}x,g^{-1}y,g^{-1}z)$$

for $g\in\mathrm{GL}(\mathbb{V})$, $(\mu,\mu')\in\mathcal{T}_n$ and for any $x,y,z\in\mathbb{V}$. Observe that the $\mathrm{GL}(\mathbb{V})$-orbit of (μ,μ'), denoted $O((\mu,\mu'))$, contains all the structural constants of the pairs isomorphic to the Lie–Yamaguti (Bol) algebra with structural constants (μ,μ').

A geometric classification of a variety of algebras consists of describing the irreducible components of the variety. Recall that any affine variety can be represented uniquely as a finite union of its irreducible components. Note that describing the irreducible components of $\mathcal{T}_n$ gives us the rigid algebras of the variety, which are those bilinear pairs with an open $\mathrm{GL}(\mathbb{V})$-orbit. This is because a bilinear pair is rigid in a variety if and only if the closure of its orbit is an irreducible component of the variety. For this reason, the following notion is convenient. Denote by $\overline{O((\mu,\mu'))}$ the closure of the orbit of $(\mu,\mu')\in\mathcal{T}_n$.

Definition 2.1. Let T and T' be two n-dimensional Lie–Yamaguti (Bol) algebras corresponding to the variety $\mathcal{T}_n$, and let $(\mu,\mu'),(\lambda,\lambda')\in\mathcal{T}_n$ be their representatives in the affine space, respectively. The algebra T is said to degenerate to T', and we write $\mathrm{T}\to\mathrm{T}'$, if $(\lambda,\lambda')\in\overline{O((\mu,\mu'))}$. If $\mathrm{T}\not\cong\mathrm{T}'$, then we call it a proper degeneration. Conversely, if $(\lambda,\lambda')\notin\overline{O((\mu,\mu'))}$ then we say that T does not degenerate to T' and we write $\mathrm{T}\not\to\mathrm{T}'$.

Furthermore, we have the following notion for a parametric family of algebras.

Definition 2.2. Let $\mathrm{T}(*)=\{\mathrm{T}(\alpha):\alpha\in I\}$ be a family of n-dimensional Lie–Yamaguti (Bol) algebras corresponding to $\mathcal{T}_n$ and let T' be another n-dimensional Lie–Yamaguti (Bol) algebra. Suppose that $\mathrm{T}(\alpha)$ is represented by the structure $(\mu(\alpha),\mu'(\alpha))\in\mathcal{T}_n$ for $\alpha\in I$ and T' is represented by the structure $(\lambda,\lambda')\in\mathcal{T}_n$. We say that the family $\mathrm{T}(*)$ degenerates to T', and write $\mathrm{T}(*)\to\mathrm{T}'$, if $(\lambda,\lambda')\in\overline{\{O((\mu(\alpha),\mu'(\alpha)))\}_{\alpha\in I}}$. Conversely, if $(\lambda,\lambda')\notin\overline{\{O((\mu(\alpha),\mu'(\alpha)))\}_{\alpha\in I}}$ then we call it a non-degeneration, and we write $\mathrm{T}(*)\not\to\mathrm{T}'$.

Observe that T' corresponds to an irreducible component of $\mathcal{T}_n$ (more precisely, $\overline{\mathrm{T}'}$ is an irreducible component) if and only if $\mathrm{T} \not\to \mathrm{T}'$ for any n-dimensional Lie–Yamaguti (Bol) algebra T and $\mathrm{T}(*) \not\to \mathrm{T}'$ for any parametric family of n-dimensional Lie–Yamaguti (Bol) algebras $\mathrm{T}(*)$. In this case, we will use the next ideas to prove that a particular algebra corresponds to an irreducible component. Firstly, since $\dim O((\mu,\mu')) = n^2 - \dim \mathfrak{Der}(\mathrm{T})$, it follows that if $\mathrm{T} \to \mathrm{T}'$ and $\mathrm{T} \not\cong \mathrm{T}'$, then $\dim \mathfrak{Der}(\mathrm{T}) < \dim \mathfrak{Der}(\mathrm{T}')$, where $\mathfrak{Der}(\mathrm{T})$ denotes the Lie algebra of derivations of T.

Secondly, to show degenerations, let T and T' be two Lie–Yamaguti (Bol) algebras represented by the structures (μ,μ') and (λ,λ') from $\mathcal{T}_n$, respectively. Let $c_{i,j}^k, c_{i,j,k}^l$ be the structure constants of (λ,λ') in a basis $e_1,\dots,e_n$ of $\mathbb{V}$. If there exist n^2 maps $a_i^j(t) : \mathbb{C}^* \to \mathbb{C}$ such that $E_i(t) = \sum_{j=1}^n a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$ and the structure constants $c_{i,j}^k(t), c_{i,j,k}^l(t)$ of (μ,μ') in the basis $E_1(t),\dots,E_n(t)$ satisfy $\lim\limits_{t\to 0} c_{i,j}^k(t) = c_{i,j}^k$ and $\lim\limits_{t\to 0} c_{i,j,k}^l(t) = c_{i,j,k}^l$, then $\mathrm{T} \to \mathrm{T}'$. In this case, $E_1(t),\dots,E_n(t)$ is called a *parametric basis* for $\mathrm{T} \to \mathrm{T}'$. If $E_{e_1}^t, E_{e_2}^t, E_{e_3}^t, E_{e_4}^t$ is a parametric basis for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by $\mathbf{A} \xrightarrow{(E_{e_1}^t, E_{e_2}^t, E_{e_3}^t, E_{e_4}^t)} \mathbf{B}$.

Thirdly, to prove non-degenerations we may use a remark that follows from the following lemma (see [4] and the references therein).

Lemma 2.3. *Consider two Lie–Yamaguti (Bol) algebras* T *and* T'*. Suppose* $\mathrm{T} \to \mathrm{T}'$*. Let* $\mathcal{Z}$ *be a Zariski closed set in* $\mathcal{T}_n$ *that is stable by the action of the invertible upper (lower) triangular matrices. Then if there is a representation* (μ,μ') *of* T *in* $\mathcal{Z}$*, then there is a representation* (λ,λ') *of* T' *in* $\mathcal{Z}$*.*

To apply this lemma, we will give the explicit definition of the appropriate stable Zariski closed $\mathcal{Z}$ in terms of the variables $c_{i,j}^k, c_{i,j,k}^l$ in each case. For clarity, we assume by convention that $c_{i,j}^k = 0$ (resp. $c_{i,j,k}^l = 0$) if $c_{i,j}^k$ (resp. $c_{i,j,k}^l$) is not explicitly mentioned in the definition of $\mathcal{Z}$.

Remark 2.4. Let T and T' be two Lie–Yamaguti (Bol) algebras represented by the structures (μ,μ') and (λ,λ') from T_n. Suppose $\mathrm{T} \to \mathrm{T}'$. Then if μ,λ represent algebras $\mathrm{T}_0, \mathrm{T}_0'$ in the affine space $\mathbb{C}^{n^3}$ and μ',λ' represent algebras $\mathrm{T}_1, \mathrm{T}_1'$ in the affine space $\mathbb{C}^{n^4}$ of algebras with a single multiplication, respectively, we have $\mathrm{T}_0 \to \mathrm{T}_0'$ and $\mathrm{T}_1 \to \mathrm{T}_1'$. So, for example, $(0,\mu)$ can not degenerate to $(\lambda,0)$ unless $\lambda = 0$.

Fourthly, to prove $\mathrm{T}(*) \to \mathrm{T}'$, suppose that $\mathrm{T}(\alpha)$ is represented by the structure $(\mu(\alpha),\mu'(\alpha)) \in \mathcal{T}_n$ for $\alpha \in I$ and T' is represented by the structure $(\lambda,\lambda') \in \mathcal{T}_n$. Let $c_{i,j}^k, c_{i,j,k}^l$ be the structure constants of (λ,λ') in a basis $e_1,\dots,e_n$ of $\mathbb{V}$. If there is a pair of maps $(f,(a_i^j))$, where $f : \mathbb{C}^* \to I$ and $a_i^j : \mathbb{C}^* \to \mathbb{C}$ are such that $E_i(t) = \sum_{j=1}^n a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$ and the structure constants $c_{i,j}^k(t), c_{i,j,k}^l(t)$ of $\big(\mu\big(f(t)\big), \mu'\big(f(t)\big)\big)$ in the basis $E_1(t),\dots,E_n(t)$ satisfy $\lim\limits_{t\to 0} c_{i,j}^k(t) = c_{i,j}^k$ and $\lim\limits_{t\to 0} c_{i,j,k}^l(t) = c_{i,j,k}^l$, then $\mathrm{T}(*) \to \mathrm{T}'$. In this case $E_1(t),\dots,E_n(t)$ and $f(t)$ are called a parametrized basis and a parametrized index for $\mathrm{T}(*) \to \mathrm{T}'$, respectively. Fithly, to prove $\mathrm{T}(*) \not\to \mathrm{T}'$, we may use an analogous of Remark 2.4 for parametric families that follows from Lemma 2.5, see [4].

Lemma 2.5. *Consider the family of Lie–Yamaguti (Bol) algebras* $\mathrm{T}(*)$ *and the Lie–Yamaguti (Bol) algebra* T'. *Suppose* $\mathrm{T}(*) \to \mathrm{T}'$. *Let* $\mathcal{Z}$ *be a Zariski closed set in* $\mathcal{T}_n$ *that is stable by the action of the invertible upper (lower) triangular matrices. Then if there is a representation* $(\mu(\alpha), \mu'(\alpha))$ *of* $\mathrm{T}(\alpha)$ *in* $\mathcal{Z}$ *for every* $\alpha \in I$, *then there is a representation* (λ, λ') *of* T' *in* $\mathcal{Z}$.

Finally, the following remark simplifies the geometric problem.

Remark 2.6. Let (μ, μ') and (λ, λ') represent two Lie–Yamaguti algebras. Suppose $(\lambda, 0) \notin \overline{O((\mu, 0))}$, (resp. $(0, \lambda') \notin \overline{O((0, \mu'))}$), then $(\lambda, \lambda') \notin \overline{O((\mu, \mu'))}$. As we construct the classification of Lie–Yamaguti (Bol) algebras from a certain class of algebras with a single multiplication which remains unchanged, this remark becomes very useful.

2.2 The geometric classification of Lie–Yamaguti algebras

In this subsection, we determine all the irreducible components of the variety of four-dimensional nilpotent Lie–Yamaguti algebras.

Theorem 2.7. *The variety of four-dimensional nilpotent Lie–Yamaguti algebras has dimension* 14 *and it has* 3 *irreducible components defined by*

$$\mathcal{C}_1 = \overline{O(\mathcal{L}_{19}^{\alpha})}, \ \ \mathcal{C}_2 = \overline{O(\mathcal{L}_{29})}, \ \ \textit{and} \ \ \mathcal{C}_3 = \overline{O(\mathcal{L}_{44}^{\alpha})}.$$

In particular, $\mathcal{L}_{29}$ *is a rigid algebra.*

Proof. After carefully checking the dimensions of orbit closures of the more important algebras, we have

$$\dim O(\mathcal{L}_{19}^{\alpha}) = \dim O(\mathcal{L}_{29}) = \dim O(\mathcal{L}_{44}^{\alpha}) = 14.$$

As the orbit dimensions of these three algebras coincide, no degeneration occurs among them.

The following degenerations are observed:

$\mathcal{L}_{44}^{0}$	$\xrightarrow{(te_1, e_2, te_3+te_4, e_4)}$	$\mathcal{L}_{01}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(t^{-1}e_1, t^3e_2, te_3+te_4, e_4)}$	$\mathcal{L}_{02}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(e_1, te_2, te_3+te_4, e_4)}$	$\mathcal{L}_{03}$
$\mathcal{L}_{15}$	$\xrightarrow{(t^2e_1, te_2, e_3, t^2e_4)}$	$\mathcal{L}_{04}$
$\mathcal{L}_{15}$	$\xrightarrow{(t^3e_1, te_2+e_3, t(t+1)e_3, t^3(t+1)e_4)}$	$\mathcal{L}_{05}$
$\mathcal{L}_{07}$	$\xrightarrow{(t^2e_1, te_2, e_3, t^2e_4)}$	$\mathcal{L}_{06}$
$\mathcal{L}_{19}^{0}$	$\xrightarrow{(t^3e_1, -t^2e_1-2t^2e_3, -te_1+te_2-te_3, -2t^5e_4)}$	$\mathcal{L}_{07}$

$\mathcal{L}_{19}^{2}$	$\xrightarrow{(-\frac{5t^2}{64}e_1, \frac{t}{16}e_1+te_3, -\frac{1}{4}e_1+\frac{1}{3}e_2+e_3, -\frac{5t^2}{48}e_4)}$	$\mathcal{L}_{08}$
$\mathcal{L}_{19}^{-1}$	$\xrightarrow{(\frac{2t^2}{1-2t}e_1, \frac{2t}{1-2t}e_1+te_3, \frac{1}{1-2t}e_1-e_2+e_3, \frac{2t^2}{2t-1}e_4)}$	$\mathcal{L}_{09}$
$\mathcal{L}_{15}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-1}e_3,t^{-3}e_4)}$	$\mathcal{L}_{10}$
$\mathcal{L}_{15}$	$\xrightarrow{(e_1,e_2,t^{-1}e_3,t^{-1}e_4)}$	$\mathcal{L}_{11}$
$\mathcal{L}_{15}$	$\xrightarrow{(-\frac{1}{2}(t+2)e_2, \frac{(t+1)(t+2)^2}{t^2}e_1+\frac{t^2+3t+2}{t}e_2, -\frac{t+2}{2t}e_3, -\frac{(t+1)(t+2)^3}{2t^3}e_4)}$	$\mathcal{L}_{12}$
$\mathcal{L}_{15}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,e_3,t^{-2}e_4)}$	$\mathcal{L}_{13}$
$\mathcal{L}_{15}$	$\xrightarrow{(-\frac{1}{2}(t+2)e_2, \frac{(t+2)^2}{t^2}e_1-\frac{t+2}{t}e_2, \frac{t+2}{2}e_3, \frac{(t+2)^3}{2t^2}e_4)}$	$\mathcal{L}_{14}$
$\mathcal{L}_{19}^{1+t}$	$\xrightarrow{(-\frac{2t}{(2+t)^2}e_1-\frac{2t}{(2+t)^2}e_2, -\frac{2}{2+t}e_1+\frac{2}{2+t}e_2, \frac{2}{2+t}e_3, -\frac{8t}{(2+t)^3}e_4)}$	$\mathcal{L}_{15}$
$\mathcal{L}_{19}^{\alpha}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-1}e_3,t^{-3}e_4)}$	$\mathcal{L}_{16}^{\alpha}$
$\mathcal{L}_{19}^{\alpha}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,e_3,t^{-2}e_4)}$	$\mathcal{L}_{17}^{\alpha}$
$\mathcal{L}_{19}^{\alpha}$	$\xrightarrow{(t^{-1}e_1,e_2,e_3,t^{-1}e_4)}$	$\mathcal{L}_{18}^{\alpha}$
$\mathcal{L}_{44}^{\frac{1}{t^2-\alpha}}$	$\xrightarrow{(te_1, \frac{t^2}{t^2-\alpha}e_2, \frac{t^3}{t^2-\alpha}e_3-\frac{\alpha t^3}{(t^2-\alpha)^2}e_4, \frac{t^4}{(t^2-\alpha)^2}e_4)}$	$\mathcal{L}_{20}^{\alpha}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(te_1,-te_2,-t^2e_3-t^2e_4,t^3e_4)}$	$\mathcal{L}_{21}$
$\mathcal{L}_{23}^{0}$	$\xrightarrow{(te_1,e_2,te_3,te_4)}$	$\mathcal{L}_{22}$
$\mathcal{L}_{29}$	$\xrightarrow{(-\frac{2t^2}{\alpha}e_1-\frac{2t^2}{\alpha^2}e_2, -\frac{t}{\alpha}e_1+\frac{t}{\alpha}e_2+\frac{2t}{\alpha^2}e_3, -\frac{2t^3}{\alpha^2}e_3-\frac{4t^3}{\alpha^3}e_4, \frac{4t^5}{\alpha^4}e_4)}$	$\mathcal{L}_{23}^{\alpha}$
$\mathcal{L}_{23}^{t}$	$\xrightarrow{(e_1+t^{-1}e_3,t^{-1}e_1+e_2,e_3-t^{-1}e_4,e_4)}$	$\mathcal{L}_{24}$
$\mathcal{L}_{23}^{t^{-1}}$	$\xrightarrow{(te_1,e_2,te_3,te_4)}$	$\mathcal{L}_{25}$
$\mathcal{L}_{23}^{1}$	$\xrightarrow{(te_1+te_3,e_1+e_2,te_3-te_4,te_4)}$	$\mathcal{L}_{26}$
$\mathcal{L}_{29}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-2}e_3,t^{-4}e_4)}$	$\mathcal{L}_{27}$
$\mathcal{L}_{29}$	$\xrightarrow{(t^{-1}e_1,e_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathcal{L}_{28}$
$\mathcal{L}_{44}^{1}$	$\xrightarrow{(t^{-1}e_1,te_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathcal{L}_{30}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(t^{-1}e_1,e_2,t^{-2}e_3,-t^{-3}e_4)}$	$\mathcal{L}_{31}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(t^{-1}e_1,-t^2e_2,-e_3,-te_4)}$	$\mathcal{L}_{32}$
$\mathcal{L}_{44}^{-t}$	$\xrightarrow{(t^{-1}e_1,te_2,t^{-1}e_3,-t^{-1}e_4)}$	$\mathcal{L}_{33}$
$\mathcal{L}_{44}^{t^2}$	$\xrightarrow{(t^{-1}e_1,-t^2e_2,-e_3,-te_4)}$	$\mathcal{L}_{34}$
$\mathcal{L}_{44}^{2t^3}$	$\xrightarrow{(t^{-1}e_1-2t^2e_2,-2t^3e_2-e_3,-2te_3-2te_4,-4t^2e_4)}$	$\mathcal{L}_{35}$
$\mathcal{L}_{44}^{\frac{2t^4+2t^3}{t-1}}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{2t^3}{1-t}e_2, \frac{2t^4}{1-t}e_2+\frac{t}{1-t}e_3, \frac{2t^2}{1-t}e_3-\frac{2t^3}{(1-t)^2}e_4, -\frac{4t^4}{(1-t)^2}e_4)}$	$\mathcal{L}_{36}$

$\mathcal{L}_{44}^{-10t^3}$	$\xrightarrow{(t^{-1}e_1+2t^2e_2,2t^3e_2+e_3,2te_3-2te_4,-4t^2e_4)}$	$\mathcal{L}_{37}$
$\mathcal{L}_{44}^{-t^2}$	$\xrightarrow{(t^{-1}e_1+te_2,t^2e_2,e_3,-e_4)}$	$\mathcal{L}_{38}$
$\mathcal{L}_{44}^{t^3}$	$\xrightarrow{(t^{-1}e_1-t^2e_2,-t^3e_2,-te_3,-t^2e_4)}$	$\mathcal{L}_{39}$
$\mathcal{L}_{44}^{1+t}$	$\xrightarrow{(t^{-1}e_1-e_2,-te_2,-t^{-1}e_3,-t^{-2}e_4)}$	$\mathcal{L}_{40}$
$\mathcal{L}_{44}^{-2t^3}$	$\xrightarrow{(t^{-1}e_1,2t^4e_2+te_3,2t^2e_3,-4t^4e_4)}$	$\mathcal{L}_{41}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(e_1,t^{-1}e_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathcal{L}_{42}$
$\mathcal{L}_{44}^{t^{-1}}$	$\xrightarrow{(e_1,t^{-1}e_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathcal{L}_{43}$
$\mathcal{L}_{44}^{-\frac{1+\alpha}{t}}$	$\xrightarrow{(e_1+\frac{\alpha}{t}e_2,e_2,e_3,-t^{-1}e_4)}$	$\mathcal{L}_{45}^{\alpha}$
$\mathcal{L}_{44}^{\frac{2}{\alpha}}$	$\xrightarrow{(e_1-\frac{1}{\alpha}e_2,-\frac{t}{\alpha}e_2,-\frac{t}{\alpha}e_3,-\frac{t}{\alpha^2}e_4)}$	$\mathcal{L}_{46}^{\alpha\neq 0}$
$\mathcal{L}_{44}^{\frac{2(\alpha+1)t}{\alpha-1}}$	$\xrightarrow{(e_1-\frac{2\alpha t}{\alpha-1}e_2,\frac{2t^2}{1-\alpha}e_2+\frac{t}{1-\alpha}e_3,\frac{2t^2}{1-\alpha}e_3-\frac{2\alpha t^2}{(1-\alpha)^2}e_4,-\frac{4t^3}{(1-\alpha)^2}e_4)}$	$\mathcal{L}_{47}^{\alpha\neq 1}$
$\mathcal{L}_{44}^{-t^{-1}}$	$\xrightarrow{(e_1+t^{-1}e_2,e_2,e_3,-t^{-1}e_4)}$	$\mathcal{L}_{48}$
$\mathcal{L}_{44}^{2t}$	$\xrightarrow{(e_1-2te_2+e_3,-2t^2e_2-te_3,-2t^2e_3,-4t^3e_4)}$	$\mathcal{L}_{49}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(t^{-1}e_1,2t^2e_2+t^{-1}e_3,2e_3,-4e_4)}$	$\mathcal{L}_{50}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(t^{-1}e_1,t^3e_2,te_3,t^2e_4)}$	$\mathcal{L}_{51}$
$\mathcal{L}_{44}^{0}$	$\xrightarrow{(e_1,2e_2+t^{-1}e_3,2e_3,-4t^{-1}e_4)}$	$\mathcal{L}_{52}$

□

2.3 The geometric classification of Bol algebras

In this section, we determine all the irreducible components of the variety of four-dimensional nilpotent Bol algebras.

Theorem 2.8. *The variety of four-dimensional nilpotent Bol algebras has dimension* 15 *and it has* 2 *irreducible components defined by*

$$\mathcal{C}_1 = \overline{O(\mathfrak{B}_{06}^{\alpha})} \quad and \quad \mathcal{C}_2 = \overline{O(\mathfrak{B}_{10})}.$$

In particular, $\mathfrak{B}_{10}$ *is a rigid algebra.*

Proof. From the proof of Theorem 2.7, we obtain that there are degenerations from the algebras $\mathcal{L}_{19}^{\alpha}$ and $\mathcal{L}_{29}$ to the algebras $\mathcal{L}_{04}$, $\mathcal{L}_{05}$, $\mathcal{L}_{06}$, $\mathcal{L}_{07}$, $\mathcal{L}_{08}$, $\mathcal{L}_{09}$, $\mathcal{L}_{10}$, $\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{13}$, $\mathcal{L}_{14}$, $\mathcal{L}_{15}$, $\mathcal{L}_{16}^{\alpha}$, $\mathcal{L}_{17}^{\alpha}$, $\mathcal{L}_{18}^{\alpha}$, $\mathcal{L}_{22}$, $\mathcal{L}_{23}^{\alpha}$, $\mathcal{L}_{24}$, $\mathcal{L}_{25}$, $\mathcal{L}_{26}$, $\mathcal{L}_{27}$, $\mathcal{L}_{28}$.

After carefully checking the dimensions of orbit closures of the more important algebras, we have

$$\dim O(\mathfrak{B}_{06}^{\alpha}) = 15 \quad \text{and} \quad \dim O(\mathfrak{B}_{10}) = 14.$$

The following degenerations are observed:

$\mathfrak{B}_{10}$	$\xrightarrow{(te_1,e_2,te_3,e_4)}$	$\mathcal{L}_{01}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1,t^4e_2,t^2e_3,e_4)}$	$\mathcal{L}_{02}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1,te_2,te_3,e_4)}$	$\mathcal{L}_{03}$
$\mathfrak{B}_{06}^{\frac{1}{1+\alpha}}$	$\xrightarrow{(-(1+\alpha)e_1,-(1+\alpha)e_2,\frac{1}{t}e_3+\frac{1}{(1+\alpha)t^2}e_4,-\frac{1+\alpha}{t}e_4)}$	$\mathcal{L}_{19}^{\alpha\neq-1}$
$\mathfrak{B}_{10}$	$\xrightarrow{(te_1+t(\alpha+t^2)e_3,te_2,t^2e_3+t^4e_4,t^3e_4)}$	$\mathcal{L}_{20}^{\alpha}$
$\mathfrak{B}_{10}$	$\xrightarrow{(te_2+te_3,2te_1+2te_3,-2t^2e_3-2t^2e_4,-4t^3e_4)}$	$\mathcal{L}_{21}$
$\mathfrak{B}_{06}^{\frac{t+1}{t}}$	$\xrightarrow{(te_1,te_2,t^2e_3,t^3e_4)}$	$\mathcal{L}_{29}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1,te_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathcal{L}_{30}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-3}e_3,t^{-5}e_4)}$	$\mathcal{L}_{31}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1-t^{-4}e_3,t^{-1}e_2,t^{-3}e_3-t^{-6}e_4,t^{-5}e_4)}$	$\mathcal{L}_{32}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1,e_2,t^{-2}e_3,t^{-3}e_4)}$	$\mathcal{L}_{33}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1-t^{-3}e_3,e_2,t^{-2}e_3-t^{-4}e_4,t^{-3}e_4)}$	$\mathcal{L}_{34}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{t}e_2,e_2+\frac{1}{2t^3}e_3,\frac{1}{t^2}e_3+\frac{1}{2t^5}e_4,\frac{1}{t^4}e_4)}$	$\mathcal{L}_{35}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+e_2+\frac{1-t}{2t^4}e_3,te_2+\frac{1}{2t^2}e_3,\frac{1}{t}e_3+\frac{1}{2t^4}e_4,\frac{1}{t^2}e_4)}$	$\mathcal{L}_{36}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{t}e_2-\frac{1}{t^4}e_3,e_2+\frac{1}{2t^3}e_3,\frac{1}{t^2}e_3-\frac{1}{2t^5}e_4,\frac{1}{t^4}e_4)}$	$\mathcal{L}_{37}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{t}e_2,e_2,\frac{1}{t^2}e_3,\frac{1}{t^4}e_4)}$	$\mathcal{L}_{38}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{t}e_2-\frac{1}{t^4}e_3,e_2,\frac{1}{t^2}e_3-\frac{1}{t^5}e_4,\frac{1}{t^4}e_4)}$	$\mathcal{L}_{39}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+e_2,te_2,\frac{1}{t}e_3,\frac{1}{t^2}e_4)}$	$\mathcal{L}_{40}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{2t^4}e_3,te_2+\frac{1}{2t^2}e_3,\frac{1}{t}e_3+\frac{1}{2t^4}e_4,\frac{1}{t^2}e_4)}$	$\mathcal{L}_{41}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+\alpha e_2,te_2,te_3,te_4)}$	$\mathcal{L}_{45}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+e_2-\alpha e_3,te_2,te_3-\alpha te_4,te_4)}$	$\mathcal{L}_{46}^{\alpha\neq0}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+\alpha e_2+\frac{1-\alpha}{2t}e_3,te_2+\frac{1}{2}e_3,te_3+\frac{1}{2}e_4,te_4)}$	$\mathcal{L}_{47}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+t^{-1}e_2,te_2,te_3,e_4)}$	$\mathcal{L}_{48}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+\frac{1}{t}e_2+\frac{t-1}{2t^2}e_3,e_2+\frac{1}{2t}e_3,e_3+\frac{1}{2t}e_4,\frac{1}{t}e_4)}$	$\mathcal{L}_{49}$
$\mathfrak{B}_{10}$	$\xrightarrow{(\frac{1}{t}e_1+\frac{1}{2t^4}e_3,e_2+\frac{1}{2t^3}e_3,\frac{1}{t^2}e_3+\frac{1}{2t^5}e_4,\frac{1}{t^4}e_4)}$	$\mathcal{L}_{50}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1+t^{-1}e_3,e_2+t^{-1}e_3,e_3+t^{-1}e_4,2t^{-1}e_4)}$	$\mathcal{L}_{52}$
$\mathfrak{B}_{10}$	$\xrightarrow{(t^{-1}e_1+t^{-4}e_3,t^5e_2,t^3e_3+e_4,-te_4)}$	$\mathcal{L}_{51}$

$\mathfrak{B}_{02}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-2}e_3,t^{-4}e_4)}$	$\mathfrak{B}_{01}$
$\mathfrak{B}_{06}^{\frac{1+2t}{2}}$	$\xrightarrow{(-2te_2,-e_1+e_2,-2te_3,4t^2e_4)}$	$\mathfrak{B}_{02}$
$\mathfrak{B}_{02}$	$\xrightarrow{(t^{-1}e_1,t^{-2}e_1+t^{-1}e_2,t^{-2}e_3,t^{-4}e_4)}$	$\mathfrak{B}_{03}$
$\mathfrak{B}_{06}^{\alpha}$	$\xrightarrow{(t^{-1}e_1,t^{-1}e_2,t^{-2}e_3,t^{-4}e_4)}$	$\mathfrak{B}_{04}^{\alpha}$
$\mathfrak{B}_{06}^{\alpha}$	$\xrightarrow{(t^{-1}e_1,e_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathfrak{B}_{05}^{\alpha}$
$\mathfrak{B}_{06}^{2+t}$	$\xrightarrow{(\frac{1}{t}e_1-\frac{1}{\alpha t^2}e_3,\frac{1}{\alpha}e_2,\frac{1}{\alpha t}e_3+\frac{1}{\alpha^2t^2}e_4,\frac{1}{\alpha^2t^2}e_4)}$	$\mathfrak{B}_{07}^{\alpha\neq 0}$
$\mathfrak{B}_{06}^{2+t}$	$\xrightarrow{(e_1-\frac{1}{\alpha t}e_3,\frac{1}{\alpha}e_2,\frac{1}{\alpha}e_3+\frac{1}{\alpha^2t}e_4,\frac{1}{\alpha^2}e_4)}$	$\mathfrak{B}_{08}^{\alpha\neq 0}$
$\mathfrak{B}_{10}$	$\xrightarrow{(e_1,t^{-1}e_2,t^{-1}e_3,t^{-2}e_4)}$	$\mathfrak{B}_{09}$

Below we list all important reasons for necessary non-degeneration.

Non-degenerations reasons			
$\mathfrak{B}_{06}^{\alpha}$	$\not\to$	$\mathfrak{B}_{10}$	$\mathcal{R}=\left\{c_{1,2,1}^{3}=0,\ c_{2,3,2}^{4}=0\right\}$

Here, the coefficients $c_{i,j}^{k}$ and $c_{i,j,k}^{l}$ are structural constants in the basis $\{e_1,e_2,e_3,e_4\}$. □

2.4 The geometric classification of compatible Lie algebras

In this subsection, we determine all the irreducible components of the variety of three and four-dimensional nilpotent compatible Lie algebras.

The notions of orbit closure, degeneration, and irreducible components for algebras with two bilinear multiplications are defined analogously to those in Section 2.1; see, for example, [5]. The following results establish that the varieties of three- and four-dimensional nilpotent compatible Lie algebras each consist of a single irreducible component

Proposition 2.9. *The variety of three-dimensional nilpotent compatible Lie algebras is irreducible and defined by $\mathcal{C}=\overline{O(\mathfrak{L}_2^{\alpha})}$, and the dimension of this variety is equal to* 4.

Proof. After carefully checking the dimensions of orbit closures of three-dimensional nilpotent compatible Lie algebras, we have $\dim O(\mathfrak{L}_2^{\alpha})=4$. It is not difficult to verify the degeneration $\mathfrak{L}_2^{t^{-1}}\xrightarrow{(te_1,e_2,e_3)}\mathfrak{L}_1$. Therefore, we get that $\mathcal{C}=\overline{O(\mathfrak{L}_2^{\alpha})}$. □

Theorem 2.10. *The variety of four-dimensional nilpotent compatible Lie algebras is irreducible and defined by $\mathcal{C}=\overline{O(L_{10}^{\alpha})}$, and the dimension of this variety is equal to* 13.

Proof. After carefully checking the dimensions of orbit closures of the more important algebras, we have $\dim O(L_{10}^{\alpha})=13$.

The following degenerations are observed:

$L_{10}^{t^{-1}}$	$\xrightarrow{(e_1,t^2e_2,te_3,e_4)}$	L_{01}
L_{10}^{α}	$\xrightarrow{(te_1,e_2,te_3,e_4)}$	L_{02}^{α}
L_{10}^{0}	$\xrightarrow{(te_1,te_2,e_3,te_4)}$	L_{03}
L_{10}^{0}	$\xrightarrow{(te_1,e_2,te_3,te_4)}$	L_{04}
$L_{10}^{t^{-1}}$	$\xrightarrow{(te_2,e_1+\alpha e_2,-e_3,-e_4)}$	L_{05}^{α}
$L_{10}^{t^{-1}}$	$\xrightarrow{(e_2,e_1,-t^{-1}e_3,-t^{-1}e_4)}$	L_{06}
$L_{10}^{t^{-1}}$	$\xrightarrow{(t^2e_2,te_1-te_3,-t^2e_3,-t^3e_4)}$	L_{07}
L_{10}^{α}	$\xrightarrow{(te_2,e_1,-te_3,-te_4)}$	L_{08}^{α}
L_{10}^{α}	$\xrightarrow{(te_2,\beta e_1+e_2,-\beta te_3,-\beta te_4)}$	$L_{09}^{\alpha,\beta\neq 0}$

Hence, the variety of four-dimensional nilpotent compatible Lie algebras has a single irreducible component $\overline{O(L_{10}^{\alpha})}$. □

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